Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2002)

Problem Set 6

1. Note that parts (a) and (b) of this problem involve the same random variables defined in problems 3 and 4, respectively, of Problem Set 5.

   (a) The random variable X is exponentially distributed with parameter $\lambda = 1$. Derive the PDF $f_Z(z)$, where $Z = e^{3X}$.

   (b) Random variables X and Y have the joint PDF

   $$f_{X,Y}(x,y) = \begin{cases} c & , \text{ if } 0 \le y < x \le 2 \\ 0 & , \text{ otherwise} \end{cases} .$$

   Derive the PDF $f_Z(z)$, where $Z = \frac{Y}{X}$.

   (c) Random variables X and Y are independent and have the following PDFs,

   $$f_X(x) = \begin{cases} \frac{1}{2} & , \ 1 \le x \le 3 \\ 0 & , \ \text{otherwise} \end{cases} \quad , \quad f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & , \ y \ge 0 \\ 0 & , \ \text{otherwise} \end{cases}$$

   where λ is some arbitrary positive constant. Derive the PDF $f_Z(z)$, where $Z = X + Y$.

2. An ambulance travels back and forth, at a constant specific speed v, along a road of length ℓ. In other words, at any moment in time, consider the location of the ambulance to be uniformly distributed over the interval $(0, \ell)$. Also at any moment in time, an accident (not involving the ambulance itself) occurs at a point uniformly distributed on the road; that is, the accident's distance from one of the fixed ends of the road is also uniformly distributed over the interval $(0, \ell)$. Assume the location of the accident and the location of the ambulance are independent.

   (a) Supposing the ambulance is capable of *immediate* U-turns, compute the CDF and PDF of the ambulance's travel time T to the location of the accident.

   (b) Repeat part (a) but now suppose U-turns are only possible at either fixed end of the road.

3. Let continuous random variables X, Y and Z be independent and identically distributed according to the uniform distribution in the unit interval $[0, 1]$.

   (a) Consider two new random variables defined by $V = XY$ and $W = Z^2$. Derive the joint PDF $f_{V,W}(v, w)$.

   (b) Show that $\mathbf{P}(XY < Z^2) = \frac{5}{9}$.

4. Consider random variable Z with transform:

   $$M_Z(s) = \frac{a - 3s}{s^2 - 6s + 8}$$

   (a) Find the numerical value for parameter a.

   (b) Find $P(Z \ge 0.5)$.

(c) Find $E[Z]$ by using the probability distribution of Z.

(d) Find $E[Z]$ by using the tranform of Z and without explicity using the probability distribution of Z.

(e) Find $\text{var}(Z)$ by using the probability distribution of Z.

(f) Find $\text{var}(Z)$ by using the tranform of Z and without explicity using the probability distribution of Z.

5. Let X and Y be two IID Gaussian random variables with zero-mean and variance σ^2.

(a) Find the PDF of $R = \sqrt{X^2 + Y^2}$ (R is said to have a Rayleigh distribution.)

(b) Find the transform of R.

G1†. Let $X_1, X_2, \ldots, X_n$, where $n \geq 2$, be independent and identically-distributed continuous random variables with CDF $F(x)$ (and PDF $f(x)$). Define $Y = \max(X_1, \ldots, X_n)$, $Z = \min(X_1, \ldots, X_n)$ and $D = Y - Z$. Show that

(a)

$$F_{Y,Z}(y, z) = \mathbf{P}(Y \leq y, Z \leq z) = \begin{cases} F(y)^n - [F(y) - F(z)]^n & , \quad y > z \\ F(y)^n & , \quad y \leq z \end{cases}$$

(b)

$$F_D(d) = \begin{cases} n \int_{-\infty}^{\infty} [F(y) - F(y - d)]^{n-1} f(y) dy & , \quad d \geq 0 \\ 0 & , \quad d < 0 \end{cases}$$

HINT: One possible approach is to find first the joint PDF of Y and Z.

G2†. For a given continuous random variable X, define the random variable Y in the following manner:

$$Y = aX^2 + b.$$

(a) Find the PDF of Y in terms of the PDF of X, a and b.

(b) Let X be a Normal random variable: $X \sim \mathcal{N}(0, \sigma^2)$. Find the PDF of $Y = X^2$, and its moment generating function (Y is said to have a *central chi-square distribution.*)

(c) Now assume that $X \sim \mathcal{N}(\mu, \sigma^2)$. Find the PDF of $Y = X^2$ and its moment generating function (Y is now said the have a *non-central chi-square distribution.*)

†Required for 6.431; optional for 6.041