Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2002)

Problem Set 3

1. **Lottery.** Suppose you choose any r of the first n positive integers, and a lottery similarly chooses a random subset L of the same size. What is the probability that:
   (a) L includes no consecutive integers?
   (b) L includes exactly one pair of consecutive integers?
   (c) the numbers in L are drawn in increasing order?
   (d) your choice of numbers is the same as L?
   (e) there are exactly k of your numbers matching members of L?

2. Suppose we are given a coin for which the probability of heads is $\theta \in (0, 1)$ and the probability of tails is $1 - \theta$. Consider a sequence of independent flips of the coin.
   (a) Given that a total of exactly six heads resulted in the first nine flips, what is the conditional probability that both the first and seventh flips were tails?
   (b) Let Y be the number of flips up to and including the flip on which the first head occurs. Determine the pmf $p_Y(y)$ for all values of $y = 1, 2, \ldots$.
   (c) Let X be the number of heads that occur on any particular flip.
      i. Determine the expectation $E[X]$.
      ii. Determine the variance $\text{var}(X)$.
   (d) Let K be the number of heads that occur on the first n flips of the coin. Determine
      i. the pmf $p_K(k)$ for all values of $k = 0, 1, \ldots, n$
      ii. the expectation $E[K]$
      iii. the variance $\text{var}(K)$

      Hint: Your results from part (c) may be helpful in determining $E[K]$ and $\text{var}(K)$.

3. Let random variable X have a given pmf $p_X(x)$. We wish to guess, or *estimate*, the value of X before performing the actual experiment. Denoting our guess by $\hat{x}$, define the *mean square estimation error* by
$$e(\hat{x}) = E\left[(X - \hat{x})^2\right].$$
   Show that $e(\hat{x})$ is minimized by $\hat{x} = E[X]$.

4. A particular circuit works if all ten of its component devices work. Each circuit is tested before leaving the factory. Each working circuit can be sold for k dollars, but each nonworking circuit is worthless and must be thrown away. Each circuit can be built with either *ordinary* devices or *ultra-reliable* devices. An ordinary device has a failure probability of $q = 0.1$ while an ultra-reliable device has a failure probability of $q/2$, independent of any other device. However, each ordinary device costs \$1 whereas an ultra-reliable device costs \$3.

   Should you build your circuit with ordinary devices or ultra-reliable devices in order to maximize your expected profit $E[R]$? Keep in mind that your answer will depend on k.

G1[†]. Suppose there are n couples and it is desirable that men and women alternate when seated at a circular table. Without loss of generality, label the seats $1, 2, \ldots, 2n$ clockwise and dictate that seat 1 is occupied by a woman; note that this determines the sex of the occupant of every other seat. For $1 \le k \le 2n$, let A_k be the event that seats k and $k+1$ are occupied by one of the couples, identifying seat $2n+1$ with seat 1. We wish to ultimately prove that the probability that nobody is sitting next to his or her partner is

$$P\left(\bigcap_{k=1}^{2n} A_k^c\right) = \frac{2}{(n-1)!}\sum_{k=0}^{n}(-1)^k \binom{2n-k}{k}\frac{(n-k)!}{2n-k} \quad .$$

The following steps guide you through the proof.

(a) Show that

$$P(A_k) = n\left(\frac{(n-1)!}{n!}\right)^2 \quad .$$

(b) Show that, if $1 \le i < j \le 2n$, then

$$P(A_i \cap A_j) = \begin{cases} n(n-1)\left(\dfrac{(n-2)!}{n!}\right)^2 & , \quad |i-j| \neq 1 \text{ and } (i,j) \neq (1,2n) \\ 0 & , \quad \text{otherwise} \end{cases} \quad .$$

(c) Generalizing part (b), determine an expression for the probability $P\left(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\right)$ where $1 \le i_1 < i_2 < \cdots < i_k \le 2n$.

(d) Letting $S_{k,n}$ denote the number of ways of choosing k non-overlapping pairs of adjacent seats out of $2n$ seats, utilize results from parts (a)-(c) and show that

$$P\left(\bigcap_{k=1}^{2n} A_k^c\right) = \sum_{k=0}^{n}(-1)^k\left(\frac{(n-k)!}{n!}\right) S_{k,n} \quad .$$

(e) Show that, as defined in part (d),

$$S_{k,n} = \binom{2n-k}{k}\frac{2n}{2n-k} \quad .$$

G2[†]. An urn initially contains one red and one blue ball. At each stage a ball is randomly chosen (uniformly) and then replaced in the urn, along with another ball of the same color (that is, if the first selected ball is red, then that ball is placed back along with another red ball, making the urn contain 2 reds and one blue.) Let X denote the selection number of the first chosen ball that is blue. For example, if the first selected ball is red and the second blue, then $X = 2$. Assume that selections are independent.

(a) Find $P(X = i), \quad i \ge 1$.

(b) Show that the event "the blue ball is eventually chosen" occurs with probability 1.

(c) What can you say about the expected value of X? (i.e. how long, on average, will it take to draw the blue ball?)

[†]Required for 6.431; optional for 6.041