THE NULLSPACE OF A: SOLVING $AX = 0$ ■ 3.2

This section is about the space of solutions to $A\boldsymbol{x} = \mathbf{0}$. The matrix A can be square or rectangular. *One immediate solution is* $\boldsymbol{x} = \mathbf{0}$—and for invertible matrices this is the only solution. For other matrices, not invertible, there are nonzero solutions to $A\boldsymbol{x} = \mathbf{0}$. Each solution $\boldsymbol{x}$ belongs to the *nullspace* of A. We want to find all solutions and identify this very important subspace.

DEFINITION The ***nullspace of*** A ***consists of all solutions to*** $A\boldsymbol{x} = \mathbf{0}$**.** These vectors $\boldsymbol{x}$ are in $\mathbf{R}^n$. The nullspace containing all solutions $\boldsymbol{x}$ is denoted by $\boldsymbol{N}(A)$.

Check that the solution vectors form a subspace. Suppose $\boldsymbol{x}$ and $\boldsymbol{y}$ are in the nullspace (this means $A\boldsymbol{x} = \mathbf{0}$ and $A\boldsymbol{y} = \mathbf{0}$). The rules of matrix multiplication give $A(\boldsymbol{x} + \boldsymbol{y}) = \mathbf{0} + \mathbf{0}$. The rules also give $A(c\boldsymbol{x}) = c\mathbf{0}$. The right sides are still zero. Therefore $\boldsymbol{x} + \boldsymbol{y}$ and $c\boldsymbol{x}$ are also in the nullspace $\boldsymbol{N}(A)$. Since we can add and multiply without leaving the nullspace, it is a subspace.

To repeat: The solution vectors $\boldsymbol{x}$ have n components. They are vectors in $\mathbf{R}^n$, so *the nullspace is a subspace of* $\mathbf{R}^n$. The column space $\boldsymbol{C}(A)$ is a subspace of $\mathbf{R}^m$.

If the right side $\boldsymbol{b}$ is not zero, the solutions of $A\boldsymbol{x} = \boldsymbol{b}$ do *not* form a subspace. The vector $\boldsymbol{x} = \mathbf{0}$ is only a solution if $\boldsymbol{b} = \mathbf{0}$. When the set of solutions does not include $\boldsymbol{x} = \mathbf{0}$, it cannot be a subspace. Section 3.4 will show how the solutions to $A\boldsymbol{x} = \boldsymbol{b}$ (if there are any solutions) are shifted away from the origin by one particular solution.

Example 1 The equation $x + 2y + 3z = 0$ comes from the 1 by 3 matrix $A = [\,1 \quad 2 \quad 3\,]$. This equation produces a plane through the origin. The plane is a subspace of $\mathbf{R}^3$. *It is the nullspace of* A. The solutions to $x + 2y + 3z = 6$ also form a plane, but not a subspace.

Example 2 Describe the nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

Solution Apply elimination to the linear equations $A\boldsymbol{x} = \mathbf{0}$:

$$\begin{bmatrix} x_1 + 2x_2 = 0 \\ 3x_1 + 6x_2 = 0 \end{bmatrix} \quad \to \quad \begin{bmatrix} x_1 + 2x_2 = 0 \\ 0 = 0 \end{bmatrix}$$

There is really only one equation. The second equation is the first equation multiplied by 3. In the row picture, the line $x_1 + 2x_2 = 0$ is the same as the line $3x_1 + 6x_2 = 0$. That line is the nullspace $\boldsymbol{N}(A)$.

To describe this line of solutions, the efficient way is to give one point on it (one special solution). Then all points on the line are multiples of this one. We choose the second component to be $x_2 = 1$ (a special choice). From the equation $x_1 + 2x_2 = 0$, the first component must be $x_1 = -2$. Then the special solution produces the whole nullspace:

$$\text{The nullspace } \boldsymbol{N}(A) \text{ contains all multiples of } \boldsymbol{s} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

This is the best way to describe the nullspace, by computing special solutions to $A\boldsymbol{x} = \boldsymbol{0}$. ***The nullspace consists of all combinations of those special solutions***. This example has one special solution and the nullspace is a line.

For the plane in Example 1 there are two special solutions:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \text{ has the special solutions } \boldsymbol{s}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{s}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

Those vectors $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ lie on the plane $x + 2y + 3z = 0$, which is the nullspace of $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. All vectors on the plane are combinations of $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$.

Notice what is special about $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ in this example. They have ones and zeros in the last two components. Those components are "free" and we choose them specially. Then the first components -2 and -3 are determined by the equation $A\boldsymbol{x} = \boldsymbol{0}$.

The first column of $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ contains the *pivot*, so the first component of $\boldsymbol{x}$ is *not free*. We only make a special choice (one or zero) of the free components that correspond to columns without pivots. This description of special solutions will be completed after one more example.

Example 3 Describe the nullspaces of these three matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \text{ and } B = \begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix} \text{ and } C = \begin{bmatrix} A & 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}.$$

Solution The equation $A\boldsymbol{x} = \boldsymbol{0}$ has only the zero solution $\boldsymbol{x} = \boldsymbol{0}$. The nullspace is $\mathbf{Z}$, containing only the single point $\boldsymbol{x} = \boldsymbol{0}$ in $\mathbf{R}^2$. To see this we use elimination:

$$\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields } \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} x_1 = 0 \\ x_2 = 0 \end{bmatrix}.$$

The square matrix A is invertible. There are no special solutions. The only vector in its nullspace is $\boldsymbol{x} = \boldsymbol{0}$.

The rectangular matrix B has the same nullspace $\mathbf{Z}$. The first two equations in $B\boldsymbol{x} = \boldsymbol{0}$ again require $\boldsymbol{x} = \boldsymbol{0}$. The last two equations would also force $\boldsymbol{x} = \boldsymbol{0}$. When we add more equations, the nullspace certainly cannot become larger. When we add extra rows to the matrix, we are imposing more conditions on the vectors $\boldsymbol{x}$ in the nullspace.

The rectangular matrix C is different. It has extra columns instead of extra rows. The solution vector $\boldsymbol{x}$ has *four* components. Elimination will produce pivots in the first two columns, but the last two nonpivot columns are "free":

$$C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \text{ becomes } U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

$$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$

pivot columns free columns

For the free variables x_3 and x_4, we make the special choices of ones and zeros. Then the pivot variables x_1 and x_2 are determined by the equation $U\boldsymbol{x} = \mathbf{0}$. We get two special solutions in the nullspace of C (and also the nullspace of U). The special solutions are:

$$\boldsymbol{s}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{s}_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \begin{matrix} \leftarrow \text{ pivot} \\ \leftarrow \text{ variables} \\ \leftarrow \text{ free} \\ \leftarrow \text{ variables} \end{matrix}$$

One more comment to anticipate what is coming soon. Elimination will not stop at the upper triangular U! We continue to make this matrix simpler, in two ways:

1. ***Produce zeros above the pivots***, by eliminating upward.

2. ***Produce ones in the pivots***, by dividing the whole row by its pivot.

Those steps don't change the zero vector on the right side of the equation. The nullspace stays the same. This nullspace becomes easy to see when we reach the ***reduced row echelon form*** R:

$$U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \quad \text{becomes} \quad R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

$$\uparrow \quad \uparrow$$

pivot columns contain I

I subtracted row 2 of U from row 1, and then I multiplied row 2 by $\frac{1}{2}$. The original two equations have simplified to $x_1 + 2x_3 = 0$ and $x_2 + 2x_4 = 0$. Those are the equations $R\boldsymbol{x} = \mathbf{0}$ with the identity matrix in the pivot column.

The special solutions are still the same $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$. They are much easier to find from the reduced system $R\boldsymbol{x} = \mathbf{0}$.

Before moving to m by n matrices A and their nullspaces $\boldsymbol{N}(A)$ and the special solutions in the nullspace, allow me to repeat one comment. For many matrices, the only solution to $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$. Their nullspaces contain only that single vector $\boldsymbol{x} = \mathbf{0}$. The only combination of the columns that produces $\boldsymbol{b} = \mathbf{0}$ is then the "zero combination" or "trivial combination". The solution is trivial (just $\boldsymbol{x} = \mathbf{0}$) but the idea is not trivial.

This case of a zero nullspace $\mathbf{Z}$ is of the greatest importance. It says that the columns of A are independent. No combination of columns gives the zero vector except the zero combination. All columns have pivots and no columns are free. You will see this idea of independence again ....

Solving $Ax = 0$ by Elimination

This is important. We solve m equations in n unknowns—and the right sides are all zero. The left sides are simplified by row operations, after which we read off the solution (or solutions). Remember the two stages in solving $A\boldsymbol{x} = \mathbf{0}$:

1. Forward elimination from A to a triangular U (or its reduced form R).

2. Back substitution in $U\boldsymbol{x} = \mathbf{0}$ or $R\boldsymbol{x} = \mathbf{0}$ to find $\boldsymbol{x}$.

You will notice a difference in back substitution, when A and U have fewer than n pivots. *We are allowing all matrices in this chapter,* not just the nice ones (which are square matrices with inverses).

Pivots are still nonzero. The columns below the pivots are still zero. But it might happen that a column has no pivot. In that case, don't stop the calculation. Go on to the next column. The first example is a 3 by 4 matrix:

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix}.$$

Certainly $a_{11} = 1$ is the first pivot. Clear out the 2 and 3 below that pivot:

$$A \to \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \end{bmatrix} \qquad \begin{matrix} \\ \text{(subtract } 2 \times \text{ row 1)} \\ \text{(subtract } 3 \times \text{ row 1)} \end{matrix}$$

The second column has a zero in the pivot position. We look below the zero for a nonzero entry, ready to do a row exchange. *The entry below that position is also zero.* Elimination can do nothing with the second column. This signals trouble, which we expect anyway for a rectangular matrix. There is no reason to quit, and we go on to the third column.

The second pivot is 4 (but it is in the third column). Subtracting row 2 from row 3 clears out that column below the pivot. We arrive at

$$U = \begin{bmatrix} \mathbf{1} & 1 & 2 & 3 \\ 0 & 0 & \mathbf{4} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} (\textbf{\textit{only two pivots}}) \\ (\textbf{\textit{the last equation}} \\ \textbf{\textit{became}}\ 0 = 0) \end{matrix}$$

The fourth column also has a zero in the pivot position—but nothing can be done. There is no row below it to exchange, and forward elimination is complete. The matrix has three rows, four columns, and *only two pivots*. The original $A\mathbf{x} = \mathbf{0}$ seemed to involve three different equations, but the third equation is the sum of the first two. It is automatically satisfied $(0 = 0)$ when the first two equations are satisfied. Elimination reveals the inner truth about a system of equations.

Now comes back substitution, to find all solutions to $U\mathbf{x} = \mathbf{0}$. With four unknowns and only two pivots, there are many solutions. The question is how to write them all down. A good method is to separate the ***pivot variables*** from the ***free variables***.

P The ***pivot*** variables are x_1 and x_3, since columns 1 and 3 contain pivots.
F The ***free*** variables are x_2 and x_4, because columns 2 and 4 have no pivots.

The free variables x_2 and x_4 can be given any values whatsoever. Then back substitution finds the pivot variables x_1 and x_3. (In Chapter 2 no variables were free. When A is invertible, all variables are pivot variables.) The simplest choices for the free variables are ones and zeros. Those choices give the *special solutions*.

Special Solutions

- Set $x_2 = 1$ and $x_4 = 0$. By back substitution $x_3 = 0$ and $x_1 = -1$.
- Set $x_2 = 0$ and $x_4 = 1$. By back substitution $x_3 = -1$ and $x_1 = -1$.

These special solutions solve $U\mathbf{x} = \mathbf{0}$ and therefore $A\mathbf{x} = \mathbf{0}$. They are in the nullspace. The good thing is that *every solution is a combination of the special solutions*.

Complete Solution
$$\mathbf{x} = x_2 \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\textbf{special}} + x_4 \underbrace{\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\textbf{special}} = \underbrace{\begin{bmatrix} -x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix}}_{\textbf{complete}}. \tag{1}$$

Please look again at that answer. It is the main goal of this section. The vector $\mathbf{s}_1 = (-1, 1, 0, 0)$ is the special solution when $x_2 = 1$ and $x_4 = 0$. The second special solution has $x_2 = 0$ and $x_4 = 1$. ***All solutions are linear combinations of $\mathbf{s}_1$ and $\mathbf{s}_2$.*** The special solutions are in the nullspace $\mathbf{N}(A)$, and their combinations fill out the whole nullspace.

The MATLAB code **nulbasis** computes these special solutions. They go into the columns of a ***nullspace matrix*** N. The complete solution to $A\mathbf{x} = \mathbf{0}$ is a combination of those columns. Once we have the special solutions, we have the whole nullspace.

There is a special solution for each free variable. If no variables are free—this means there are n pivots—then the only solution to $U\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{0}$ is the trivial solution $\boldsymbol{x} = \mathbf{0}$. All variables are pivot variables. In that case the nullspaces of A and U contain only the zero vector. With no free variables, and pivots in every column, the output from **nulbasis** is an empty matrix.

Example 4 Find the nullspace of $U = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 0 & 9 \end{bmatrix}$.

The second column of U has no pivot. So x_2 is free. The special solution has $x_2 = 1$. Back substitution into $9x_3 = 0$ gives $x_3 = 0$. Then $x_1 + 5x_2 = 0$ or $x_1 = -5$. The solutions to $U\boldsymbol{x} = \mathbf{0}$ are multiples of one special solution:

$$\boldsymbol{x} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$$

The nullspace of U is a line in $\mathbf{R}^3$.
It contains multiples of the special solution.
One variable is free, and $N =$ **nulbasis** (U) has one column.

In a minute we will continue elimination on U, to get *zeros above the pivots and ones in the pivots*. The 7 is eliminated and the pivot changes from 9 to 1. The final result of this elimination will be R:

$$U = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 0 & 9 \end{bmatrix} \text{ reduces to } R = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This makes it even clearer that the special solution is $\boldsymbol{s} = (-5, 1, 0)$.

Echelon Matrices

Forward elimination goes from A to U. The process starts with an m by n matrix A. It acts by row operations, including row exchanges. It goes on to the next column when no pivot is available in the current column. The m by n "staircase" U is an ***echelon matrix***.

Here is a 4 by 7 echelon matrix with the three pivots highlighted in boldface:

$$U = \begin{bmatrix} \boldsymbol{x} & x & x & x & x & x & x \\ 0 & \boldsymbol{x} & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{x} & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Three pivot variables x_1, x_2, x_6
Four free variables x_3, x_4, x_5, x_7
Four special solutions in $\boldsymbol{N}(U)$

Question What are the column space and the nullspace for this matrix?

Answer The columns have four components so they lie in $\mathbf{R}^4$. (Not in $\mathbf{R}^3$!) The fourth component of every column is zero. Every combination of the columns—every vector in the column space—has fourth component zero. *The column space $\boldsymbol{C}(U)$ consists of all vectors of the form* $(b_1, b_2, b_3, 0)$. For those vectors we can solve $U\boldsymbol{x} = \boldsymbol{b}$ by back substitution. These vectors $\boldsymbol{b}$ are all possible combinations of the seven columns.

The nullspace $\boldsymbol{N}(U)$ is a subspace of $\mathbf{R}^7$. The solutions to $U\boldsymbol{x} = \mathbf{0}$ are all the combinations of the four special solutions—*one for each free variable*:

1. Columns 3, 4, 5, 7 have no pivots. So the free variables are x_3, x_4, x_5, x_7.

2. Set one free variable to 1 and set the other free variables to zero.

3. Solve $U\boldsymbol{x} = \mathbf{0}$ for the pivot variables x_1, x_2, x_6.

4. This gives one of the four special solutions in the nullspace matrix N.

The nonzero rows of an echelon matrix come first. The pivots are the first nonzero entries in those rows, and they go down in a staircase pattern. The usual row operations (in the Teaching Code **plu**) produce a column of zeros below every pivot.

Counting the pivots leads to an extremely important theorem. Suppose A has more columns than rows. ***With*** $n > m$ ***there is at least one free variable.*** The system $A\boldsymbol{x} = \mathbf{0}$ has at least one special solution. This solution is *not zero*!

3B If $A\boldsymbol{x} = \mathbf{0}$ has more unknowns than equations (A has more columns than rows), then it has nonzero solutions.

In other words, a short wide matrix ($n > m$) always has nonzero vectors in its nullspace. There must be at least $n - m$ free variables, since the number of pivots cannot exceed m. (The matrix only has m rows, and a row never has two pivots.) Of course a row might have *no* pivot—which means an extra free variable. But here is the point: When there is a free variable, it can be set to 1. Then the equation $A\boldsymbol{x} = \mathbf{0}$ has a nonzero solution.

To repeat: There are at most m pivots. With $n > m$, the system $A\boldsymbol{x} = \mathbf{0}$ has a free variable and a nonzero solution. Actually there are infinitely many solutions, since any multiple $c\boldsymbol{x}$ is also a solution. The nullspace contains at least a line of solutions. With two free variables, there are two special solutions and the nullspace is even larger.

The nullspace is a subspace. Its "dimension" is the number of free variables. This central idea—the ***dimension*** of a subspace—is defined and explained in this chapter.

The Reduced Echelon Matrix R

From the echelon matrix U we can go one more step. Continue onward from

$$U = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can divide the second row by 4. Then both pivots equal 1. *We can subtract* 2 *times this new row* $\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$ *from the row above.* That produces a zero above the second pivot as well as below. The ***reduced row echelon matrix*** is

$$R = \begin{bmatrix} \mathbf{1} & 1 & 0 & 1 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

R has 1's as pivots. It has 0's everywhere else in the pivot columns. Zeros above pivots come from ***upward elimination***.

If A is invertible, its reduced row echelon form is the identity matrix $R = I$. This is the ultimate in row reduction.

The zeros in R make it easy to find the special solutions (the same as before):

1. Set $x_2 = 1$ and $x_4 = 0$. Solve $R\boldsymbol{x} = \mathbf{0}$. Then $x_1 = -1$ and $x_3 = 0$.

2. Set $x_2 = 0$ and $x_4 = 1$. Solve $R\boldsymbol{x} = \mathbf{0}$. Then $x_1 = -1$ and $x_3 = -1$.

The numbers -1 and 0 are sitting in column 2 of R (with plus signs). The numbers -1 and -1 are sitting in column 4 (with plus signs). By reversing signs we can read off the special solutions from the matrix R. The general solution to $A\boldsymbol{x} = \mathbf{0}$ or $U\boldsymbol{x} = \mathbf{0}$ or $R\boldsymbol{x} = \mathbf{0}$ is a combination of those two special solutions: ***The nullspace*** $\boldsymbol{N}(A) = \boldsymbol{N}(U) = \boldsymbol{N}(R)$ ***contains***

$$\boldsymbol{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = (\textbf{\textit{complete solution of }} A\boldsymbol{x} = \mathbf{0}).$$

The next section of the book moves firmly from U to R. The MATLAB command $[\,R, pivcol\,] = \mathbf{rref}(A)$ produces R and also a list of the pivot columns.

■ REVIEW OF THE KEY IDEAS ■

1. The nullspace $\boldsymbol{N}(A)$ contains all solutions to $A\boldsymbol{x} = \mathbf{0}$.

2. Elimination produces an echelon matrix U, or a row reduced R, with pivot columns and free columns.

3. Every free column leads to a special solution to $A\boldsymbol{x} = \mathbf{0}$. The free variable equals 1 and the other free variables equal 0.

4. The complete solution to $A\boldsymbol{x} = \mathbf{0}$ is a combination of the special solutions.

5. If $n > m$ then A has at least one column without pivots, giving a special solution. So there are nonzero vectors $\boldsymbol{x}$ in the nullspace of this A.