**Field-theoretic RG. Wilson-Fisher fixed point.**

**1. Wick theorem.** Consider a real-valued $n$-component field $\eta_i(\mathbf{x})$ with gaussian probability distribution

$$P(\eta) \propto \exp(-\mathcal{H}), \quad \mathcal{H}(\phi) = \int \frac{1}{2}\left(\tau\eta^2 + K(\nabla\eta)^2\right) d^dx \tag{1}$$

We are interested in statistics of Fourier components of this field,

$$\eta_i(\mathbf{x}) = \int \eta_i(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{x}} d^dx \tag{2}$$

a) Show that

$$\langle\eta_i(\mathbf{q})\rangle = 0\,, \quad \langle\eta_i(\mathbf{q})\eta_{i'}(\mathbf{q}')\rangle = \frac{\delta_{ii'}\delta(\mathbf{q}+\mathbf{q}')}{\tau + K\mathbf{q}^2}, \tag{3}$$

b) Show that

$$\langle\eta_i(\mathbf{q})\eta_{i'}(\mathbf{q}')\eta_{i''}(\mathbf{q}'')\rangle = 0\,, \quad \langle\eta_{i_1}(\mathbf{q}_1)\eta_{i_2}(\mathbf{q}_2)\eta_{i_3}(\mathbf{q}_3)\eta_{i_4}(\mathbf{q}_4)\rangle = \langle\eta_{i_1}(\mathbf{q}_1)\eta_{i_2}(\mathbf{q}_2)\rangle\langle\eta_{i_3}(\mathbf{q}_3)\eta_{i_4}(\mathbf{q}_4)\rangle$$
$$+\langle\eta_{i_1}(\mathbf{q}_1)\eta_{i_3}(\mathbf{q}_3)\rangle\langle\eta_{i_2}(\mathbf{q}_2)\eta_{i_4}(\mathbf{q}_4)\rangle + \langle\eta_{i_1}(\mathbf{q}_1)\eta_{i_4}(\mathbf{q}_4)\rangle\langle\eta_{i_3}(\mathbf{q}_3)\eta_{i_2}(\mathbf{q}_2)\rangle \tag{4}$$

c) Generalize the results for an average of an arbitrary product $\langle\eta_{i_1}(\mathbf{q}_1)...\eta_{i_n}(\mathbf{q}_n)\rangle$.

**2. Field statistics. Cummulants.** Consider a partition function for a field $\phi(\mathbf{x})$ defined by a Hamiltonian

$$\mathcal{H}(\phi) = \int F(\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi, ...)d^dx \tag{5}$$

where $F$ is a polynomial in the field and its gradients. (For example, $F = \frac{1}{2}\left(\tau\phi^2 + K(\nabla\phi)^2\right) + g\phi^4$ in Landau theory.) The Hamiltonian $\mathcal{H}$ defines a probability distribution $P \propto e^{-\beta\mathcal{H}}$ of fields and can be used to find an average of any quantity expressed in terms of $\phi(\mathbf{x})$.

Let us consider an average of an exponential of an extensive quantity of the form

$$P_V(\lambda) = \langle\exp(\lambda G(\phi))\rangle_{\mathcal{H}}\,, \quad G = \int U(\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi, ...)d^dx \tag{6}$$

where $\lambda$ is a parameter, $U$ is a polynomial and the integral over $\mathbf{x}$ is taken over a domain of a very large volume $V$.

a) Show that, if correlations of $\phi$ are finite range or decrease with distance sufficiently rapidly, the quantity $P_V(\lambda)$ is exponential in $V$. (Hint: Divide volume $V$ into $V_1$ and $V_2$, so that $V_1 + V_2 = V$, and think of a relation between $P_V(\lambda)$, $P_{V_1}(\lambda)$, and $P_{V_2}(\lambda)$.)

b) Consider an expansion of $\ln P_V(\lambda)$ in powers of $\lambda$ and show that

$$\ln P_V(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n}\lambda^n\langle\langle G^n\rangle\rangle \tag{7}$$

where all *cummulants* $\langle\langle G^n \rangle\rangle$, also known as *irreducible moments*, are proportional to system volume $V$.

c) By expanding the exponent in Eq.(6) and the logarithm in Eq.(7), relate the first four cummulants with the moments $\langle G^n \rangle$ by showing that

$$\langle\langle G \rangle\rangle = \langle G \rangle\,, \quad \langle\langle G^2 \rangle\rangle = \langle \delta G^2 \rangle\,, \quad \langle\langle G^3 \rangle\rangle = \langle \delta G^3 \rangle\,, \quad \langle\langle G^4 \rangle\rangle = \langle \delta G^4 \rangle - 3\langle \delta G^2 \rangle^2 \qquad (8)$$

where $\delta G = G - \langle G \rangle$. One can view cummulants as generalized moments of the distribution with a nontrivial property that the cummulants of all orders are proportional to system volume, $\langle\langle G^n \rangle\rangle \propto V$, while the moments $\langle G^n \rangle \propto V^n$

**3. Cubic anisotropy.** Consider Landau Hamiltonian for an $n$-component field $\mathbf{m}$ with a quartic anisotropy term:

$$\mathcal{H} = \int \left[ \frac{1}{2} \left( \tau \mathbf{m}^2 + K(\nabla \mathbf{m})^2 \right) + u((\nabla \mathbf{m})^2)^2 + v \sum_{i=1}^{n} m_i^4 \right] d^d x \qquad (9)$$

This problem with $n = 3$ describes a critical point of Heisenberg ferromagnet in a cubic crystal with a weak spin-orbital coupling of strength $v$ that tends to align spins with crystal axes.

Following the steps that led us in class to the Wilson-Fisher fixed point, derive RG flow equations for the two couplings:

$$\frac{du}{dl} = \epsilon u - 4C\left[(n+8)u^2 + 6uv\right]\,, \quad \frac{dv}{dl} = \epsilon v - 4C\left[12uv + 9v^2\right] \qquad (10)$$

where $C = K_d \Lambda^d / (\tau + K\Lambda^2)^2 \approx K_4/K^2$ is a constant.

Analyze the RG flow in the $u - v$ plane. Find fixed points for $n < 4$ and $n > 4$, and discuss relevance of the cubic anisotropy at the critical point in each case.