Random Matrices, Quantum Chaos and Irreversible Classical Dynamics

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by

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B.S. Physics, Moscow Institute of Physics and Technology (1990)
M.S. Physics, The Johns Hopkins University (1993)

Submitted to the Department of Physics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

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Abstract

The main result of this thesis is a novel semiclassical description of quantum chaos. It is based on a field theoretic approach in the form of a supersymmetric nonlinear σ-model and relates quantum spectral correlators to the underlying irreversible classical dynamics. This method provides justification for use of random matrix models in problems of quantum chaos. I also present several applications of the supersymmetry method to the study of correlators which appear in the the context of mesoscopics and quantum chaos.

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Chapter 1

Introduction

1.1 Background and Motivation

The problem of quantum description of classically chaotic systems has interested researchers since the early days of quantum mechanics. It was Einstein [1] who first pointed out the problems that arise in semiclassical quantization of nonintegrable systems. He tried to obtain the analogue of the Born-Sommerfeld quantization rule for multidimensional systems and realized that a straightforward generalization can be done only in the integrable case.

A long time after his paper the interest in semiclassical approach to quantization of chaotic systems was revived by Gutzwiller (for references see [63]). He recognized the importance of periodic orbits as the "skeleton" of classical dynamics. The intuition behind this is as follows [2]: eigenvalues of a quantum Hamiltonian correspond to stationary states of the system and should, in turn, be related to the invariant manifolds of the classical counterpart. The latter are defined as regions of the phase space of the system, which remain invariant under classical evolution. In a nonintegrable system there are only two types of invariant manifolds: the whole constant energy surface and the periodic orbits of the sytems. This should be contrasted to the integrable case, where due to the presence of invariant tori there is a whole spectrum invariant manifolds whose dimension ranges from unity (periodic orbits) to the dimension of the constant energy shell.

This is the main idea of the periodic orbit theory (POT) approach to quantum chaos. This method is applied to the study of quantum spectra of chaotic systems whose classical dynamics are very simple but nonetheless chaotic. The spectrum of a quantum system is related to the periodic orbits of the underlying classical dynamics, i.e. the density of states is expressed as an infinite sum over the periodic orbits of the system. This method describes well the large scale structure of the spectrum which corresponds to the short time dynamics, but is faced with problems when it is used to study the small frequency behavior, for example for the probability to find two levels close to each other. The difficulty arises from the exponential proliferation of the number of orbits with their period.

A different approach to quantum chaotic systems was pioneered by Wigner [5] who studied crossings of close levels in complex molecules. One of the manifestations of quantum chaos is absence of degeneracies in the quantum spectrum. They usually arise due to some symmetry transformations which leave the Hamiltonian invariant. As was shown by Noeter [3] these symmetries lead to the existence of integrals of motion. In the absence of integrals of motion the quantum Hamiltonian posesses no continuous symmetries and is expected to have no degeneracies. This manifests itself in level statistics through the suppressed probability of finding two levels next to each other. Such a phenomenon is known as "level repulsion" and is the most sailient characteristic of spectra of chaotic systems. The distribution of spacings between nearest neighbours is described extremely accurately by the Wigner surmize (see for example [7]).

Dyson [54] discovered that at small energy separations level statistics of chaotic systems become unversal, i.e. independent of the details of the Hamiltonian. Most of chaotic sytems are rather complex, e.g. big molecules, heavy nuclei etc, and our knowledge of their dynamics is minimal. Dyson proposed to describe spectral statistics of such systems by modeling their quantum Hamiltonians by an ensemble of hermitean matrices whose entries are independently distributed random quantities. Random matrix theory (RMT) description of quantum chaos applies to small energy scales (of order of the mean level spacing in the system) and has proved remarcably

successful. It has originally been applied to the study of spectra of nuclear resonances [54]. Bohigas Giannoni and Schmit [87] proposed that RMT also should describe spectral statistics of rather simple nonintegrable systems whose Hamilonians are very well known (BGS conjecture), such as Sinai billiards or Rydberg atoms in a magnetic field. Extensive numerical and experimental data [4] provide strong support for this conjecture. However, the theoretical understanding of the success of RMT in quantum chaos has been lacking.

Recently RMT was used for the study of electron transport in mesoscopic disordered conductors at low temperatures. In these systems the electron phase coherence length can excede the size of the sample and quantum mechanical interference becomes important. It gives rise to extreme sensitivity of transport coeffecients (or energy levels if one speaks about a closed metallic grain) on the details of the disorder potential, sample shape etc. Therefore the sample-to-sample variations of transport coeffecients (energy levels) are surprisingly large and known as mesoscopic fluctuations. One typically studies ensemble averaged properties of mesoscopic systems, where averaging is performed over different realizations of the disorder potential. The resistance in this regime can be expressed through the quantum transmission matrix by use of the Landauer formula (see [26]). The ensemble averaged quantities can be obtained by averaging over a random matrix ensemble of transmission matrices. There is an extensive literature on the applications of RMT to quantum chaos and mesoscopic transport (see for example [7], [65]), [26]).

The microscopic theory which describes ensemble averaged quantities is based on the field theory in the form of the σ-model. It was originally proposed by Wegner [6] in the replica form. The supersymmetric version of it was developed by Efetov [20]. The use of replicas or supersymmetry facilitates ensemble averaging in these approaches. The supersymmetric nonlinear σ-model [20] provides microscopic justification for the existence of universal limit in spectral statistics when ensemble average is performed. This technique is very useful whenever one is interested in computing quenched averages and can also be applied to the study of various correlators of random matrix ensembles [21].

1.2 Organization of the material

The organization of the thesis is as follows:

In chapter 2 we apply RMT to the problem of coherent electron transport through a mesoscopic quantum wire. To facilitate ensemble averaging we use a new form of the supersymmetric nonlinear σ-model developed in [12]. This σ-model is now used in the study of disordered mormal metal grains surrounded by superconductor [60].

In chapter 3 we generalize the supersymmetry method of Guhr [35] to compute correlators of spectral determinants of systems with broken time reversal invariance in the universal regime. We obtain algebraic expressions for spectral determinant correlators of arbitrary order. As an application of the method we compute several correlators of physical observables which arise in the study of problems of quantum chaos.

In chapter 4 we present non-perturbative results [51] for the two-point density of states correlator of disordered metallic grains. The results show that quantum spectral statistics can be expressed through the spectral determinant of the classical diffusion operator in the grain. We also obtain an expression for the spectral structure factor close to the Heisenberg time. These results cannot be obtained by the usual semeclassical methods and require the use of the σ-model.

In chapter 5 we show how to generalize the results obtained in chapter 4 to a general chaotic system [62]. The classical operator which plays the role of the diffusion operator for a general system is identified with the Perron-Frobenius operator. The latter describes the irreversible classical dynamics.

In chapter 6 we present a new semiclassical description of quantum chaos [74] based on a field theory in the form of a nonlinear σ-model. It relies only on energy averaging and accounts for individual features of a given system. This approach is free from the drawbacks of periodic orbit theory, it provides the theoretical justification for the BGS conjecture [87] and the results presented in chapter 5. The main ingredients of the theory are eigenmodes of the irreversible classical dynamics in the phase space of the system rather than individual periodic orbits. It relates quantum spectral

statistics to the Ruelle resonances [9] which characterize chaotic classical dynamics. This theory is used to obtain new results for the two-point density of states correlator.

Chapter 2

Supersymmetry Applied to the Spectrum Edge of Random Matrix Ensembles

A new matrix ensemble has recently been proposed to describe the transport properties in mesoscopic quantum wires. Both analytical and numerical studies have shown that the ensemble of Laguerre or of chiral random matrices provides a good description of scattering properties in this class of systems. Until now only conventional methods of random matrix theory have been used to study statistical properties within this ensemble. In this chapter we demonstrate that the supersymmetry method, already employed in the study Dyson ensembles, can be extended to treat this class of random matrix ensembles. We follow closely Ref. [12]. In developing this approach we investigate both new, as well as verify known statistical measures. Although we focus on ensembles in which T-invariance is violated this approach lays the foundation for future studies of T-invariant systems. The supersymmetric treatment presented here can be generalized to treat islands of normal metal imbedded in a superconductor [13].

2.1 Introduction

Recently Slevin and Nagao [10] proposed the existence of a new group structure associated with scattering in disordered mesoscopic quantum wires. Further numerical investigation [11] verified the utility of this description. In the case of broken time-reversal invariance the maximum entropy hypothesis for this new group gives rise to the Laguerre unitary ensemble of random matrices (LUE). The same random matrix ensemble appears in the context of QCD [14], and taking our notation from there, we will refer to it as the chiral gaussian unitary ensemble (chGUE). The characteristic feature of this ensemble which distinguishes it from conventional Dyson ensembles is the presence of a "hard edge [15]" or boundary to the spectrum. This has a dramatic effect on eigenvalue correlations in the vicinity of the edge which we will explorein this paper.

Although there exist a variety of approaches directed towards the study of the statistical properties of Dyson random matrix ensembles, so far only the "orthogonal polynomial method [7]" has found success in describing the chGUE [10, 11, 14, 15, 16, 17, 18, 19]. At the same time, approaches based on the "supersymmetry method" introduced by Efetov [20] have found considerable success in treating both spectral and scattering properties [21, 22, 23] of mesoscopic quantum dots and compound nuclei. In this chapter we present the supersymmetry approach developed in [12] for the study of parametric correlations of the chGUE in response to an external perturbation. A first task will be to determine properties of eigenvalue correlations already studied using orthogonal polynomials [11] and the Brownian motion method [24]. In doing so we will reveal new types of Goldstone modes which are associated with the structure of chGUE at the hard-edge. Later we will turn to a new characterization based on matrix element correlations of the chGUE. Although we focus on the unitary ensemble, we remark that the supersymmetry approach developed here seems at present to be the only one capable of treating parametric correlations in T-invariant orthogonal and symplectic ensembles.

To emphasize the validity of this ensemble in the problem of scattering and mo-

tivate our calculation we begin with a brief review following the studies of Ref. [11]. Consider a scatterer S (which might be a quantum dot, a chaotic cavity, or a mesoscopic quantum wire) connected to two identical ideal leads L and R (see Fig. 2-1). Although the leads contain both propagating and evanescent modes, the latter do not contribute to the total current and can be disregarded [11]. Normalizing the propagating modes to unit flux and denoting the amplitudes of the incoming waves in L and R respectively as i and i', and those of the outgoing waves in L and R as o and o', the amplitudes o and o' can be uniquely expressed through i and i' with the use of the scattering matrix S

$$S\begin{pmatrix} i \\ i' \end{pmatrix} = \begin{pmatrix} o \\ o' \end{pmatrix}; \qquad S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}, \tag{2.1}$$

where r (r'), t (t') are the $N \times N$ reflection and transmission matrices on the left (right) hand side.

Equivalently, instead of using S, we can reexpress Eq. (2.1) using the transfer matrix M which directly connects the amplitudes in the lead R with those in the lead L

$$M\begin{pmatrix} i \\ o \end{pmatrix} = \begin{pmatrix} o' \\ i' \end{pmatrix}. \tag{2.2}$$

From the definitions in Eqs. (2.1,2.2) it is possible to determine the conductance through the Landauer relation [25, 26],

$$G = \frac{2e^2}{h}\mathrm{tr}\, tt^\dagger = \frac{2e^2}{h}\mathrm{tr}\, \frac{4}{MM^\dagger + (MM^\dagger)^{-1} + 2}, \tag{2.3}$$

where the factor of 2 takes account of the spin degeneracy of each channel. With the chosen normalization of the wave functions in the leads the current conservation implies the unitarity of the S-matrix and pseudo-unitarity of the transfer matrix M, i. e.

$$M\Sigma_z M^\dagger = \Sigma_z \mathbf{1}_N \equiv \begin{pmatrix} \mathbf{1}_N & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_N \end{pmatrix}, \tag{2.4}$$

where Σ_z denotes the corresponding Pauli matrix. ($\Sigma_{x,y}$ are defined analogously below.)

To establish the connection to the chGUE, Slevin and Nagao introduced the matrix $\Omega = \ln(MM^\dagger)$. As shown in Eqn. (2.3) this matrix contains enough information to define the transmission properties, and the conductance can be expressed through it as $G = \frac{2e^2}{h}\mathrm{tr}\frac{2}{\cosh(\Omega)+1}$. Upon multiplying Eqn. (2.4) by $M^\dagger$ on the left and by M on the right we arrive at $\Sigma_z \exp(\Omega)\Sigma_z = \exp(-\Omega)$. Using the fact that $\Sigma_z^2 = 1$ we immediately obtain $\Sigma_z\Omega\Sigma_z = -\Omega$. In other words the matrix Ω can be written as

$$\Omega = \begin{pmatrix} 0 & \omega \\ \omega^\dagger & 0 \end{pmatrix}, \tag{2.5}$$

where ω is an $N \times N$ arbitrary complex matrix. From now on we refer to Ω as "chiral" and to the blocks comprising it as "chiral" blocks (the name is borrowed from QCD [14], where matrices of the same structure appear). The structure constrains the eigenvalues of Ω to appear in pairs of opposite sign, i.e., $\{\nu_i, -\nu_i\}$.

The matrices Ω form a group under addition. The invariant measure for this group is [11] $d\mu(\Omega) = \Pi_{m,n} d\mathrm{Im}\omega_{m,n} d\mathrm{Re}\omega_{m,n}$. The global maximum entropy hypothesis (GMEH) introduced by Slevin and Nagao [10] proposes that only symmetry and the variance of $tr\Omega^2$ determine the statistical properties of the ensemble. This fixes the support for the spectrum of Ω to be Gaussian, and we obtain a probability distribution defined by the Laguerre unitary ensemble (LUE),

$$P(\{\nu_i\}) \propto \prod_i \nu_i \, e^{-c\nu_i^2} \prod_{i<j} |\nu_i^2 - \nu_j^2|^2, \tag{2.6}$$

where c fixes the average density.

The validity of this ensemble for the study of mesoscopic quantum wires was recently established both analytically and numerically in Ref. [11]. There results

were compared to alternative approaches based on the Local and Global Maximum Entropy Hypotheses.

2.1.1 The Statistical Ensemble of Ω.

The discussion above provides the motivation for the study of spectral properties of chiral random matrices Ω. Before determining explicit correlation functions, let us first examine the features of the spectrum which are implied by the symmetry of the matrix Ω. For this purpose it is convenient to reparametrize Ω making use of Pauli matrices as a basis for the chiral blocks. We define,

$$\Omega = \begin{pmatrix} 0 & \omega_1 - i\omega_2 \\ \omega_1 + i\omega_2 & 0 \end{pmatrix} \equiv \omega_1 \Sigma_x + \omega_2 \Sigma_y, \tag{2.7}$$

where ω_1 and ω_2 denote $N \times N$ random hermitean matrices and the Pauli matrices are defined by,

$$\begin{aligned} \Sigma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \Sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{2.8}$$

To examine the structure of the spectrum it is helpful to perform a unitary rotation of Ω in the chiral block,

$$\Omega = \omega_1 \Sigma_x + \omega_2 \Sigma_y \rightarrow \begin{pmatrix} \omega_1 & -i\omega_2 \\ i\omega_2 & -\omega_1 \end{pmatrix}. \tag{2.9}$$

If Ω contains only ω_1, the spectrum separates into two sets of eigenvalues which differ only by a sign. Since there is no level repulsion between the blocks, accidental degeneracies of levels from different sets are not excluded. The average eigenvalue density, or density of states (DOS), is equivalent to that of two independent Gaussian unitary ensembles. A finite ω_2 couples the two spectra and generates repulsion

between the levels of the two sets producing a vanishing DOS at the band center (or hard edge [15]). However the spectrum is still comprised of eigenvalue pairs of opposite sign. Far from the band center, the distribution converges rapidly to that the usual Dyson unitary ensemble. In this chapter, we will focus on the features of the spectrum close to the center of the band where the behavior is both universal and characteristic of chaotic quantum scattering.

Since the universal properties of random matrix ensembles are known to be independent of the support of the spectrum, therefore we are at liberty to consider a Gaussian ensemble of chiral random matrices

$$P(\Omega) \propto \exp[-\frac{N}{\lambda^2}\mathrm{tr}\Omega^2] \tag{2.10}$$

Our main task will be to determine the response of the eigenvalues of Ω to an arbitrary external perturbation,

$$\Omega(X) = \Omega_0 + X\Theta \tag{2.11}$$

where Ω_0 is chosen randomly from chGUE of Eq. (2.10) and $\Theta = \Theta_1\Sigma_x + \Theta_2\Sigma_y$ is some fixed matrix taken from the same ensemble. To prevent a drift of the spectrum we impose the additional constraints that tr Θ_1 = tr Θ_2 = 0. For convenience we also choose Θ_1 and Θ_2 to be orthogonal such that tr $(\Theta_1\Theta_2) = 0$. Although it is more natural to imagine applying the perturbation directly to the Hamiltonian of the scatterer S in Fig. 2-1, it seems likely that the universal response of the transmission eigenvalues will be reflected by the model above.

2.1.2 Universality

Previous studies of spectral correlations in random matrix ensembles revealed that the response of the spectrum to arbitrary perturbations also displayed a large degree of universality [27]. Starting from a random matrix model of the form defined by Eq. (2.10) and (2.11), but with Ω belonging to one of the usual Dyson ensembles,

spectral properties were shown to depend only on the mean-level spacing, Δ and the mean-square gradient of the levels,

$$C(0) = \frac{1}{\Delta^2} \left\langle \left(\frac{\partial \nu_i}{\partial X} \right)^2 \right\rangle . \tag{2.12}$$

where $\langle \cdots \rangle$ denotes statistical averaging over a range of energy or X. In both cases Δ and $C(0)$ were assumed to be constant over the range of consideration. After rescaling or "unfolding" the energy levels in the manner,

$$\epsilon_i \equiv \nu_i / \Delta, \qquad x \equiv X \sqrt{C(0)}, \tag{2.13}$$

properties of the entire random functions $\epsilon_i(x)$ become universal dependent only on the symmetry of the Dyson ensemble. In that case, $C(0)$ found a physical interpretation through a "fluctuation-dissipation" theorem as the generalized conductance defining the rate of energy dissipation in response to a time-dependent perturbation. We will demonstrate that in the case of the chGUE the rescaling still applies everywhere, but Δ is taken to the mean level spacing in the bulk (outside the vicinity of the hard edge).

2.1.3 Correlators

In this chapter we will be concerned with correlators of DOS,

$$\rho(E, X) = \text{tr } \delta(E - \Omega(X)). \tag{2.14}$$

Using a field theoretic approach based on the non-perturbative formalism introduced by Efetov [20] and developed by Verbaarschot et al. [21] to treat random matrix ensembles, we will determine both one and two-point correlators of DOS. This task requires a substantial generalization of the existing approach to treat the chiral symmetry. Since such generalizations are of important pedagogical interest we will emphasize the aspects of our calculation which depart from the usual approach.

In section 2.2 we will determine the one-point function or average DOS, $\langle \rho(E) \rangle$.

Although for usual Dyson ensembles this function is simply equal to the inverse level spacing, taken to be constant, its calculation here is highly non-trivial. The description near the hard-edge of the spectrum, where the DOS vanishes, requires the introduction of additional degrees of freedom or "Goldstone modes".

In section 2.3 we will determine two types of parametric correlation functions. The first is equal to the dimensionless autocorrelator of DOS fluctuations measured both as a function of changing E, and of external perturbation, X,

$$\begin{aligned} k(\epsilon_1, \epsilon_2, x) &= \Delta^2 \Big\langle \rho(\epsilon_1, \bar{x} - x/2)\ \rho(\epsilon_2, \bar{x} + x/2) \Big\rangle \\ &- \Delta^2 \Big\langle \rho(\epsilon_1, \bar{x} - x/2) \Big\rangle \Big\langle \rho(\epsilon_2, \bar{x} + x/2) \Big\rangle, \end{aligned} \tag{2.15}$$

the second determines the dimensionless response function [28],

$$n(\epsilon_1, \epsilon_2, x) = \Delta^2 \Big\langle \mathrm{tr} \Big[\ \delta(\epsilon_1 - \Omega(\bar{x} - x/2)/\Delta)\ \delta(\epsilon_2 - \Omega(\bar{x} + x/2)/\Delta) \Big] \Big\rangle. \tag{2.16}$$

We note that the former measures the correlation of the eigenvalues of the chiral matrix while the second measures the correlation of matrix elements.

Finally, we remark that the method of determining eigenvalue correlations is not unique. In fact, if we are interested just in eigenvalue correlations it is possible to exploit a one-dimensional Brownian motion or fermionic model in which the eigenvalues describe the position of particles and the parameter of the perturbation is related to time [29, 30]. Formally this connection can be made by mapping a one-dimensional matrix field theory onto the same effective non-linear σ-model which controls the spectral problem.

Importantly, for unitary ensembles, the quantum mechanical model describes non-interacting fermions and can typically be solved exactly. For the case considered here, results for the two-point correlator of DOS fluctuations have recently been obtained by Macédo [24]. We remark that within this approach it does not seem possible to

determine correlators of matrix elements such as $n(\epsilon_1, \epsilon_2, x)$. Since one of our aims is to develop some new technology for the field theoretic approach we proceed from a first principles calculation. We note that, as yet, this approach seems the only one capable, in principle, of treating orthogonal and symplectic symmetry which both translate to interacting models.

2.2 Average density of states.

To determine the spectral correlators of the transmission matrix we will make use of the supersymmetry approach in which Green functions are expressed in the form of integrals over supervectors. This method facilitates simple ensemble averaging. In particular the DOS can be expressed through the Green function,

$$G^R(E,X) \equiv [G^A(E,X)]^\dagger = \frac{1}{E^+ - \Omega(X)}, \tag{2.17}$$

($E^+ \equiv E + i0$) in the form

$$\rho(E,X) = -\frac{1}{\pi}\text{Im tr } G^R(E,X). \tag{2.18}$$

In fact the average DOS is independent of X since the perturbation is assumed to be small so that $\Omega(X)$ belongs to the same chGUE for any X. Therefore, for the purpose of this section we omit the perturbation.

Instead of treating DOS explicitly, it is convenient to define a generating function for $G^R(E)$. In this way, we can most readily adapt our formalism to study two-point functions in the next section. Following the notation of Ref. [21] let us define the following generator of one-point functions,

$$\mathcal{Z}_1(E,\hat{J}) = \int d[\psi] \exp\Big[- i\psi^\dagger\omega_1\Sigma_z\psi - i\psi^\dagger\omega_2\Sigma_y\psi + i\psi^\dagger(E^+ + \hat{J}k)\psi\Big]. \tag{2.19}$$

where ψ is the $4 \times N$-component supervector

$$\psi^T = (\Phi_i[1], \chi_i[1], \Phi_i[2], \chi_i[2]) \tag{2.20}$$

with $\Phi_i[c]$ and $\chi_i[c]$ denoting commuting (Bosonic) and Grassmann (Fermionic) elements respectively. The subscript $i = 1, \cdots, N$ indexes the components of the chiral matrix while $c = 1, 2$ references the chiral blocks of Ω. the matrix $\hat{J}$ acts as a source and is coupled to the supersymmetry breaking 2×2 matrix k, which in BF components is given by

$$k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.21}$$

The matrices that act on ψ are taken to be $4N \times 4N$ supermatrices where, for concreteness, we specify the ordering of the "chiral" and BF block in the manner,

$$M = \begin{pmatrix} M[11] & M[12] \\ M[21] & M[22] \end{pmatrix}, \qquad M[12] = \begin{pmatrix} M_{BB}[12] & M_{BF}[12] \\ M_{FB}[12] & M_{FF}[12] \end{pmatrix}, \tag{2.22}$$

reflecting the order of the components in Eq. (2.20). The small imaginary part of E^+ ensures the convergence of the Bosonic integration.

The utility of $\mathcal{Z}_1$ can be seen by taking derivatives with respect to the source $\hat{J}$.

$$\frac{1}{2}\frac{\partial \mathcal{Z}_1}{\partial \hat{J}_{ij}}\Big|_{\hat{J}_{ij}=0} = [G^R(E)]_{ij}. \tag{2.23}$$

In this way we can express $\rho(E)$ through the generating function $\mathcal{Z}_1$. We note that the use of supervectors with equal numbers of commuting and anticommuting elements obviates the need for explicit normalization. This has the important virtue of eliminating the need for replicas in treating ensemble averages [21] and greatly simplifies the present calculation.

Averaging over the ensemble of random matrices Ω Eq. (2.10) leads to the following expression for the generating function,

$$\langle \mathcal{Z}_1(E,\hat{J})\rangle = \int d[\psi] \exp\Big[-\frac{1}{2N}\mathrm{trg}(A_z^2 + A_y^2) + i\psi^\dagger(E^+ + k\hat{J})\psi], \tag{2.24}$$

where trg denotes the graded trace [21] ($\mathrm{trg}M = \mathrm{tr}M_{BB} - \mathrm{tr}M_{FF}$), and the 4×4 supermatrices A_x and A_y are defined by,

$$\begin{aligned} A_z &= \lambda\Sigma_z^{1/2}\sum_i \psi_i \otimes \psi_i^\dagger \Sigma_z^{1/2}, \\ A_y &= \lambda\Sigma_y^{1/2}\sum_i \psi_i \otimes \psi_i^\dagger \Sigma_y^{1/2}. \end{aligned} \tag{2.25}$$

For the square roots of Pauli matrices we adopt the following conventions

$$\Sigma_z^{1/2} = (\tfrac{1}{2i})^{1/2}(1 + i\Sigma_z) \quad , \quad \Sigma_y^{1/2} = (\tfrac{1}{2i})^{1/2}(1 + i\Sigma_y) \tag{2.26}$$

The two interaction terms $\mathrm{trg}(A_z^2)$ and $\mathrm{trg}(A_y^2)$ generated by the ensemble averaging are both invariant under an associated group of transformations $U(1,1/1,1)$ [31]. Using the identity $\mathrm{trg}(A_y)^2 \equiv \mathrm{trg}(A_z\Sigma_y)^2$, and introducing the supermatrix $\tilde{A} = \frac{1}{\sqrt{2}}(A_z - \Sigma_y A_z \Sigma_y)$ it is possible to rewrite the interaction as a single dyadic product [32]

$$\mathrm{trg}(A_z^2 + A_y^2) = \mathrm{trg}(\tilde{A}^2). \tag{2.27}$$

In this representation it is straightforward to decouple the interaction by means of a Hubbard-Stratonivich transformation with the introduction of only a single 4×4 supermatrix, Q. The supermatrix introduced by the transformation is required to have the same symmetry as $\tilde{A}$. Since $\tilde{A}$ contains only those components which anticommute with Σ_y, namely Σ_z and Σ_x, we impose the constraint,

$$[Q, \Sigma_y]_+ = 0, \tag{2.28}$$

and introduce the following decomposition,

$$\langle \mathcal{Z}_1(E,\hat{J})\rangle = \int d[Q]d[\psi]\exp\Big[-\frac{N}{2}\mathrm{trg}Q^2 - i\sqrt{2}\mathrm{trg}(QA_z) + i\psi^\dagger(E^+ + k\hat{J})\psi\Big]. \quad (2.29)$$

The dependence of $\mathcal{Z}_1$ on ψ is now Gaussian and it is straightforward to perform the integration which yields,

$$\langle \mathcal{Z}_1(E,\hat{J})\rangle = \int d[Q]\exp\Big[-\frac{N}{2}\mathrm{trg}Q^2 - \mathrm{trg}_N \ln\Big(E^+\Sigma_z + k\hat{J}\Sigma_z - \sqrt{2}\lambda Q\Big)\Big], \quad (2.30)$$

where $\mathrm{trg_N}$ denotes a graded trace of a $4N \times 4N$ supermatrix.

So far we have made no approximation and Eq. (2.30) provides an exact expression for the ensemble average of the generating function. To proceed further we will take advantage of large N and apply a saddle-point approximation, in which we are at liberty to neglect the source term. The variation of the effective Lagrangian in Q leads to the saddle-point equation,

$$Q\Big(Q - \frac{E^+\Sigma_z}{\sqrt{2}\lambda}\Big) = -1. \quad (2.31)$$

The appropriate solution depends on the part of the spectrum in which we are interested. Firstly, if we are far from the center of the spectrum ($|E| \sim O(\lambda N^0)$) all modes are massive with a mass of order N and the saddle-point manifold (SPM) consists only of the single point,

$$Q = \Sigma_z\left[\frac{E}{2\sqrt{2}\lambda} - i\sqrt{1 - \frac{E^2}{8\lambda^2}}\right]. \quad (2.32)$$

A minus sign is chosen in front of the square root so that the deformation of the integration contour be consistent with the sign of $\mathrm{Im}E^+$ (see Ref. [21]). Physically, Eq. (2.31) corresponds to the mean-field equation for Q and the sign is chosen to give the correct form for the self-energy. In this case, the contribution from the massive modes give a correction of order N^{-1} and can be neglected. Expanding Eq. (2.30)

in J and making use of Eq. (2.23) we obtain the following expression for the average DOS

$$\langle \rho(E) \rangle = -\frac{N}{2\sqrt{2}\pi\lambda} \text{Im trg}(\Sigma_z Q k). \tag{2.33}$$

Inserting the saddle-point solution for Q we find an average DOS that coincides with that of the usual Dyson ensembles taking the form of a semi-circle. With the average DOS in the vicinity of the center of the band equal to $\Delta = \pi\lambda/\sqrt{2}N$ we obtain the final expression,

$$\langle \rho(E) \rangle = \frac{1}{\Delta}\sqrt{1 - \left(\frac{\pi E}{4N\Delta}\right)^2}. \tag{2.34}$$

In contrast, if we are concerned with the spectrum very close to the center of the DOS distribution $E \sim O(N^0\Delta)$, the saddle-point becomes degenerate and governed by "Goldstone modes". To leading order in N, the saddle-point equation (2.31) then reads

$$Q^2 = -1. \tag{2.35}$$

The solution is now described by a degenerate manifold with the energy E itself playing the role of a symmetry breaking parameter. We remark that this contrasts with the usual case of Dyson ensembles where the Goldstone modes are preserved by changing energy.

Expanding the effective Lagrangian in Eq. (2.30) and retaining terms of order N^0 we obtain

$$\langle \rho(E) \rangle = -\frac{1}{4\Delta} \text{Im} \int d[Q] \text{ trg}(\Sigma_z k Q) \exp\Big[-\frac{\pi E}{2\Delta} \text{trg}(\Sigma_z Q)\Big]. \tag{2.36}$$

The degeneracy of the SPM is intimately related to the existence of a family of transformations leaving both interaction terms in Eq. (2.24) invariant. As a ubiquitous feature of transmission matrices it contrasts with the behavior of usual Dyson ensembles and deserves some consideration.

The new class of Goldstone modes which characterize the SPM represent the

main new feature of the problem and are present even in the average DOS. We can gain a better understanding of their origin by examining a model describing particles moving in a random, *off-diagonal* δ-correlated potential. Although, perhaps unphysical, we might consider a model of superconductivity in the presence of a random order parameter, $\Delta(r)$ with $\langle\Delta(\vec{r})\Delta^*(\vec{r'})\rangle = \gamma\delta(\vec{r}-\vec{r'})$. The Schrödinger equation for each of the "chiral" or spinor components can be written as a set of Bogoliubov-de Gennes (BdG) equations in which the order parameter does not obey a self-consistency condition (a form of self-consistency condition may be imposed on the fluctuation of the order parameter γ),

$$\begin{aligned}\Big(\frac{\hat{p}^2}{2m}-E\Big)u(\vec{r})+\Delta(\vec{r})v(\vec{r})=0,\\ \Delta^*(\vec{r})u(\vec{r})+\Big(-\frac{\hat{p}^2}{2m}-E\Big)v(\vec{r})=0.\end{aligned} \tag{2.37}$$

This model possesses two types of gauge invariance: (i) $u \to u\exp(i\phi)$, $v \to v\exp(i\phi)$. (ii) $u \to u\exp(i\phi)$, $v \to v\exp(-i\phi)$, $\Delta(r) \to \Delta(r)\exp(i2\phi)$. Note that the gauge invariance considered here is global as we do not wish to elongate derivatives in the BdG equations. $\Delta(r)$ enters the Hamiltonian as Re $\Delta(r)\Sigma_x$ – Im $\Delta(r)\Sigma_y$ and the second type of gauge invariance corresponds to rotations in the Σ_x — Σ_y subspace. It is the second type of gauge invariance that gives rise to new Goldstone modes. If we follow the method of Efetov [20] and construct a nonlinear σ-model for this problem, the zero mode is identical to the chiral random matrix ensemble that we are considering. This closely parallels the equivalence of the random matrix theory with the statistics of weakly disordered metallic grains demonstrated in Ref. [20].

Now let us turn to the parametrization of the SPM. To perform the integration over Q it is necessary to analyze further the symmetry transformations $\psi \to U\psi$, which preserve both of the interaction terms in Eq. (2.24). The transformations U_z, U_y preserving $\mathrm{trg}(A_z^2)$, $\mathrm{trg}(A_y)^2$ obey

$$U_z^\dagger\Sigma_z U_z = \Sigma_z, \quad U_y^\dagger\Sigma_y U_y = \Sigma_y. \tag{2.38}$$

Writing the infinitesimal transformations in terms of the generators $U_z = 1 + G_z$ and $U_y = 1 + G_y$, we require that

$$(i) \quad G_z^\dagger \Sigma_z + \Sigma_z G_z = 0, \qquad (ii) \quad G_y^\dagger \Sigma_y + \Sigma_y G_y = 0, \tag{2.39}$$

which implies

$$\begin{aligned} G_z &= ig_0\Sigma_0 + g_x\Sigma_x + g_y\Sigma_y + ig_z\Sigma_z, \\ G_y &= ig_0\Sigma_0 + g_x\Sigma_x + ig_y\Sigma_y + g_z\Sigma_z, \end{aligned} \tag{2.40}$$

where $\{g_i\}$ denote 2×2 hermitean supermatrices. From Eq. (2.40) we see that the transformations U preserving both $\mathrm{trg}(A_z^2)$ and $\mathrm{trg}(A_y^2)$ are of the form,

$$U = \exp(G) \quad , \quad G = ig_0\Sigma_0 + g_x\Sigma_x, \tag{2.41}$$

which induce the following transformations on A_z,

$$A_z \to T A_z T^{-1} \quad , \quad T = \Sigma_z^{1/2} U \Sigma_z^{-1/2}. \tag{2.42}$$

The intersection of the two groups U_z and U_y contains transformations that change the saddle-point (generators g_x) This fact leads to the degeneracy of the saddle-point manifold and to a new family of Goldstone modes. We note that were we to consider the usual $2N \times 2N$ GUE we could rewrite it in terms of Σ matrices as well. However in this case we would have four interaction terms, four groups of transformations (U_0, U_x, U_y, U_z) and their intersection would consist only of transformations that leave the saddle-point invariant (generators g_0). As such, the saddle-point manifold would be nondegenerate and the evaluation of the one-point function would be trivial. It is this restrictedness of the chGUE that gives rise to the new Goldstone modes.

Continuing, it is necessary to ensure that both terms in $\tilde{A}$ transform in the same manner under the action of Eq. (2.41)

$$\begin{aligned} \Sigma_y A_z \Sigma_y &\to \tilde{T} \Sigma_y A_z \Sigma_y \tilde{T}^{-1}, \\ \tilde{T} = \Sigma_y T \Sigma_y &= \Sigma_y \Sigma_z^{1/2} U \Sigma_z^{-1/2} \Sigma_y. \end{aligned} \tag{2.43}$$

From Eq. (2.41) we see that U can be written as $U = a\Sigma_0 + b\Sigma_x$, where a and b are 2×2 supermatrices. Then

$$T = \Sigma_z^{1/2}(a\Sigma_0 + b\Sigma_x)\Sigma_z^{-1/2} = a\Sigma_0 - b\Sigma_y, \tag{2.44}$$

from which it follows that $\tilde{T} = \Sigma_y T \Sigma_y = T$. Thus, both A_z and $\Sigma_y A_z \Sigma_y$ are transformed through the same matrices. $\tilde{A}$ is therefore transformed as

$$\tilde{A} \to T\tilde{A}T^{-1}, \tag{2.45}$$

which, in turn, induces the transformations on the Q matrix as $Q \to T^{-1}QT$. However, since not all of these transformations change the saddle-point $Q = -i\Sigma_z$, we draw the distinction by separating T in the manner $T = RT_0$. R contains the transformations that leave the saddle-point unchanged while T_0 contains only the Goldstone modes. The massive degrees of freedom reside in $R^{-1}Q_D R$ (Q_D is diagonal).

From the form of U in Eq. (2.41) it is clear that the parametrization of Q is noncompact. This feature, while necessary for the BB block, is undesirable for the FF block. So we have to compactify the transformation matrices T_0 first. In the usual square root parametrization (see for example Refs. [21, 31]) the compactified matrix T_0 is given by

$$T_0 = \begin{pmatrix} (1 + w_1^2)^{1/2} & iw_1 \\ -iw_1 & (1 + w_1^2)^{1/2} \end{pmatrix}, \tag{2.46}$$

where w_1 is a 2×2 supermatrix given by

$$w_1 = \begin{pmatrix} w_{1B} & \alpha_1 \\ \alpha_3 & iw_{1F} \end{pmatrix}. \tag{2.47}$$

The compactification of the FF block is imposed by introducing the factor of i in front of w_{1F}. Herein lies an important difference from the usual ensembles that deserves an explanation.

As we see the [12] and [21] blocks of T_0 in Eq. (2.46) are identical. Following from

the usual Dyson ensembles [31], it might seem sensible to compactify T_0 as

$$T_0 = \begin{pmatrix} (1+w[12]w[21])^{1/2} & iw[12] \\ -iw[21] & (1+w[21]w[12])^{1/2} \end{pmatrix}, \tag{2.48}$$

with

$$w[12] = \begin{pmatrix} w_{1B} & i\alpha_1 \\ \alpha_3 & iw_{1F} \end{pmatrix}, \qquad w[21] = \begin{pmatrix} w_{1B} & \alpha_1 \\ i\alpha_3 & iw_{1F} \end{pmatrix}. \tag{2.49}$$

However, the condition that T_0 contain only Σ_0 and Σ_y is violated by this parametrization. For unitary ensembles (see Ref. [31]) the presence of two distinct matrices $w[12]$ and $w[21]$ with a simple relation $w[21] = k\ w[12]^\dagger$ between them allowed the introduction of two "diagonalization" matrices u and v. Thus each matrix (u and v) contained only half as many Grassmann variables as there were in the problem. In addition their symmetry properties under hermitean conjugation were known. In the present case w_1 is neither hermitean nor antihermitean and the relation between α_1^* and α_3 is unknown. However if we assume that α_1 and α_3 are independent variables w_1 can be diagonalized by a single matrix (unless its eigenvalues are equal to each other, in which case it can be treated in the same manner as Dyson ensembles [21]). Thus the behavior of the matrix under hermitean conjugation is actually irrelevant to the question of its diagonalization. We write the diagonalization transformation as

$$\begin{aligned} w_1 &= u_1 \hat{\rho}_1 u_1^{-1}; \quad \hat{\rho}_1 = \begin{pmatrix} \rho_{1B} & 0 \\ 0 & i\rho_{1F} \end{pmatrix} = \sinh(\hat{\theta}_1/2); \\ \hat{\theta}_1/2 &= \begin{pmatrix} \theta_{1B}/2 & 0 \\ 0 & i\theta_{1F}/2 \end{pmatrix}; \quad u_1 = \exp \begin{pmatrix} 0 & \xi_1 \\ \xi_3 & 0 \end{pmatrix} \end{aligned} \tag{2.50}$$

The saddle-point equation (2.35) is invariant under the transformation $Q \to T_0^{-1} Q T_0$, and consequently the SPM can be parametrized as

$$Q = -iT_0^{-1}\Sigma_z T_0. \tag{2.51}$$

Using Eqs. (2.46), (2.47), (2.50) and (2.51) we obtain the following expressions for the effective Lagrangian in Eq. (2.36) and the invariant one-point function $\mathrm{trg}(\Sigma_z Qk)$ that enter the definition of the average DOS,

$$\begin{aligned} \mathrm{trg}(\Sigma_z Q) &= -2i(\cosh\theta_{1B} - \cos\theta_{1F}), \\ \mathrm{trg}(\Sigma_z Qk) &= -2i\Big[\cosh\theta_{1B} + \cos\theta_{1F} + 2\xi_1\xi_3(\cosh\theta_{1B} - \cos\theta_{1F})\Big]. \end{aligned} \tag{2.52}$$

The final step in the parametrization is to find the invariant measure on the SPM. Since this follows a straightforward calculation, we present the final result and refer to appendix A.1 for a more detailed discussion.

$$d\mu(Q) = \frac{\cosh\theta_{1B}\cos\theta_{1F} - 1}{2(\cosh\theta_{1B} - \cos\theta_{1F})^2} d\theta_{1B} d\theta_{1F} d\xi_1 d\xi_3 \tag{2.53}$$

All integrations over the saddle-point manifold can be performed after which we obtain (see appendix A.2)

$$\langle\rho(E)\rangle = \frac{\pi^2 E}{2\Delta^2}\Big[J_0^2(\pi E/\Delta) + J_1^2(\pi E/\Delta)\Big]. \tag{2.54}$$

This expression, which clearly can not be obtained within perturbation theory, is most relevant for energies which lie within a few average level spacings of the center of the density of states distribution. It also coincides with results found previously from random matrix theory [10, 11, 14, 15, 16, 17, 18, 19]. The dimensionless form of the average DOS, $\Delta\langle\rho(\epsilon)\rangle$ is shown as a function of the dimensionless energy $\epsilon = E/\Delta$ in Fig. 2-2.

As discussed earlier, the chiral structure of the matrix gives rise to a vanishing of the average density of states at the band center. The presence of the hard-edge generates oscillations which decay into the bulk where the density of states is uniform. For energies which are of order $N\Delta$ the expression for the average density of states

shown in Eq. (2.34) becomes valid and we obtain the usual semi-circular distribution.

In the next section we generalize the approach developed here to examine the fluctuation in the DOS.

2.3 Two-point correlation function.

2.3.1 Density Correlator.

Turning now to two-point correlation functions, we will examine the fluctuation in DOS in response to a change in an external perturbation characterized by the parameter X. In particular our goal will be to find an expression for

$$K(E_1, E_2, X) = \langle \rho(E_1, X/2)\rho(E_2, -X/2)\rangle - \langle \rho(E_1)\rangle\langle \rho(E_2)\rangle. \tag{2.55}$$

Since the eigenvalues of Ω arise in pairs of opposite sign, $\mathrm{Im}\left[\mathrm{tr}\ G^{R,A}(E)\right]$ is an even function of energy and $\mathrm{Re}\left[\mathrm{tr}\ G^{R,A}(E)\right]$ is odd. This implies a simple relation between the *traces* of retarded and advanced Green functions which holds even before averaging (E is assumed to be real) :

$$\mathrm{tr}\ G^A(E) = -\mathrm{tr}\ G^R(-E). \tag{2.56}$$

Thus the average $\left\langle \mathrm{tr}\ G^R(E_1)\mathrm{tr}\ G^R(E_2)\right\rangle$ contains all necessary information, contrary to our experience with usual random matrix ensembles (see for example [20]). If we determine

$$W(E_1, E_2, X) = \left\langle \mathrm{tr}\ G^R(E_1, X/2)\ \mathrm{tr}\ G^R(E_2, -X/2)\right\rangle, \tag{2.57}$$

and make use of Eq. (2.56) we can obtain K through the relation

$$K(E_1, E_2, X) + \langle \rho(E_1)\rangle\langle \rho(E_2)\rangle = -\frac{1}{(2\pi)^2}\left[W(E_1, E_2, X) + W(-E_1, E_2, X)\right.$$
$$\left. + W(E_1, -E_2, X) + W(-E_1, -E_2, X)\right]. \tag{2.58}$$

Therefore, by obtaining an expression for W in the form of an integral over the saddle-

point manifold and evaluating its even part (as a function of E_1 and E_2) we obtain K. The complete function W can then, in principle, be obtained from K with the aid of Kramers-Kronig relations.

As in the previous section, we proceed by defining the generating function for two-point correlators of retarded Green functions,

$$\mathcal{Z}_2^{RR}(E_1, E_2, X) = \int d[\Psi] \exp\Big[-i\Psi^\dagger \omega_1 \Sigma_z \Psi - i\Psi^\dagger \omega_2 \Sigma_y \Psi + i\Psi^\dagger \Big(\hat{E} + k\hat{J} - \frac{1}{2} X \Theta \Lambda\Big)\Psi\Big], \tag{2.59}$$

where Ψ denotes the $8 \times N$-component supervector

$$\Psi^T = \Big(\Phi_i^R[1], \chi_i^R[1], \Phi_i^A[1], \chi_i^A[1], \Phi_i^R[2], \chi_i^R[2], \Phi_i^A[2], \chi_i^A[2]\Big). \tag{2.60}$$

Although $\mathcal{Z}_2^{RR}$ strictly generates two-point correlators of retarded Green functions we find it convenient to refer to the block corresponding to energy E_1 as RR and that corresponding to E_2 as AA. Then, as in Ref. [21] the matrix Λ breaks the symmetry between the blocks corresponding to different energies (RA blocks), $\Lambda = \mathrm{diag}(1, -1, 1, -1)$, $\hat{E} = \mathrm{diag}(E_1^+, E_2^+, E_1^+, E_2^+)$ and $\hat{J}$ represents the supersymmetry breaking source.

The $8N \times 8N$ component supermatrices that now act on supervectors Ψ have the following block structure; chiral, retarded-advanced (RA), BF.

$$M = \begin{pmatrix} M[11] & M[12] \\ M[21] & M[22] \end{pmatrix}; M[12] = \begin{pmatrix} M^{RR}[12] & M^{RA}[12] \\ M^{AR}[12] & M^{AA}[12] \end{pmatrix}, \tag{2.61}$$

$$M^{RR}[12] = \begin{pmatrix} M_{BB}^{RR}[12] & M_{BF}^{RR}[12] \\ M_{FB}^{RR}[12] & M_{FF}^{RR}[12] \end{pmatrix}. \tag{2.62}$$

Differentiating $\mathcal{Z}_2^{RR}$ with respect to the source we obtain the following identity,

$$W(E_1, E_2, X) = \frac{1}{4}\mathrm{tr}\,\Big(\frac{\partial}{\partial \hat{J}^{RR}}\Big)\ \mathrm{tr}\,\Big(\frac{\partial}{\partial \hat{J}^{AA}}\Big)\mathcal{Z}_2^{RR}\Big|_{\hat{J}=0} \tag{2.63}$$

Proceeding as before, the ensemble average generates interactions which can be

decoupled by a Hubbard-Stratonovich transformation. Integration over Ψ gives the effective theory,

$$\begin{aligned}
&\langle \mathcal{Z}_2^{RR}(E_1, E_2, X)\rangle \\
&= \int d[Q] \exp\Big[-\frac{N}{2}\mathrm{trg}Q^2 \\
&-\mathrm{trg}_N \ln\Big(\hat{E}\Sigma_z + k\hat{J}\Sigma_z - \sqrt{2}\lambda Q - \frac{1}{2}X\Sigma_z^{-1/2}\hat{\Theta}\Sigma_z^{1/2}\Big)\Big]. \qquad (2.64)
\end{aligned}$$

where Q denote 8×8 supermatrices.

Just as with the average DOS, far from the center of the eigenvalue distribution, statistical properties of Ω coincide with usual Dyson ensembles and can be found in Refs. [7]. We therefore restrict attention to fluctuations near the center of the distribution. The appropriate saddle-point equation is found by setting $\hat{E} = 0$, $\hat{J} = 0$, and $X = 0$. Doing so, we obtain the same condition as in Eq. (2.35) with the base saddle-point still given by $-i\Sigma_z$.

For $X^2\mathrm{tr}\Theta^2 \sim O(N^2\Delta^2)$ and $\hat{E} \sim O(N^0\Delta)$ an expansion to leading order in N around the saddle-point generates the effective non-linear supermatrix σ-model,

$$\begin{aligned}
W(E_1, E_2, X) = \Big(\frac{\pi}{8\Delta}\Big)^2 \int d[Q]\mathrm{trg}(Q\Sigma_z(1+\Lambda)k)\mathrm{trg}(Q\Sigma_z(1-\Lambda)k) \\
\times \exp\Big\{ -\frac{\pi}{2\Delta}\mathrm{trg}\Big[Q\hat{E}\Sigma_z\Big] + \frac{(\pi X)^2}{8(N\Delta)^2}\mathrm{trg}_N\Big[Q\Lambda(\Theta_1 + i\Theta_2\Sigma_y)\Big]^2\Big\}. \qquad (2.65)
\end{aligned}$$

Using the properties of Θ (see the paragraph below Eq. 2.11) and Eq. (2.28) we take the trace over Θ_1 and Θ_2 and obtain

$$\begin{aligned}
W(E_1, E_2, X) &= \Big(\frac{\pi}{8\Delta}\Big)^2 \int d[Q]\mathrm{trg}(Q\Sigma_z(1+\Lambda)k)\mathrm{trg}(Q\Sigma_z(1-\Lambda)k) \\
&\times \exp(-\mathcal{L}), \qquad (2.66) \\
\mathcal{L} &= \frac{\pi}{2\Delta}\mathrm{trg}(Q\hat{E}\Sigma_z) - \frac{\mathrm{tr}(\Theta^2)X^2}{(4\lambda)^2}\mathrm{trg}(Q\Lambda)^2. \qquad (2.67)
\end{aligned}$$

To proceed further it is necessary to find an appropriate parametrization for the SPM. The inclusion of addition blocks to account for the two-point correlation requires

further consideration. As before the Q matrices on the SPM can be parametrized as $Q = -iT_0^{-1}\Sigma_z T_0$. The generators of the transformations preserving both interaction terms and changing the base saddle-point (i.e. the Goldstone mode generators) can be again written as $T_0 = 1 + t$ and

$$t = \begin{pmatrix} 0 & ig \\ -ig & 0 \end{pmatrix}, \tag{2.68}$$

where g is a hermitean (before compactification) 4×4 supermatrix

$$g = \begin{pmatrix} w_1 & w \\ w^\dagger & w_2 \end{pmatrix}. \tag{2.69}$$

The compactification of these matrices can be done in the following way: w_1 is parametrized as before (Eq. (2.50)), and w_2 is parametrized in the same manner,

$$\begin{aligned} w_2 = u_2\hat{\rho}_2 u_2^{-1}; \quad \hat{\rho}_2 = \begin{pmatrix} \rho_{2B} & 0 \\ 0 & i\rho_{2F} \end{pmatrix} = \sinh(\hat{\theta}_2/2); \\ \hat{\theta}_2/2 = \begin{pmatrix} \theta_{2B}/2 & 0 \\ 0 & i\theta_{2F}/2 \end{pmatrix}; \quad u_2 = \exp\begin{pmatrix} 0 & \xi_2 \\ \xi_4 & 0 \end{pmatrix} \end{aligned} \tag{2.70}$$

Following Ref. [31] matrices w and $w^\dagger$ are compactified as

$$w \to w_{12} = u\begin{pmatrix} \nu_1 & 0 \\ 0 & i\nu_2^* \end{pmatrix} v^{-1} \quad , \quad w^\dagger \to w_{21} = v\begin{pmatrix} \nu_1^* & 0 \\ 0 & i\nu_2 \end{pmatrix} u^{-1}, \tag{2.71}$$

where u and v are given by

$$u = \exp\begin{pmatrix} 0 & \alpha \\ \alpha^* & 0 \end{pmatrix} \quad , \quad v = \exp\begin{pmatrix} 0 & i\beta \\ i\beta^* & 0 \end{pmatrix}. \tag{2.72}$$

Note that w_1 and w_2 correspond to "chiral transformations". That is they leave the $[RA]$ blocks invariant and transform the variables only between chiral blocks

within each $[RA]$ block. The generators w_{12} and w_{21}, on the other hand, transform the variables between R and A. These transformations are present in the σ-model for the usual unitary ensemble and we call them "unitary".

Below we make use of a parametrization which is similar to that introduced by Altland et al. [32], representing T_0 as $T_0 = T_u T_{ch}$, where T_u and T_{ch} respectively correspond to the unitary and chiral transformations

$$T_u = \begin{pmatrix} (1+w_{12}w_{21})^{1/2} & 0 & 0 & iw_{12} \\ 0 & (1+w_{21}w_{12})^{1/2} & iw_{21} & 0 \\ 0 & -iw_{12} & (1+w_{12}w_{21})^{1/2} & 0 \\ -iw_{21} & 0 & 0 & (1+w_{21}w_{12})^{1/2} \end{pmatrix}, \tag{2.73}$$

$$T_{ch} = \begin{pmatrix} (1+w_1^2)^{1/2} & 0 & iw_1 & 0 \\ 0 & (1+w_2^2)^{1/2} & 0 & iw_2 \\ -iw_1 & 0 & (1+w_1^2)^{1/2} & 0 \\ 0 & -iw_2 & 0 & (1+w_2^2)^{1/2} \end{pmatrix}. \tag{2.74}$$

We introduce the eigenvalue parametrization as

$$T_u = \hat{V}_u \hat{\Upsilon}_u \hat{V}_u^{-1} = \hat{V}_u \begin{pmatrix} \cosh(\hat{\Omega}/2) & 0 & 0 & ie^{i\hat{\phi}}\sinh(\hat{\Omega}/2) \\ 0 & \cosh(\hat{\Omega}/2) & ie^{-i\hat{\phi}}\sinh(\hat{\Omega}/2) & 0 \\ 0 & -ie^{i\hat{\phi}}\sinh(\hat{\Omega}/2) & \cosh(\hat{\Omega}/2) & 0 \\ -ie^{-i\hat{\phi}}\sinh(\hat{\Omega}/2) & 0 & 0 & \cosh(\hat{\Omega}/2) \end{pmatrix} \hat{V} \tag{2.75}$$

$$T_{ch} = \hat{U}_{ch} \hat{\Upsilon}_{ch} \hat{U}_{ch}^{-1} = \hat{U}_{ch} \begin{pmatrix} \cosh(\hat{\theta}_1/2) & 0 & i\sinh(\hat{\theta}_1/2) & 0 \\ 0 & \cosh(\hat{\theta}_2/2) & 0 & i\sinh(\hat{\theta}_2/2) \\ -i\sinh(\hat{\theta}_1/2) & 0 & \cosh(\hat{\theta}_1/2) & 0 \\ 0 & -i\sinh(\hat{\theta}_2/2) & 0 & \cosh(\hat{\theta}_2/2) \end{pmatrix} \hat{U}_{ch}^{-1}, \tag{2.76}$$

where

$$\hat{\Omega} = \begin{pmatrix} \Omega_B & 0 \\ 0 & i\Omega_F \end{pmatrix}; \qquad \hat{\phi} = \begin{pmatrix} \phi_B & 0 \\ 0 & \phi_F \end{pmatrix}, \tag{2.77}$$

and are related to ν_{12} in Eq. (2.71) as $\nu_1 = \sinh(\Omega_B/2)\exp(i\phi_B)$, $\nu_2 = \sin(\Omega_F/2)\exp(-i\phi_F)$.

Finally, now that the parametrization of SPM has been established we can calculate the invariant measure. This can be done following Altland et al. [32]. The calculation is carried out in appendix A.3 and yields

$$\begin{aligned} d\mu(Q) = & \frac{4s_1 s_2}{(s_1+s_2)^2}\frac{(2\pi)^2}{(s_1-s_2)^2}\frac{(\cosh\theta_{1B}\cos\theta_{1F}-1)}{2(\cosh\theta_{1B}-\cos\theta_{1F})^2} \\ & \times\frac{(\cosh\theta_{2B}\cos\theta_{2F}-1)}{2(\cosh\theta_{2B}-\cos\theta_{2F})^2} ds_1 ds_2 d\theta_{1B} d\theta_{1F} d\theta_{2B} d\theta_{2F} D_G \end{aligned} \tag{2.78}$$

where we used the notation $s_1 = \cosh\Omega_B$, $s_2 = \cos\Omega_F$, and D_G denotes the differentials of all Grassmann variables in the parametrization.

Using Eqs. (2.66), (2.67), together with the invariant measure shown in Eq. (2.78), the parametrization of Eqs. (2.73),(2.74), (2.75), (2.76) and the integration limits established in appendix A.4 we can obtain an exact analytical expression for $K(E_1, E_2, X)$. Although technical, the definite integration is straightforward and the details are presented in appendix A.5. Here we present the result for the connected part of the DOS correlator

$$\begin{aligned} K(E_1,E_2,X) = & \frac{\pi^4}{\Delta^4}\int_0^1 ds_1 \int_1^\infty ds_2 E_1 E_2 s_1 s_2 \exp\left[-\left(\frac{\pi X}{2N\Delta}\right)^2 \mathrm{tr}(\Theta^2)(s_1^2-s_2^2)\right] \\ & \times J_0(\pi E_1 s_1/\Delta) J_0(\pi E_1 s_2/\Delta) J_0(\pi E_2 s_1/\Delta) J_0(\pi E_2 s_2/\Delta) \end{aligned} \tag{2.79}$$

which coincides with the result found by Macédo [24] using the mapping to non-interacting fermions. We remark that for $X = 0$ further integration is possible after which we obtain the result found earlier by Verbaarschot and Zahed in the context of QCD [14] from random matrix theory.

$$K(E_1,E_2,X=0) = \delta(E_1-E_2)\frac{\pi^2 E_1}{2\Delta^2}\left[J_0^2(\pi E_1/\Delta) + J_1^2(\pi E_1/\Delta)\right]$$

$$-\frac{\pi^2 E_1 E_2}{\Delta^2 (E_1^2 - E_2^2)^2}\Big[E_1 J_0(\pi E_1/\Delta) J_1(\pi E_2/\Delta)$$
$$-E_2 J_0(\pi E_2/\Delta) J_1(\pi E_1/\Delta)\Big]^2 \tag{2.80}$$

From Eq. (2.79) we can extract the level velocity distribution function $P(V,E) = \langle \delta(V - \partial E_i/\partial X)\delta(E - E_i)\rangle / \langle \rho(E)\rangle$ using the formula (Ref. [33])

$$P(V,E) = \lim_{X\to 0} \frac{X}{\langle \rho(E)\rangle} K(E, E + VX, X). \tag{2.81}$$

A straightforward calculation yields the result

$$P(V,E) = \frac{1}{\sqrt{2\pi C(0)}} \exp\left(-\frac{V^2}{2C(0)}\right) \quad , \quad C(0) = \frac{\mathrm{tr}(\Theta^2)}{2N^2\Delta^2} \tag{2.82}$$

Thus, as with usual Dyson ensembles, the velocity distribution is Gaussian everywhere, even in the vicinity of the hard edge. Applying the rescaling of Eq. (2.13) we obtain the dimensionless two-point density correlation function $k = \Delta^2 K$ in the form,

$$\begin{aligned} k(\epsilon_1, \epsilon_2, x) &= \pi^4 \int_0^1 ds_1 \int_1^\infty ds_2 \epsilon_1 \epsilon_2 s_1 s_2 \exp(-\frac{\pi^2 x^2}{2}[s_1^2 - s_2^2]) \\ &\quad \times J_0(\pi\epsilon_1 s_1) J_0(\pi\epsilon_1 s_2) J_0(\pi\epsilon_2 s_1) J_0(\pi\epsilon_2 s_2) \end{aligned} \tag{2.83}$$

Equation (2.83) depends in a non-trivial way on the dimensionless parameters, ϵ_1, ϵ_2 and x and provides a signature of quantum chaotic scattering. We remark that for ϵ_1, ϵ_2 large the Bessel functions can be expanded and we obtain the results found for usual Dyson ensembles [27].

2.3.2 Response Function

Let us now turn to the second type of correlation function which can be presented in terms of the Green functions in the form

$$N(E_1,E_2,X) = -\frac{1}{(2\pi)^2}\langle \mathrm{tr}\{[G^R(E_1,X/2)-G^A(E_1,X/2)] \times [G^R(E_2,-X/2)-G^A(E_2,-X/2)]\}\rangle. \quad (2.84)$$

Its evaluation requires the calculation of the following quantities

$$F^{R\star}(E_1,E_2,X) = \left\langle \mathrm{tr}\,[G^R(E_1,X/2)G^\star(E_2,-X/2)]\right\rangle = \frac{1}{4}\mathrm{tr}\,\frac{\partial^2\,\mathcal{Z}_2^{R\star}(E_1,E_2,X)}{\partial\hat{J}^{RA}\partial\hat{J}^{AR}}\bigg|_{\hat{J}=0}, \quad (2.85)$$

where $\star$ denotes A or R, $\mathcal{Z}_2^{RR}(E_1,E_2,X)$ is given by Eq. (2.59) and $\mathcal{Z}_2^{RA}(E_1,E_2,X)$ is the generating function for the product of retarded and advanced Green functions and will be defined in Eq. (2.87) below. To calculate $F_2^{RR}(E_1,E_2,X)$ we can use the parametrization introduced in the previous subsection for the two-point density correlator. Using Eq. (2.85) we find

$$F^{RR}(E_1,E_2,X) = \left(\frac{\pi}{8\Delta}\right)^2\int d[Q]\mathrm{trg}(Q\Sigma_z(1+\Lambda)kQ\Sigma_z(1-\Lambda)k)\exp(-\mathcal{L}), \quad (2.86)$$

where $\mathcal{L}$ is given in Eq. (2.67).

For $F_2^{RA}(E_1,E_2,X)$ some basic formulas change. We adopt the same block hierarchy Eqs. (2.60), (2.62) that was used for K. Then the generating function becomes

$$\mathcal{Z}_2^{RA}(E_1,E_2,X) = \int d[\Psi]\exp\Big[-i\Psi^\dagger\omega_1\Sigma_z\Lambda\Psi - i\Psi^\dagger\omega_2\Sigma_y\Lambda\Psi + i\Psi^\dagger(\hat{E}+\hat{J}-X\Theta)\Psi\Big], \quad (2.87)$$

and now $\hat{E} = \mathrm{diag}(E_1^+,-E_2^-,E_1^+,-E_2^-)$

From Eqs. (2.85), (2.87) after the ensemble averaging and the Hubbard-Stratonovich transformation we obtain the following expression

$$F^{RA}(E_1,E_2,X) = \left(\frac{\pi}{8\Delta}\right)^2\int d[Q]\mathrm{trg}(Q\Sigma_z(1+\Lambda)kQ\Sigma_z(1-\Lambda)k)\exp[-\mathcal{L}_{\mathcal{RA}}] \quad (2.88)$$

where $\mathcal{L}_{\mathcal{RA}}$ is the corresponding effective Lagrangian, which we now discuss. We make a change of basis in which we permute columns 2 and 4, and of rows 2 and 4 of matrices. We will distinguish the matrices in this representation by the addition of a tilde. It straightforward to show that,

$$\tilde{\Sigma}_y = \Sigma_y \Lambda; \quad \tilde{\Sigma}_z = \Sigma_z \Lambda; \quad \tilde{\Lambda} = \Lambda; \quad \tilde{\hat{E}} = \hat{E}. \tag{2.89}$$

Then in this new basis the Lagrangian $\mathcal{L}_{\mathcal{RA}}$ simply coincides with that in Eq. (2.67) if we change the sign of E_2. Moreover, the prefactor in Eq. (2.88) keeps the same form, but with change of sign. Thus, we conclude that $F^{RA}(E_1, E_2, X) = -F^{RR}(E_1, -E_2, X)$, and to find $N(E_1, E_1, X)$ we can take the expression for $F^{RR}(E_1, E_2, X)$ and take its even part in E_1 and E_2. The corresponding intermediate formulas are more cumbersome than those for K and, for brevity we omit many intermediate steps in the derivation. However we emphasize that the calculation proceeds in a completely analogous manner. After collecting the terms which are nonvanishing after Grassmann integration and performing the Grassmann integrals we obtain the expression for $N(E_1, E_1, X)$ in the form of an integral over s_1, s_2 and θ's. We deform the integration contours for θ's as explained in appendix A.2 (see Fig. 2-3) and note that only legs 2 and 2' of the integral contribute to the even part of $F^{RR}(E_1, E_2, X)$. These steps bring us to the following expression

$$\begin{aligned} N(E_1, E_2, X) = &- \left(\frac{1}{2\Delta}\right)^2 \int_1^\infty ds_1 \int_0^1 ds_2 \frac{4 s_1 s_2}{(s_1^2 - s_2^2)^2} \exp\left[- \left(\frac{\pi X}{2N\Delta}\right)^2 \mathrm{tr}(\Theta^2)(s_1^2 - s_2^2)\right] \\ &\times \left\{ (s_1^2 - 1) B_2(E_1, s_1, s_2) B_2(E_2, s_1, s_2) + (1 - s_2^2) B_1(E_1, s_1, s_2) B_1(E_2, s_1, s_2) \right\} \end{aligned} \tag{2.90}$$

where

$$\begin{aligned} B_{1(2)}(E_1, s_1, s_2) &= \cos(\pi E_1 (s_2 - s_1)/\Delta) \\ &+ \frac{1}{4\pi} \int_{-\pi}^{\pi} d\theta_{1B} \int_{-\pi}^{\pi} d\theta_{1F} \exp\left[\frac{i\pi E_1}{\Delta} (s_1 \cos\theta_{1B} - s_2 \cos\theta_{1F}) \right] \\ &\times \frac{(\cos(\theta_{1B}) \cos(\theta_{1F}) - 1)}{(\cos(\theta_{1B}) - \cos(\theta_{1F}))} \left(1 + i \frac{\pi E_1}{\Delta} [s_1 - s_2] \cos(\theta_{1F(1B)}) \right). \end{aligned} \tag{2.91}$$

Parentheses in the subscripts in the last equation mean that it holds for both choices of the index. We note that $B_j(E_1, s_1, s_2)$ $(j = 1, 2)$ can be expressed through the function $L(E_1, s_1, s_2)$ in Eq. (A.36) as

$$B_j(E_1, s_1, s_2) = L(E_1, s_1, s_2) - \frac{iE_1 s_j}{4\Delta} \int_{-\pi}^{\pi} d\theta_{1B} \int_{-\pi}^{\pi} d\theta_{1F}$$
$$\exp\left(\frac{i\pi E_1}{\Delta}[s_1 \cos(\theta_{1B}) - s_2 \cos(\theta_{1F})]\right) \times (\cos(\theta_{1B}) \cos(\theta_{1F}) - 1) \qquad (2.92)$$

With the substitution of E_2, θ_2 for E_1, θ_1 Eqs. (2.91), (2.92) hold for $B_j(E_2, s_1, s_2)$. Equation (2.92) enables us to take the integrals over θ's and to write the final answer in the form of an integral over s_1 and s_2 only. Making use of Eq. (2.16) and rescaling (2.13) we obtain

$$\begin{aligned} n(\epsilon_1, \epsilon_2, x) \;=\; & \pi^4 \int_1^\infty ds_1 \int_0^1 ds_2 \frac{s_1 s_2 \epsilon_1 \epsilon_2}{s_1^2 - s_2^2} \exp(-\frac{\pi^2 x^2}{2}[s_1^2 - s_2^2]) \\ & \times \Big[(s_1^2 + s_2^2 - 1) J_0(\pi\epsilon_1 s_1) J_0(\pi\epsilon_1 s_2) J_0(\pi\epsilon_2 s_1) J_0(\pi\epsilon_2 s_2) \\ & + J_1(\pi\epsilon_1 s_1) J_1(\pi\epsilon_1 s_2) J_1(\pi\epsilon_2 s_1) J_1(\pi\epsilon_2 s_2) \\ & + s_1 s_2 \{J_0(\pi\epsilon_1 s_1) J_0(\pi\epsilon_1 s_2) J_1(\pi\epsilon_2 s_1) J_1(\pi\epsilon_2 s_2) \\ & + (\epsilon_1 \leftrightarrow \epsilon_2)\}\Big] \end{aligned} \qquad (2.93)$$

In contrast to $k(\epsilon_1, \epsilon_2, x)$, the correlator $n(\epsilon_1, \epsilon_2, x)$ measures fluctuations in the matrix elements. As with $k(\epsilon_1, \epsilon_2, x)$, for $\epsilon_1, \epsilon_2 >> 1$, the expression for $n(\epsilon_1, \epsilon_2, x)$ coincides with the result found for usual Dyson ensembles [28].

2.4 Discussion and Conclusion

In this chapter we have examined statistical correlations of the chiral unitary ensemble.In determining the average level density and the two-point density correlation functions we have utilized some new technology for the supersymmetry method developed in [12]. In both cases we have obtained results which coincide with the findings of the orthogonal polynomial method [10, 11, 14, 15, 16, 17, 18, 19]. and the Brownian motion model [24]. These results demonstrate that the universal rescaling introduced

in Ref. [27] applies equally well to the ensemble of chiral random matrices. In addition to the DOS correlators we have obtained a second type of two-point function describing the correlation of matrix elements in response to the action of some external perturbation, a result which does not seem to be accessible from the Brownian motion model. We believe that, in contrast to other methods, this approach can be generalized to treat matrices which belong to chiral orthogonal and symplectic ensembles.

From a technical point of view the new feature of the calculation is the inclusion of Goldstone modes related to the chiral transformations in addition to the usual unitary transformations. The method presented in this chapter proved useful in the study of disordered Andreev billiards [13]. These billiards are comprised of a piece of normal metal imbedded in a superconductor.

The description of Macédo, shows that the chGUE corresponds to a gas of non-interacting one-dimensional Fermions whose wave functions are Bessel functions $J_0(kr)$ [24]. In this interpretation the average over the chGUE is equivalent to the average over the ground state corresponding to the filled Fermi sea. Eq. (2.79) can be recognized as the irreducible part of time dependent density-density correlation function of the Fermi gas with the parameter of the perturbation related to the Euclidean time, τ through the relation $x^2 = 2|\tau|$. The integration can be interpreted as a sum over the particle-hole excitations, with the eigenvalues of the "unitary" rotations, s_1 and s_2 corresponding to the momenta of holes and particles. Integration over the "chiral" variables θ generates the wave functions.

We remark that this class of matrices has a wider application encompassing the spectral properties of the Dirac operator of QCD [14]. In the QCD partition function, the eigenvalues show fluctuations over the ensemble of gauge field configurations. As such, the microscopic correlations are universal and can be described by random matrix theory. The symmetries of the QCD Dirac operator dictate chiral random matrix ensembles. In particular, if the topological charge and the number of flavors are zero, the results presented above for the average and fluctuation of the density of states can be compared with expressions from Ref. [14]. Whether the matrix ele-

ment correlations embodied in the response function $n(\epsilon_1, \epsilon_2, x)$, or in the parametric dependencies can find application in the field of QCD is left for the subject of future investigation.

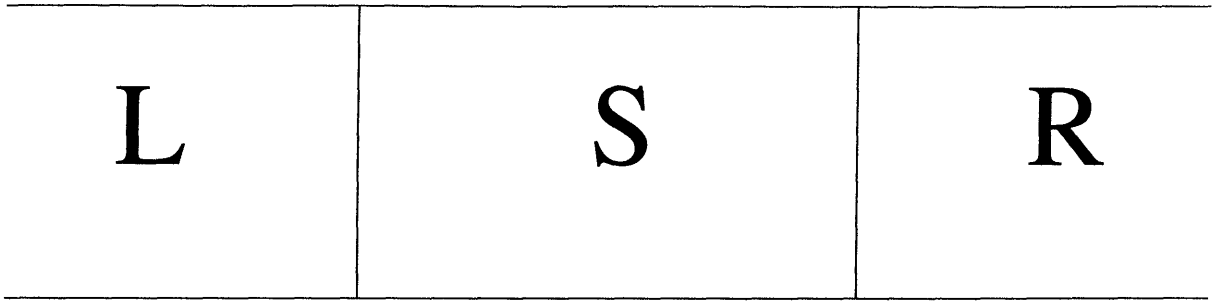

Figure 2-1: A schematic representation of the scattering system: two identical ideal leads L and R are connected to the scatterer S.

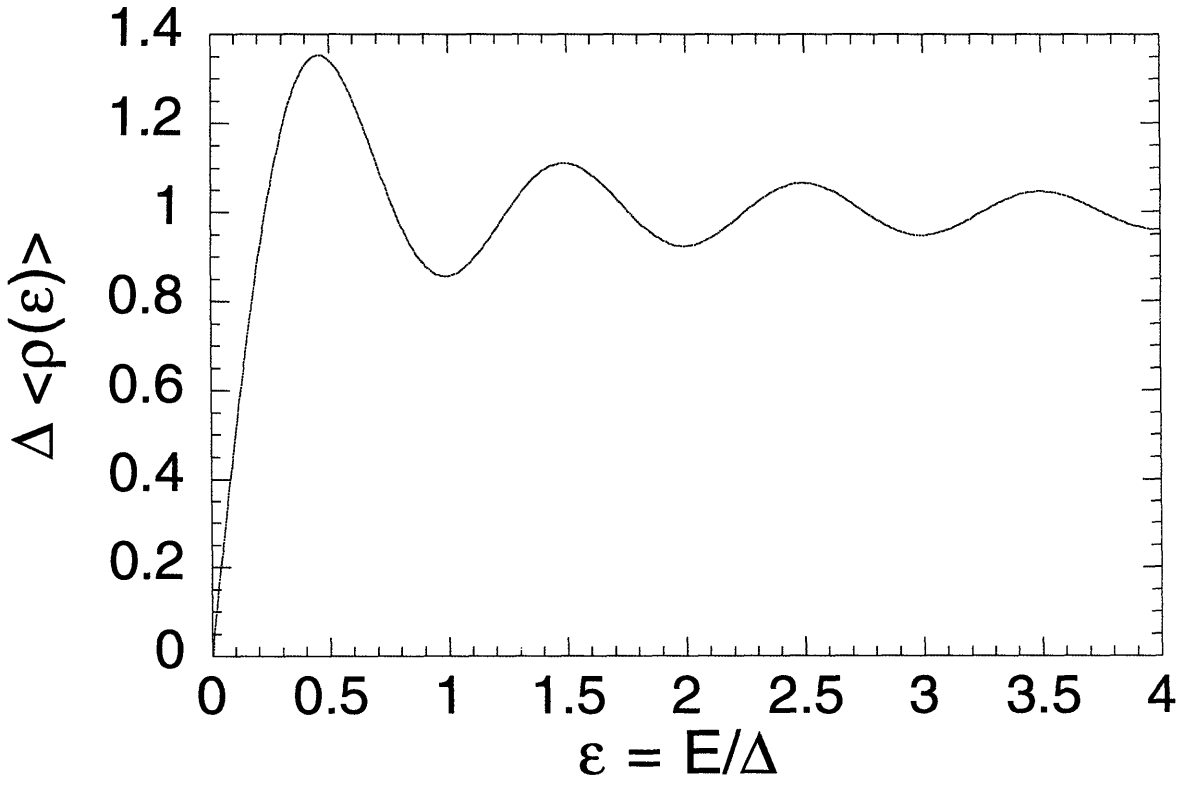

Figure 2-2: The dimensionless average DOS $\Delta\langle\rho(\epsilon)\rangle$ is plotted versus the dimensionless energy $\epsilon = E/\Delta$.

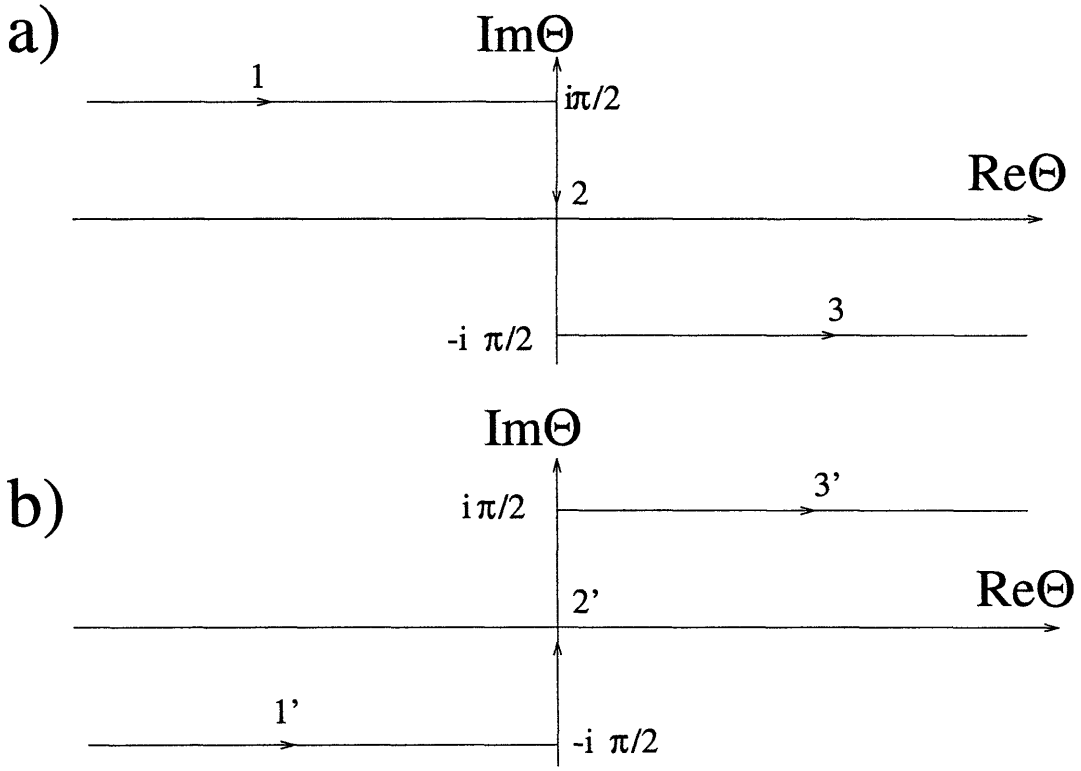

Figure 2-3: The integration contour after deformation for variable θ_{1B} a) for the first exponential and b) for the second exponential in Eqn. (A.12).

Chapter 3

Correlators of Spectral Determinants in Quantum Chaos

In this chapter we present the generalization [34] of the recently introduced approach [35] and use it to study arbitrary correlators of spectral determinants in quantum chaotic systems with broken T-invariance. The utility of this method for obtaining generating functions for a variety of universal correlators is illustrated with several applications.

Despite the success of the supersymmetry approach [20] in describing the universal properties that characterize the phenomena of quantum chaos, its application is still at present restricted to a subset of correlators that involve at most two-points. More traditional methods of random matrix theory [7] provide a complementary approach, although their application again brings some restrictions. In this chapter we will generalize a third approach, originally introduced by Guhr [35], to study a whole class of universal correlation functions applicable to quantum chaotic systems with broken T-invariance (unitary symmetry). We closely follow Ref. [34]. The virtue of this approach, which relies on a superalgebraic construction, lies in its straightforward application. Since, in contrast to Efetov's supersymmetry method, final expressions are not presented in terms of an integration over a restricted saddle-point manifold (the non-linear σ-model), this technique draws no distinction between two and higher-point functions. All are equally tractable.

The function that we consider involves the general many-point correlation of spectral determinants

$$W(\{U;V\}) = a_{lm}\left\langle \frac{\prod_{i=1}^{m}\det(V_i - H)}{\prod_{j=1}^{l}\det(U_j - H)}\right\rangle, \tag{3.1}$$

where the normalization $a_{lm} = \langle|\det H|^{l-m}\rangle$ ensures that W remains finite as the number of levels $N \to \infty$. We will restrict attention to even values of $l+m$ where the correlation function becomes universal on scales of $\{U\}$ and $\{V\}$ comparable to the average level spacing. In the same limit the correlator for $l+m$ odd is vanishingly small.

Apart from its role as a generating function for density of states (DoS) fluctuations [35], W is related to a number of distribution functions. One example involves the curvature distribution derived in Ref. [36]. A straightforward generalization leads to the following expression for the generating function of the joint curvature distribution

$$\begin{aligned} K(s,\omega) &= \sum_{\mu}\left\langle \exp\left[-\frac{is}{2}\frac{\partial^2\epsilon_\mu}{\partial x^2}\right]\delta(\epsilon_\mu)\nu(\omega)\right\rangle \\ &= \frac{\pi^2}{3}\frac{\omega^3}{\omega - is}W(is;0,0,0,\omega,\omega), \end{aligned} \tag{3.2}$$

where $\nu(\epsilon) = \mathrm{Tr}\,\delta(\epsilon - H/\Delta)$ denotes the dimensionless operator for DoS. Here we have introduced $\epsilon_\mu = E_\mu/\Delta$ with $\Delta = \langle E_{\mu+1} - E_\mu\rangle$, and $x = X\sqrt{C(0)}$ which parametrizes an arbitrary external perturbation with $C(0) = \langle(\partial\epsilon_\mu/\partial X)^2\rangle$ [37, 27]. A second example concerns the generating function for the joint distribution of local DoS

$$\begin{aligned} L(s,\omega) &= \left\langle \exp\left[2is\gamma\sum_{\mu}\frac{|\psi_\mu(\eta)|^2}{\epsilon_\mu^2+\gamma^2}\right]\nu(\omega)\right\rangle \\ &= \frac{\omega^2+\gamma^2}{\omega^2+\alpha^2}W(i\alpha,-i\alpha;i\gamma,-i\gamma,\omega,\omega), \end{aligned} \tag{3.3}$$

where $\alpha^2 = \gamma^2 - is\gamma$, Re $\alpha > 0$, and $\psi_\mu(\eta)$ denotes the μ-th wavefunction at site η. Eqs. (3.2) and (3.3) are both straightforwardly obtained by exploiting the statistical independence of the spectra and wavefunctions and making use of the Porter-Thomas distribution [38].

Without the additional correlation to the DoS, both have been examined in the recent literature. The curvature distribution has been used as the first indicator of universality in statistics that depend on an external parameter [36, 37, 39], while the distribution of local DoS, measured through NMR, has been studied as a signature of chaotic behavior in mesoscopic metallic grains [40, 41].

Although Eq. (3.1) seems amenable to traditional orthogonal polynomial methods of random matrix theory [7] such as that used by Ref. [41], the theory rapidly becomes intractable as the number of points in the correlator increases. At the same time, as will become clear, the disparity of the order of the determinants in the numerator and denominator rules out conventional supersymmetry approaches. We will show that for unitary symmetry a third approach, which generalizes a method introduced by Guhr [35], yields the following exact expression involving the dimensionless parameters $u_i = U_i/\Delta$ and $v_i = V_i/\Delta$,

$$\begin{aligned} W(\{U;V\}) &= A_N \sum_{P[\{v\}]} F_{lm}(\{u;v\})\, e^{i\pi\left(\sum_{i=1}^{l_*} u_i - \sum_{i=l_*+1}^{l} u_j - \sum_{r=1}^{m_*} v_r + \sum_{s=m_*+1}^{m} v_s\right)}, \\ F_{lm}(\{u;v\}) &= \frac{\prod_{r=1}^{m_*}\prod_{j=l_*+1}^{l}(u_j - v_r)\prod_{i=1}^{l_*}\prod_{s=m_*+1}^{m}(u_i - v_s)}{\prod_{i=1}^{l_*}\prod_{j=l_*+1}^{l}(u_i - u_j)\prod_{r=1}^{m_*}\prod_{s=m_*+1}^{m}(v_r - v_s)}. \end{aligned} \tag{3.4}$$

A_N denotes a normalization constant, and for convenience, we have made the ordering such that $\{u_1, \cdots u_{l_*}\}$ represent the l_* parameters for which Im $u_i > 0$ while $\{u_{l_*+1}, \cdots u_l\}$ denote those elements with Im $u_i < 0$. With $m_* = l_* + (m-l)/2$, the summation is performed over the ${}^mC_{m_*} = m!/m_*!(m-m_*)!$ permutations which interchange elements v_r and v_s between the two summations in the exponential. Our goal will be to obtain this expression using a Gaussian distribution of Hermitian random matrices with unitary symmetry, and to demonstrate its utility by obtaining explicit expressions for $K(s,\omega)$ and $L(s,\omega)$. A more detailed discussion of this general approach together with some applications can be found in a longer paper [42]. The coincidence of the statistical properties of random matrix ensembles with the universal properties of quantum chaos is well studied in the literature (see, for example, Refs. [43]), and we will not discuss it further here.

The starting point, which is common to the supersymmetry method, involves constructing an expression for W in the form of a Gaussian integral,

$$W(\{U;V\}) = a_{lm}(-1)^{Nl_*}(2\pi i)^{N(m-l)} \int d[\psi] \left\langle e^{-S_0-S_1} \right\rangle,$$
$$S_0 = -i\psi^\dagger g\hat{Z}\psi, \quad S_1 = i\psi^\dagger gH\psi, \tag{3.5}$$

where $\hat{Z} = \mathrm{diag}(U_1,\ldots,U_l,V_1,\ldots,V_m)$, and $\psi^T = (\vec{S}_1,\ldots,\vec{S}_l,\vec{\chi}_1,\ldots,\vec{\chi}_m)$ denotes the $(l+m)\times N$ component fields with complex bosonic $\vec{S}$ and fermionic $\vec{\chi}$ variables [44]. Formally, the convergence of Eq. (3.5) is assured by the inclusion of the metric $g = \mathrm{diag}(\mathbf{1}_{l_*}, -\mathbf{1}_{m+l-l_*})$ [21].

The ensemble average over the Gaussian distribution of H

$$P(H)\, dH = C_N \exp\left[-\frac{\pi^2}{2N\Delta^2}\mathrm{Tr}H^2\right]\, dH, \tag{3.6}$$

where C_N is the normalization constant [7], leads to an effective action,

$$\langle e^{-S_1}\rangle = e^{-S_{\mathrm{eff}}}, \quad S_{\mathrm{eff}} = -\frac{N\Delta^2}{2\pi^2}\mathrm{STr}\left[i\sum_\mu g^{1/2}\psi_\mu \otimes \psi_\mu^\dagger g^{1/2}\right]^2. \tag{3.7}$$

The trace or supertrace operation for supermatrices follows the convention $\mathrm{STr}M = \mathrm{Tr}M_{\mathrm{BB}} - \mathrm{Tr}M_{\mathrm{FF}}$, where M_{BB} and M_{FF} respectively denote the boson-boson and fermion-fermion block of M. The quartic interaction of the fields can be decoupled by the Hubbard-Stratonovich transformation

$$e^{-S_{\mathrm{eff}}} = A_{lm}\int d[Q]\exp\left[-\frac{N}{2}\mathrm{STr}\ Q^2 + \frac{iN\Delta}{\pi}\psi^\dagger g^{1/2}Qg^{1/2}\psi\right], \tag{3.8}$$

where Q denote $(l+m)\times(l+m)$ supermatrices with a block structure reflecting that of $\psi\otimes\psi^\dagger$, and $A_{lm} = 2^{-(l+m)/2}(N/\pi)^{(l-m)^2/2}$. Combining Eq. (3.8) with Eq. (3.5), integrating over ψ, and shifting the integration variables $Q \to Q - \pi\hat{z}/N$, we obtain

$$W(\{U;V\}) = a_{lm}A_{lm}\int d[Q]\exp\left[-\frac{N}{2}\mathrm{STr}\ \left(Q-\frac{\pi\hat{z}}{N}\right)^2 - N\mathrm{STr}\ \ln\left(\frac{N\Delta}{\pi}Q\right)\right], \tag{3.9}$$

where $\hat{z} = \hat{Z}/\Delta$.

Thus far, the method departs from the conventional supersymmetry approach [20, 21, 32] only in that it allows for a different number of bosonic and fermionic variables. At this stage, however, instead of forming the usual expansion around the saddle-point of Eq. (3.9) to obtain a non-linear σ-model, we will exploit the fact that the expression is of the form of an Itzykson-Zuber integral [45] over the full pseudounitary supergroup that diagonalizes the supermatrices Q. This approach was introduced by Guhr [35] who used equal number of bosonic and fermionic variables to examine the high-point correlator of DoS fluctuations in unitary ensembles (see also Ref. [46]).

The interaction $S_{\rm eff}$ in Eq. (3.7) is invariant under the action of the pseudounitary supergroup $\mathrm{SU}(l_*, l - l_*/m)$ [21]. This is reflected in the structure of the Hubbard-Stratonovich field Q. The supermatrix Q can be diagonalized by a matrix $T \in \mathrm{SU}(l_*, l - l_*/m)$ such that $Q = T^{-1}ST$, where $S = \mathrm{diag}(b_1, \ldots b_l, if_1, \ldots, if_m)$ denotes the matrix of eigenvalues with b_i and f_i taking values on the range $-\infty$ to ∞. The integration measure is then given by

$$\begin{aligned} d[Q] &= \text{Const. } B_{lm}^2(S) d[S] d\mu[T], \\ B_{lm}(S) &= \frac{\prod_{r<s}^{l}(b_r - b_s)\prod_{p<q}^{m}(if_p - if_q)}{\prod_{i=1}^{l}\prod_{j=1}^{m}(b_i - if_j)}. \end{aligned} \tag{3.10}$$

Previous studies have demonstrated the extension of the Itzykson-Zuber integral to the superunitary group $\mathrm{SU}(l/m)$ [47, 48]. Similar considerations suggest a further extension to the pseudounitary supergroup $\mathrm{SU}(l_*, l - l_*/m)$. However, since these arguments are somewhat technically involved we will reserve their discussion to a longer paper [42] and make use of several applications of Eq. (3.4) to justify the validity of this approach. The result of the psuedounitary integration closely parallels that of superunitary group and leads to the expression [47, 48]

$$\int d[T]\; e^{[\pi \mathrm{STr}(T^{-1}ST\hat{z})]} = \text{Const.} \times \frac{e^{[\pi \mathrm{STr}(S\hat{z})]}}{B_{lm}(S)B_{lm}(\pi\hat{z})}. \tag{3.11}$$

Combining Eqs. (3.9), (3.10), and (3.11), and shifting back the integration variables

$S \to S + \pi\hat{z}/N$, we obtain

$$W(\{U;V\}) = \text{Const.} \times \int d[S] \, \frac{B_{lm}(S + \pi\hat{z}/N)}{B_{lm}(\hat{z})} \times \exp\left[-\frac{N}{2}\text{STr}S^2 - N\text{STr}\ln\left\{\frac{N\Delta}{\pi}\left(S + \frac{\pi\hat{z}}{N}\right)\right\}\right]. \quad (3.12)$$

The problem of evaluating the correlator of spectral determinants has been reduced to a set of $l + m$ real integrations over the eigenvalues of S. It is at this stage that we make use of large N to estimate the integral by means of the saddle-point approximation as described in Ref. [35]. For problems of physical interest the dimensionless source $\hat{z}$ is of order unity and does not affect the saddle-points: $S_0 = \text{diag}(b_{01}, \ldots b_{0l}, if_{01}, \ldots, if_{0m})$ where the elements $\{b_{oi}\}$ and $\{if_{oi}\}$, in principle, take values of $\pm i$. However, the saddle-point values of the bosonic variables $b_{0i} = \pm i$ lie off the real axis, and a deformation of the integration contour is required to reach them. This has to be done in such a way that singularities of the integrand are not crossed. In particular, the signs of the eigenvalues, Im b_{io} must be chosen to be consistent with those of Im u_i. This implies that $[S_0]_{\text{BB}} = ig_{\text{BB}}$. Conversely, the saddle-points associated with the fermionic degrees of freedom f_{0i} lie on the real axis, and must all be taken into account.

The leading order contribution to the integral comes from the value of the integrand at the saddle-points. Fluctuations give corrections which are small as $1/N$ [35]. As a result, we obtain

$$W(\{U;V\}) = \text{Const.} \times (-1)^{N(m_*-l_*+(l-m)/2)} \times \sum_{\{S_0\}} \frac{B_{lm}(S_0 + \pi\hat{z}/N)}{B_{lm}(\hat{z})} e^{i\pi\left(\sum_{i=1}^{l_*} u_i - \sum_{i=l_*+1}^{l} u_j - \sum_{r=1}^{m_*} v_r + \sum_{s=m_*+1}^{m} v_s\right)}, \quad (3.13)$$

where m_* is the number of $+i$'s in the fermionic block of S_0, and $\sum_{\{S_0\}}$ denotes the sum over all possible saddle-points. This involves the interchange of all possible signs of if_{oi} in the fermionic sector of S_0. However, although the number of such terms is 2^m, not all of them are of the same order. Some are small as $1/N$ significantly reducing the number of terms that must be taken into account. This can be seen by considering

a typical factor arising from the numerator of the integrand: $(if_{0i} + \pi y_i/N - if_{0j} - \pi y_j/N)$. If $if_{0i} = if_{0j}$ this factor is proportional to $1/N$, while if $if_{0i} = -if_{0j}$, y_i and y_j can be neglected and it becomes of order unity. If there are l_* terms which take the value of i in the bosonic sector of S_0, and m_* in the fermionic sector, there is a relative factor of $N^{(m-l-m_*+l_*)(m_*-l_*)}$ multiplying the contribution of this point to Eq. (3.13). The maximum of this factor is achieved when $m_* = l_* + (m-l)/2$. Then only ${}^mC_{m_*}$ saddle-points have to be taken into account. Applying this condition we arrive at the expression shown in Eq. (3.4).

Eq. (3.4) represents the central result of this chapter. As a generating function, the correlator of spectral determinants allows access to a number of useful correlation functions. To conclude, we will apply Eq. (3.13) to determine algebraic expressions for several examples. As a simple application we begin by considering the generating function for local DoS

$$\begin{aligned} P(s) &= \left\langle \exp\left[2is\gamma \sum_\mu \frac{|\psi_\mu(\eta)|^2}{\epsilon_\mu^2 + \gamma^2}\right]\right\rangle \\ &= W(i\alpha, -i\alpha; i\gamma, -i\gamma), \end{aligned} \tag{3.14}$$

where the notation is taken from Eq. (3.3). As a supersymmetric combination, this average can be compared with the known result first obtained by Efetov and Prigodin [40]. Adopting the approach above, $l_* = 1$, $m_* = 1$, and we need consider saddle-point contributions from ${}^2C_1 = 2$ terms: $S_0 = \mathrm{diag}(i, -i; i, -i)$ and $S_0 = \mathrm{diag}(i, -i; -i, i)$. Applying Eq. (3.4) we obtain

$$P(s) = P_1(\alpha, \gamma), \qquad P_\mu(\alpha, \gamma) = \sum_{\sigma=\pm 1} \frac{(\alpha + \sigma\gamma)^2}{4\sigma^\mu \alpha\gamma} e^{-2\pi(\alpha - \sigma\gamma)}. \tag{3.15}$$

This result coincides with that obtained in Ref. [40] (see also Ref. [41]).

Having verified this approach with a known example, let us consider the two generating functions defined previously in Eqs. (3.2) and (3.3). Both cases require the application of a non-supersymmetric construction. Beginning with the joint curvature

distribution, applying Eq. (3.4) to the case where $s > 0$, we obtain

$$K(s,\omega) = \frac{\omega e^{-\pi s}}{\omega - is}\Big[\frac{(is-\omega)^2}{\omega^2}R_2(\omega) - \frac{is}{\omega^3}\left(\omega\left(\frac{1}{\pi}+s\right) - \frac{2is}{\pi}\right)e^{-\pi i\omega}\sin(\pi\omega) + \frac{s}{\omega^2}\left(\omega(i+\pi\omega)+s(2-i\pi\omega)\right)\Big], \tag{3.16}$$

where $R_2(\omega) = 1 - \sin^2(\pi\omega)/(\pi\omega)^2$ denotes the two-point correlator of DoS [7]. (An expression for values of $s < 0$ can be found by complex conjugation of Eq. (3.16).)

The validity of Eq. (3.16) can be tested by considering two limiting cases. Firstly, for $s = 0$, $K(0,\omega)$ describes the two-point correlator of DoS fluctuations [7]. Secondly, as $\omega \to \infty$, the generating function should collapse to the disconnected average involving the average DoS and the Fourier transform of the known curvature distribution [39, 36]. An inspection confirms that both limits are realised by Eq. (3.16). We remark that, in the limit of $\omega \to 0$, $K(s,\omega) = R_2(\omega)\,\omega/(\omega - is)$. Its Fourier transform implies a joint curvature distribution which vanishes for $\omega\ \partial^2\epsilon_\mu/\partial x^2 > 0$ and decays exponentially at a rate proportional to ω for $\omega\ \partial^2\epsilon_\mu/\partial x^2 < 0$. This contrasts with the power law decay of the uncorrelated curvature distribution ($\omega \to \infty$).

Turning to the second generating function, Eq. (3.4) implies

$$L(s,\omega) = i\pi\omega\frac{\gamma^2-\alpha^2}{\gamma^2+\omega^2}P_2(\alpha,i\omega) + \frac{1}{\pi}\frac{\pi\omega^2+\pi\alpha^2+\alpha}{\gamma^2+\omega^2}P_1(\alpha,\gamma) - \frac{\gamma}{\pi}\frac{\omega^2+\alpha^2}{(\gamma^2+\omega^2)^2}P_2(\alpha,\gamma). \tag{3.17}$$

where $P_{1,2}$ are defined in Eq. (3.15). In this case, the validity of Eq. (3.17) can also be checked in the same limits. For $s = 0$ ($\alpha = \gamma$), $L(0,\omega)$ corresponds to the average DoS, which is independent of ω. On the other hand, as $\omega \to \infty$ the decoupling of the average recovers the generating function for the local DoS already obtained in Eq. (3.15). Again, both limits bare inspection.

Eqs. (3.16) and (3.17) represent just two examples of where the average W can be exploited. Further examples include generalisations of the distribution of resonance conductance peaks in quantum dots [49] as well as the sensitivity of chaotic wavefunction intensities to changes in an external perturbation [50]. The extension of this approach to orthogonal and symplectic symmetry relies on the construction

of the appropriate Itzykson-Zuber integral corresponding to Eq. (3.11). However, to our knowledge, for the pseudoorthogonal and pseudosymplectic supergroup, such a generalisation has yet to be found.

To conclude, in this chapter we have obtained an exact analytical expression for a whole class of correlators that characterize quantum chaos for systems without T-invariance. The utility of this approach has been demonstrated with the derivation of two distribution functions.

Chapter 4

Spectral Statistics Beyond Random Matrix Theory.

In this chapter we follow Ref. [51]. Using a nonperturbative approach we examine the large frequency asymptotics of the two-point level density correlator in weakly disordered metallic grains. We find that the singularities of the structure factor at the Heisenberg time (present for random matrix ensembles) are washed out when conductance is finite. The results are nonuniversal (they depend on the shape of the grain and on its conductance), though they suggest a generalization for any system with finite Heisenberg time.

A great variety of physical systems are known to exhibit quantum chaos. The common examples are atomic nuclei, Rydberg atoms in a strong magnetic field, electrons in disordered metals, etc [52]. Chaotic behavior manifests itself in the energy level statistics. It was a remarkable discovery of Wigner and Dyson, that these statistics in a particular system can be approximated by those of an ensemble of random matrices (RM). Here we consider deviations from the RM theory taking an ensemble of weakly disordered metallic grains with a finite conductance g as an example. The results seem to be extendible to general chaotic systems.

There are two characteristic energy scales associated with a particular system: a classical one E_c and a quantum one. The quantum energy scale is the mean level spacing Δ. In a chaotic billiard, for example, E_c is set by the frequency of the shortest

periodic orbit. Well developed chaotic behavior can take place only if $E_c \gg \Delta$.

In a disordered metallic grain the classical energy is the Thouless energy $E_c = D/L^2$, where D is the diffusion constant, and L is the system size. For a weakly disordered grain the two scales are separated by the dimensionless conductance $g = E_c/\Delta \gg 1$ [53]. For frequencies $\omega \ll E_c$ the behavior of the system becomes universal (independent of particular parameters of the system). In this regime in the zeroth approximation the level statistics depend only on the symmetry of the system and are described by one of the RM ensembles: unitary, orthogonal or symplectic [7].

One of the conventional statistical spectral characteristics is the two-point level density correlator

$$K(\omega, x) = \langle \rho(\epsilon + \omega, \hat{H} + x\delta\hat{H})\rho(\epsilon, \hat{H})\rangle - \Delta^{-2}, \tag{4.1}$$

where $\hat{H}$ is the Hamiltonian of the system, $\delta\hat{H}$ is a perturbation, x is the dimensionless perturbation strength and $\rho(\epsilon, \hat{H} + x\delta\hat{H}) = \mathrm{Tr}\delta(\epsilon - \hat{H} - x\delta\hat{H})$ is the x-dependent density of states at energy ϵ. It is convenient to introduce the dimensionless frequency $s = \omega/\Delta$ and the dimensionless correlator $R(s, x) = \Delta^2 K(\omega, x)$. Dyson [54] determined $R(s, x = 0)$ for RM. For example, $R(s, o)$ in the unitary case plotted in the insert in Fig. 4-1 equals to

$$R(s, 0) = \delta(s) - \sin^2(\pi s)/(\pi s)^2. \tag{4.2}$$

Perhaps the most striking signature of the Wigner-Dyson statistics is the rigidity of the energy spectrum [43]. Among the major consequences of this phenomenon are: a) the probability to find two levels separated by $\omega \ll \Delta$ vanishes as $\omega \to 0$; b) the level number variance in an energy strip of width $N\Delta$ is proportional to $\ln N$ rather than N; and c) oscillations in the correlator $R(s, 0)$ in Eq. (4.2) decay only algebraically.

In the two level structure factor [55] $S(\tau, x) = \int_{-\infty}^{\infty} ds \exp(i\tau s) R(s, x)$ the reduced fluctuations of the level number manifest themselves in the vanishing of $S(\tau, 0)$ at $\tau = 0$, and the algebraic decay of the oscillations in $R(s, 0)$ leads to the singularity

in $S(\tau,0)$ at the Heisenberg time $\tau = 2\pi$. In the unitary case, e. g. $S(\tau,0) = \min\{|\tau|/(2\pi),1\}$. At $\tau \ll 2\pi$ this Dyson result was obtained by Berry [56] for a generic chaotic system by use of a semiclassical approximation. To the best of our knowledge nobody succeeded in analyzing the behavior of $S(\tau,0)$ around $\tau = 2\pi$ using this formalism.

Wigner-Dyson statistics become exact in the limit $g = E_c/\Delta \to \infty$. We consider corrections to these statistics for finite g. One of the better understood systems in this respect is a weakly disordered metallic grain. For frequencies much smaller than E_c the statistics are close to universal ones, the corrections being small as $(s/g)^2$ [57]. At $s \gg 1$ the monotonic part of $R(s,x)$ can be obtained perturbatively [58]

$$R_{\rm p}(s,x) = \Re \sum_{\mu} \left[\alpha\pi^2(-is + x^2 + \epsilon_\mu)^2\right]^{-1}, \tag{4.3}$$

where ϵ_μ are eigenvalues (in units of Δ) of the diffusion equation in the grain, $\alpha = 2$ for the unitary ensemble and $\alpha = 1$ for the orthogonal and symplectic ensembles [59]. At this point we can define

$$E_c = \epsilon_1\Delta/\pi^2, \qquad g = \epsilon_1/\pi^2, \tag{4.4}$$

where ϵ_1 is the smallest nonzero eigenvalue. Perturbation theory allows one to determine $S(\tau,0)$ at small times $\tau \ll 1$. Since the oscillatory part of $R(s,x)$ is non-analytic in $1/s$ it can not be obtained perturbatively.

In this chapter we obtain the leading $s \gg 1$ asymptotics of $R(s,x)$ retaining the oscillatory terms [60] and monitor how the singularity in $S(\tau,0)$ at the Heisenberg time is modified by the finite conductance g. We make use of the nonterturbative approach [20] that is valid for arbitrary relation between s and g. The oscillatory part $R_{\rm osc}(s,x) \equiv R(s,x) - R_{\rm p}(s,x)$ for the unitary (u), orthogonal (o) and symplectic (s) cases equals to

$$R^{u}_{osc}(s,x) = \frac{\cos(2\pi s)}{2\pi^2|y|^2}P(s,x), \tag{4.5}$$

$$R^{o}_{osc}(s,x) = \frac{\cos(2\pi s)}{2\pi^4|y|^4}P^2(s,x), \tag{4.6}$$

$$R^{s}_{osc}(s,x) = \frac{\cos(\pi s)}{2|y|}P(s,x) - \frac{\cos(2\pi s)}{2\pi^4|y|^4}P^2(s,x), \tag{4.7}$$

where $y = x^2 - is$, and $P(s,x)$ is the spectral determinant of the diffusion operator

$$P(s,x) = \prod_{\mu,\epsilon_\mu\neq 0}\left[\left(\frac{s}{\epsilon_\mu}\right)^2 + \left(1+\frac{x^2}{\epsilon_\mu}\right)^2\right]^{-1}. \tag{4.8}$$

Note that Eq. (4.3) expresses $R_p(s,x)$ through the Green function of this operator. Thus, regardless of the spectrum ϵ_μ, $R_p(s,x)$ and $R_{osc}(s,x)$ are related:

$$R_p(s,x) = \Re\frac{1}{\alpha\pi^2 y^2} - \frac{1}{2\alpha\pi^2}\frac{\partial^2 \ln[P(s,x)]}{\partial s^2}. \tag{4.9}$$

It follows from Eq. (4.8) that $P(s,x)$ decays exponentially at $s \gg g$. As a result, the singularity in $S(\tau,0)$ at the Heisenberg time is washed out: $S(\tau,0)$ becomes *analytic* around $\tau = 2\pi$. The scale of smoothening of the singularity is $1/E_c$ (see Fig. 4-1). At $1 \ll s \ll g$ the sums of Eqs. (4.5), (4.6) or (4.7) and Eq. (4.3) give the leading high frequency asymptotics of the universal results, for $s \gg g$ they coincide with the perturbative result $R_P(s,x)$ of Ref. [58].

In a closed (Dirichlet boundary conditions) d-dimensional cubic sample $\epsilon_\mu = g\pi^2\vec{n}^2$, where $\vec{n} = (n_1,\ldots,n_d)$ and n_i are non-negative integers. For $s \gg g$ and $d < 4$ we have $P(s,0) \approx \exp\{-\pi(s/\pi g)^{d/2}/[\Gamma(d/2)d\sin(\pi d/4)]\}$. At $1 \ll s \ll g$

$$R(s,0) = -\frac{\sin^2(\pi s)}{(\pi s)^2} + \frac{\sin^2(\pi s)}{\pi^2 g^2}\sum_{\vec{n}}\frac{1}{(\pi^2\vec{n}^2)^2}, \tag{4.10}$$

This result was shown in Ref. [57] to be valid even for $s < 1$. One can assume that the sum of Eq. (4.5) and Eq. (4.3) gives the correct $g \gg 1$ asymptotics at arbitrary frequency for the unitary ensemble. Recall that the lowest order of perturbation

theory for $\tau < 2\pi$ gives the exact result $S(\tau, 0) \propto \tau$.

Now we sketch the derivation of our results. Consider a quantum particle moving in a random potential $V(\vec{r})$. The perturbation acting on the system is a change in the potential $\delta V(\vec{r})$. Both $V(\vec{r})$ and $\delta V(\vec{r})$ are taken to be white noise random potentials with variances $\langle V(\vec{r})V(\vec{r'})\rangle = \delta(\vec{r}-\vec{r'})/2\pi\nu\tau$ and $\langle \delta V(\vec{r})\delta V(\vec{r'})\rangle = x^2\Delta\delta(\vec{r}-\vec{r'})/(4\pi\nu)$, $\Delta\tau \ll 1$, $\langle V(\vec{r})\delta V(\vec{r'})\rangle = 0$, where $\langle\rangle$ denotes ensemble averaging and ν is the density of states per unit volume. The dimensionless perturbation strength x^2 is assumed to be of order unity.

We use the supersymmetric nonlinear σ-model introduced by Efetov [20], and follow his notations everywhere. One can show that for the system under consideration the σ-model expression for $K(\omega, x)$ is given by

$$K(\omega, x) = -\frac{1}{\pi^2}\Re\frac{\partial^2}{\partial J^2}\int \mathcal{D}Q \exp\{-F_J(\Lambda)\}\Big|_{J=0}, \tag{4.11}$$

The 8×8 supermatrix $Q(\vec{r})$ obeys the constraint $Q^2 = 1$ and takes on its values on a symmetric space $\mathbf{H} = \mathbf{G}/\mathbf{K}$, where $\mathbf{G}$ and $\mathbf{K}$ are groups [21]. In the unitary case $\mathbf{H} = \mathbf{U(1,1/2)}/\mathbf{U(1/1)} \otimes \mathbf{U(1/1)}$ [31]. The integration measure for Q in Eq. (4.11) is the invariant measure on $\mathbf{H}$ and

$$F_J(\Lambda) = \frac{\pi\nu}{8}\int d\vec{r}\mathrm{STr}\left\{D(\nabla Q)^2 + 2i\omega\Lambda Q + iJ\Lambda kQ - \frac{x^2\Delta}{2}(\Lambda Q)^2\right\}. \tag{4.12}$$

The hierarchy of blocks of supermatrices is as follows: advanced-retarded (A-R) blocks, fermion-boson (F-B) blocks, and blocks corresponding to time-reversal. $\Lambda = \mathrm{diag}\{1,1,1,1,-1,-1,-1,-1\}$ is the matrix breaking the symmetry in the advanced-retarded (A-R) space, $k = \mathrm{diag}\{1,1,-1,-1,1,1,-1,-1\}$ is the symmetry breaking matrix in the Fermion-Boson (F-B) space.

The large frequency asymptotics of $K(\omega, x)$ can be obtained from Eq. (4.11) by use of the stationary phase method. Perturbation theory corresponds to integrating over the small fluctuations of Q around Λ [20],

$$Q = \Lambda(1+iP)(1-iP)^{-1}, P = \begin{pmatrix} 0 & B \\ \bar{B} & 0 \end{pmatrix}, \tag{4.13}$$

where the matrix P describes these small fluctuations.

$Q = \Lambda$ is not the only stationary point on $\mathbf{H}$. This fact to the best of our knowledge was not appreciated in the literature. The existence of other stationary points makes the basis for our main results.

It is possible to parameterize fluctuations around a point Q_0 in the form $Q = Q_0(1+iP_0)(1-iP_0)^{-1}$. Expanding the Free Energy F_J in Eq. (4.12) in P_0 we would obtain the stationarity condition $\partial F_J/\partial P_0 = 0$. This route however is inconvenient because the parametrization of P_0 will depend on Q_0. Instead we perform a global coordinate transformation on $\mathbf{H}$ that maps Q_0 to Λ, $Q_0 \to T_0^{-1}Q_0T_0 = \Lambda$. We note that the matrices Λ and $-\Lambda k$ belong to $\mathbf{H}$, and the corresponding terms in Eq. (4.12) can be viewed as symmetry breaking sources. This transformation changes the sources, but allows us to keep the parametrization of Eq. (4.13) and preserves the invariant measure. Introducing the notation $Q_\Lambda = T_0^{-1}\Lambda T_0$ and $Q_{\Lambda k} = T_0^{-1}\Lambda k T_0$ we write $K(\omega, x)$ in the form of Eq. (4.11) if $F_J(\Lambda)$ is substituted by $F_J(Q_\Lambda)$ given by

$$F_J(Q_\Lambda) = \frac{\pi\nu}{8}\int d\bar{r}\mathrm{STr}\left\{D(\nabla Q)^2 + 2i\omega Q_\Lambda Q + iJQ_{\Lambda k}Q - \frac{x^2\Delta}{2}(Q_\Lambda Q)^2\right\} \tag{4.14}$$

The stationarity condition $\partial F_J(Q_\Lambda)/\partial P|_{P=0} = 0$ implies that all the elements of Q_Λ in the AR and RA blocks should vanish (this can be seen from Eq. (4.13)).

Here we discuss in detail only the calculation for the unitary ensemble. The calculation for the other cases proceeds analogously, and we just point out the important differences from the unitary case.

In the unitary case the only matrix besides Λ that satisfies the stationarity condition is $Q_\Lambda = -k\Lambda = \tilde{\Lambda}$. In this case $Q_{\Lambda k} = -\Lambda$. All other matrices from $\mathbf{H}$ contain nonzero elements in the AR and RA blocks. Both stationary points contribute substantially to $K(\omega, x)$.

Consider the contribution of $Q_\Lambda = \tilde{\Lambda}$ to $K(\omega, x)$ first. We substitute $Q_\Lambda = -k\Lambda$

and $Q_{\Lambda k} = -\Lambda$ into Eq. (4.14), expand $F(Q_\Lambda)$ to the second order in B and $\bar{B}$ and substitute it in Eq. (4.11). Expanding $B(\vec{r})$ in the eigenfunctions of the diffusion operator: $B(\vec{r}) = \sum_\mu \phi_\mu(\vec{r}) B_\mu$ we obtain

$$R^{\rm u}_{\rm osc}(s,x) = \Re \int \mathcal{D}B (\sum_\mu A_\mu)^2 \\ \times \exp(-2\pi\{-is + \sum_\mu [\epsilon_\mu A_\mu + y^* |B^{11}_\mu|^2 + y|B^{33}_\mu|^2]\}), \tag{4.15}$$

where $y = x^2 - is$, $y^* = x^2 + is$ and $A_\mu = {\rm STr}(B_\mu \bar{B}_\mu)/2$. We have to keep x^2 finite to avoid the divergence of the integral over B^{11}_0 caused by the presence of the infinitesimal imaginary part in s. One can take the $x^2 \to 0$ limit only after the integral in Eq. (4.15) is evaluated.

Since the Free Energy in Eq. (4.15) contains no Grassmann variables in the zero mode they have to come from the pre-exponent. Therefore out of the whole square of the sum in the pre-exponent only the terms containing all four zero mode Grassmann variables contribute. In these terms the prefactor does not contain any variables from non-zero modes. Thus, the evaluation of the Gaussian integrals over non-zero modes yields the superdeterminant of the quadratic form in the exponent. Supersymmetry around $\tilde{\Lambda}$ is broken by s, therefore this superdeterminant differs from unity and is given by $P(s,x)$ of Eq. (4.8). Evaluating the integral we arrive at Eq. (4.5).

In quasi-1D for closed boundary conditions and $x = 0$ the spectral determinant $P(s,0)$ can be evaluated exactly, and from Eq. (4.5) we obtain

$$R^{\rm u,osc}_{1D}(s,0) = \frac{s}{2g\pi^2 s^2} \frac{\cos(2\pi s)}{\sinh^2\left(\sqrt{\frac{s}{2g}}\right) + \sin^2\left(\sqrt{\frac{s}{2g}}\right)}. \tag{4.16}$$

For $Q_\Lambda = \Lambda$ the same procedure as used above leads to Eq. (4.3), which coincides with the result of Ref. [58].

The behavior of $S(\tau,0)$ at $\tau = 0$ and $\tau = 2\pi$ is associated respectively with $R_{\rm p}(s,0)$ (Eq. (4.3)) and $R^{\rm u}_{\rm osc}(s,0)$ (Eq. (4.5)). In other words the singularity at the Heisenberg time is determined by the contribution to $R(s,0)$ from $\tilde{\Lambda}$. It is clear that the cusp in $S(\tau,0)$ at $\tau = 2\pi$ will be rounded off because $R^{\rm u}_{\rm osc}(s,0)$ decays exponentially at large

s. The scale of the smoothening is of order $1/g$.

The Fourier transform of Eq. (4.16) (see Fig. 4-1) is

$$S^{\rm u}_{1D}(2\pi+t,0)_{\tilde\Lambda}=\sum_{n=1}^{\infty}\frac{(-1)^n\exp(-\pi^2n^2g|t|)}{\pi^2gn\sinh(\pi n)}-\frac{|t|}{4\pi}. \tag{4.17}$$

Even though $S^{\rm u}_{1D}(2\pi+t,0)_{\tilde\Lambda}$ appears to be a function of $|t|$, it is regular at $t=0$.

We can also estimate $S^{\rm u}(2\pi,0)_{\tilde\Lambda}$ in any dimension. It is proportional to $1/g$ of Eq. (4.4) and is given by

$$S^{\rm u}(2\pi,0)_{\tilde\Lambda}=\frac{1}{4\pi^4g}\int_{-\infty+i\eta}^{\infty+i\eta}\frac{dz}{z^2}\prod_{\mu\epsilon_\mu\neq0}\left(1+\left[\frac{z\epsilon_1}{\epsilon_\mu}\right]^2\right)^{-1}.$$

Consider now T-invariant systems. For the orthogonal ensemble there are still only two stationary points on $\mathbf{H}$: Λ and $\tilde\Lambda$. To determine the contribution of the $\tilde\Lambda$-point we use the formula Eq. (4.14) with $Q_\Lambda=\tilde\Lambda$ and $Q_{k\Lambda}=-\Lambda$ and Efetov's parametrization for the perturbation theory [20]. The calculations are analogous to those for the unitary ensemble and lead to Eq. (4.6). The contribution of $Q_\Lambda=\Lambda$ gives Eq. (4.3). At $\tau=2\pi$ the third derivative of $S(\tau,0)$ for the orthogonal ensemble has a jump. This singularity also disappears at finite g.

In the symplectic case there are three types of stationary points which correspond to singularities in the structure factor $S(\tau,0)$ at $\tau=0,\pi,2\pi$ [7]. The $\tau=2\pi$ singularity corresponds to $Q_\Lambda=\tilde\Lambda$, and its contribution to $R(s,x)$, given by the second term in Eq. (4.7), is exactly the same as $R^{\rm o}_{\rm osc}(s,x)$. The stationary point $Q_\Lambda=\Lambda$ corresponds to the $\tau=0$ singularity in $S(\tau,0)$ and leads to Eq. (4.3). The $\tau=\pi$ singularity corresponds to a degenerate manifold of matrices Q_Λ on $\mathbf{H}$ $Q_\Lambda=\mathrm{diag}(\tau_{\vec m},\mathbf{1}_2,-\tau_{\vec n},-\mathbf{1}_2)$, $Q_{k\Lambda}=-kQ_\Lambda$, where $\mathbf{1}_2$ is a 2×2 unit matrix, $\tau_{(\vec m,\vec n)}=(m,n)_x\tau_x+(m,n)_y\tau_y$, $\vec m^2=1$, $\vec n^2=1$ and $\tau_{x,y}$ are Pauli matrices in the time-reversal block. The calculation proceeds as before and leads to the first term in Eq. (4.7). In quasi-1D we can obtain the leading contribution to the structure factor $S(\tau,0)$ around $\tau=\pi$

$$S^{\rm s}(t+\pi,0)=\int_0^\infty\frac{-4\sin^2(g|t|z)dz}{\sinh^2\sqrt{z}+\sin^2\sqrt{z}}+\ln(1.9g)+O(1/g). \tag{4.18}$$

The result is plotted in Fig. 4-2. In all dimensions the logarithmic divergence in the zero mode result is now cut off by finite g, and $S^s(\pi,0) \propto \ln g$.

In conclusion we mention several points about our results. 1) Equations (4.5), (4.6) and (4.7) together with (4.3) describe the deviation of the level statistics of a weakly disordered chaotic grain from the universal ones. This deviation is controlled by the diffusion operator. This operator is purely classical. It seems plausible that the nonuniversal part of spectral statistics of any chaotic system can be expressed through a spectral determinant of some classical system-specific operator. If so, the relation Eq. (4.9) should be universally correct!

2) The formalism used here should be applicable even to the systems weakly coupled to the outside world (say through tunnel contacts). As long as the level broadening Γ ($\omega = \Re\omega + i\Gamma$) is smaller than Δx^2 the integration over the zero mode variables in Eq. (4.15) is convergent. The integral over the other modes is always convergent provided $\Gamma < E_c$. Thus, the presence of a perturbation can effectively "close" a weakly coupled system. Under these conditions Eqs. (4.5), (4.6) and (4.7) remain valid after the substitution $\cos(2\pi s) \to \exp(-2\pi\Gamma/\Delta)\cos(2\pi s)$ and $x^2 \to x^2 - \Gamma/\Delta$.

3) The classification of physical systems into the three universality classes (unitary, orthogonal and symplectic) may be an oversimplification. A system subjected to a magnetic field remains orthogonal for short times and has the unitary long time behavior. The crossover time is set by the strength of the magnetic field. For a disordered metallic grain in a magnetic field this characteristic time is $\hbar c/(eHD)$. For $\omega > DeH/(\hbar c)$ the system effectively becomes orthogonal. This implies that even if we neglect the spatially nonuniform fluctuations of the Q-matrix the cusp in $S(\tau,0)$ at $\tau = 2\pi$ will be washed out on the scale of $\Delta\hbar c/(eHD)$ (the jump in the third derivative of $S(\tau,0)$ will still remain). For the system to behave as unitary at $\omega \approx E_c$ the magnetic length $\hbar c/(eH)$ has to be shorter than the size of the system. Spin-orbit interaction that causes the orthogonal-to-symplectic crossover can be considered analogously.

4) The rounding off of the singularity in $S(2\pi,0)$ is also present in the RM model

with preferred basis [61]. Our results differ from those in Ref. [61] substantially. Thus, finite g is not equivalent to finite temperature for the corresponding Calogero-Sutherland model [29].

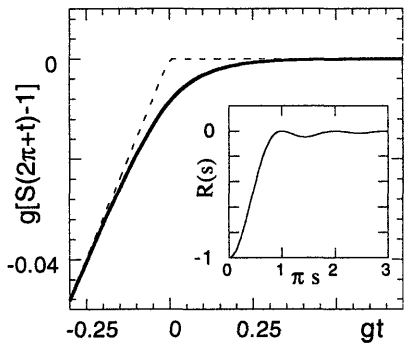

Figure 4-1: Structure factor in quasi-1D case for unitary symmetry (solid line) and the universal structure factor (dashed line). Inset: the two level correlator as a function of level separation.

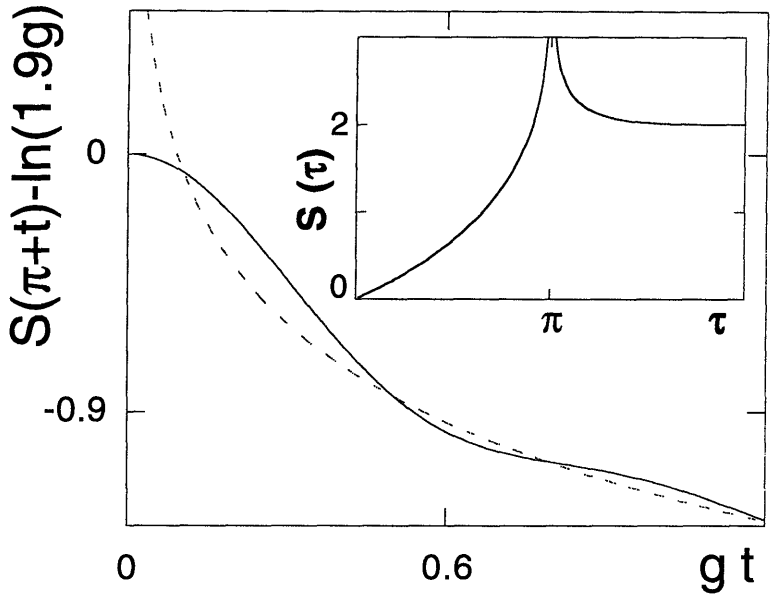

Figure 4-2: The structure factor for the symplectic case in quasi-1D (solid line) and the universal result (dashed line). Inset: the universal stucture factor.

Chapter 5

Spectral Statistics: From Disordered to Chaotic Systems

In this chapter the relation between disordered and chaotic systems is investigated. It is obtained by identifying the diffusion operator of the disordered systems with the Perron-Frobenius operator in the general case. This association enables us to extend results obtained in the diffusive regime to general chaotic systems. In particular, the two–point level density correlator and the structure factor for general chaotic systems are calculated and characterized. The behavior of the structure factor around the Heisenberg time is quantitatively described in terms of short periodic orbits. We closely follow Ref. [62] here.

The statistical description of the quantum spectra of systems which exhibit chaotic dynamics in their classical limit, has been conducted mainly along two routes. One is to study an *ensemble* of similar systems, such as disordered metallic grains, where electrons experience scattering by a random potential. In this approach, ensemble averaging is a crucial step done at an early stage of the calculation. The results of such a calculation apply to an individual member of the ensemble, provided the time of observation is long enough. The second route is to characterize the properties of *individual* systems by means of the periodic orbit theory [63]. This is possible for a system with chaotic dynamics governed by a Hamiltonian that is simple enough, so that the parameters of the periodic orbits needed for semiclassical spectral analysis

can be calculated explicitly. Averaging in this case is usually done over an energy interval which consists of many energy levels [56]. This approach is very powerful in describing the short time behavior of the system, but is faced with significant problems when applied to times of order or bigger than the Heisenberg time $\tau_H = h/\Delta$ or to energies much smaller than the mean level spacing Δ [64]. Despite the obvious differences between the two approaches it is believed that to a large extent both describe the same physics. In this chapter we are applying results of the first approach in order to extend the periodic orbit theory to times close to τ_H.

The relation between ensembles of metallic grains and ensembles of random matrices (RM) [7] is now well understood. The supersymmetric non-linear σ model [20], actually, provides a microscopic justification for the use of RM theory in order to describe the universal features of these systems. This formalism offers a routine way of calculation of a variety of universal correlation functions for all Dyson pure symmetry RM ensembles and for crossovers between them [65]. In view of the growing interest in applying the supersymmetry approach to the investigation of deviations from universality [57, 51, 66], it becomes important to understand the connection between the two approaches described above.

The object that we analyse is the dimensionless two–point level density correlator,

$$R(s) = \Delta^2 \langle \rho(E)\rho(E + s\Delta)\rangle - 1. \tag{5.1}$$

Here $\rho(E)$ is the density of states at energy E, Δ is the mean level spacing, and $\langle \cdots \rangle$ represents ensemble averaging in the case of disordered systems, or averaging over some interval of energy E if an individual chaotic system is considered. The universal form of $R(s)$ is especially simple in the unitary case. It is the sum of smooth and oscillating parts [7]: $R(s) = \delta(s) - [1 - \cos(2\pi s)]/(2\pi^2 s^2)$. The conventional perturbation theory for disordered metals [67] can provide only the smooth part of $R(s)$ [58]. The $s \gg 1$ asymptotics of $R(s)$ in which the oscillatory term, non-analytic in $1/s$, is retained was recently evaluated in Ref. [51]. This result (for $s \neq 0$) can be

still presented as a sum,

$$R(s) = R_P(s) + R_{osc}(s), \tag{5.2}$$

of a perturbative term $R_P(s)$ and oscillatory one $R_{osc}(s)$. We rewrite the expression for $R_P(s)$ (see Ref. [58]) as

$$R_P(s) = -\frac{1}{2\alpha\pi^2}\frac{\partial^2}{\partial s^2}\ln[\mathcal{D}(s)], \tag{5.3}$$

where $\alpha = 2$ for the unitary ensemble and $\alpha = 1$ for T-invariant ensembles. $\mathcal{D}(s)$ is the spectral determinant of a classical operator, namely the diffusion operator:

$$\mathcal{D}(s) = \prod_{\mu} A(\epsilon_\mu)\left(s^2 + \epsilon_\mu^2\right)^{-1}. \tag{5.4}$$

Here ϵ_μ are eigenvalues (in units of Δ) of the diffusion equation in the grain, and $A(\epsilon_\mu)$ is a regularization factor which equals ϵ_μ^2 for $\epsilon_\mu \neq 0$ and unity otherwise [68]. Surprisingly the oscillatory term $R_{osc}(s)$, which cannot be obtained by a perturbative calculation, is also governed by the same classical spectral determinant $\mathcal{D}(s)$. E. g., in the unitary case it has the form

$$R_{osc}(s) = \frac{\cos(2\pi s)}{2\pi^2}\mathcal{D}(s). \tag{5.5}$$

Since $\mathcal{D}(s)$ is purely classical, it is plausible that for any chaotic system there exists a classical operator whose spectral determinant can be identified with $\mathcal{D}(s)$. In what follows we shall identify this operator for general chaotic systems by a semiclassical analysis of relation (5.3). For the sake of simplicity we shall consider a two dimensional system which belongs to the unitary ensemble.

The semiclassical analysis begins with Gutzwiller's trace formula [63], which expresses the density of states $\rho(E)$ as a sum over the classical periodic orbits

$$\rho(E) = \frac{1}{\Delta} + \Re\frac{1}{\pi\hbar}\sum_p T_p \sum_r \frac{e^{\frac{i}{\hbar}S_p(E)r - i\nu_p r}}{|\det(M_p^r - I)|^{1/2}}, \tag{5.6}$$

where p labels a primitive orbit that is characterized by a period T_p, action $S_p(E)$,

and Maslov phase ν_p; r stands for the number of the repetitions of this orbit. M_p is the monodromy matrix associated with the linearized dynamics on the Poincarè section perpendicular to the orbit. From here on, energy and time will be measured in units of Δ ($\epsilon = E/\Delta$), and $\hbar/\Delta$ respectively. One can substitute (5.6) into (5.1) and represent $R(s)$ in the form of a double sum over the periodic orbits. $R_P(s)$ is given by the diagonal part of this sum. Expanding $S_p(\epsilon + s)$ up to the linear order in s: $S_p(\epsilon + s) \simeq S_p(\epsilon) + T_p s$, we obtain

$$R_P(s) = \Re \frac{1}{2\pi^2} \sum_p T_p^2 \sum_{r=1}^{\infty} \frac{e^{isT_p r}}{|\det(M_p^r - I)|}. \tag{5.7}$$

The traditional way to deal with the above sum is to approximate it by an integral:

$$\sum_p \frac{f(T_p)}{|\det(M_p - I)|} \to \int \frac{dt}{t} f(t) \tag{5.8}$$

for any sufficiently smooth function $f(t)$. This approximation, known as the Hannay and Ozorio de Almeida (H&OA) sum rule, holds in the limit $t \to \infty$ where long periodic orbits which explore the whole energy shell uniformly are considered. In employing it for the calculation of $R_P(s)$, the time t should be restricted to the regime where it is much larger than the shortest periodic orbits but still smaller than the Heisenberg time τ_H. The result associated with it is therefore the universal one $R_P(s) = -1/2\pi^2 s^2$ which holds as long as $s \gg 1$ [56]. Below we present a more careful treatment of the sum (5.7) that keeps the non-universal part of $R_P(s)$.

Let Λ_p be the eigenvalue ($|\Lambda_p| > 1$) of the monodromy matrix M_p. The area preserving property of the latter implies that the second eigenvalue of M_p is $1/\Lambda_p$. Hence,

$$|\det(M_p^r - I)|^{-1} = |\Lambda_p|^{-r} \sum_{k=0}^{\infty} (k+1) \Lambda_p^{-rk}, \tag{5.9}$$

and we can rewrite (5.7) in the form of a triple sum

$$R_P(s) = \frac{-1}{2\pi^2} \frac{\partial^2}{\partial s^2} \Re \sum_{p,k} (k+1) \sum_{r=1}^{\infty} \frac{1}{r^2} t_{pk}^r, \tag{5.10}$$

where

$$t_{pk} = |\Lambda_p|^{-1}\Lambda_p^{-k}e^{isT_p}. \tag{5.11}$$

Using the relation (5.3) we can determine the spectral determinant $\mathcal{D}(s)$ up to a normalization constant:

$$\mathcal{D}(s) = |\mathcal{N}\tilde{Z}(is)|^2. \tag{5.12}$$

Upon evaluation of the sum over the repetitions in Eq. (5.10), the expression for $\tilde{Z}(is)$ takes the form

$$1/\tilde{Z}(is) = \prod_p \prod_{k=0}^{\infty} \exp\left[(k+1)\phi(t_{pk})\right], \tag{5.13}$$

where $\phi(x) = \int_0^x t^{-1}\ln(1-t)dt$. Notice that the normalization constant $\mathcal{N}$ plays no role in the perturbative part of the two–point correlator. We therefore postpone its determination.

Suppose now that all the periodic orbits are very unstable, namely $|\Lambda_p| \gg 1$ for all p-s. In this case $t_{pk} \to 0$, $\phi(t_{pk}) \to -t_{pk}$ and $\tilde{Z}(z)$ reduces to the dynamical zeta function [9],

$$1/Z(z) = \prod_p \prod_{k=0}^{\infty} \left(1 - \frac{e^{zT_p}}{|\Lambda_p|\Lambda_p^k}\right)^{k+1}. \tag{5.14}$$

This function is the spectral determinant associated with the Perron-Frobenius (PF) operator $\mathcal{L}^t$ (also known as Ruelle-Araki or the transfer operator) [70]. $\mathcal{L}^t$ is the classical evolution operator which propagates phase space density for a time $t > 0$. Its kernel is therefore given by

$$\mathcal{L}^t(\vec{y},\vec{x}) = \delta[\vec{y} - \vec{u}(\vec{x};t)], \tag{5.15}$$

where $\vec{y}$ and $\vec{x}$ are phase space vectors representing coordinates and momenta, and $\vec{u}(\vec{x};t)$ is the point in phase space to which a particle that starts its motion at $\vec{x}$ arrives after time t. The eigenvalues of the PF operator are of the form $e^{-\gamma_\mu t}$. They are associated with the decaying modes of a disturbance in the density of classical particles exhibiting chaotic dynamics, analogous to the diffusion modes of disordered system. Yet, the difference is that, unlike in the latter case, here γ_μ-s can appear

also in complex conjugate pairs $\gamma = \gamma' \pm i\gamma''$ where $\gamma' \geq 0$. The leading eigenvalue of the PF operator, $\gamma_0 = 0$, corresponds to the conservation of the number of particles. The dynamical zeta function (5.14) is the spectral determinant associated with the eigenvalues γ_μ. Up to a normalization constant it is given by the product

$$1/Z(z) = \prod_\mu B_\mu(z - \gamma_\mu), \tag{5.16}$$

where B_μ are regularization factors introduced to make the product converge.

Unlike the periodic orbit theory in quantum mechanics which gives only the leading asymptotics in the limit $\hbar \to 0$, the periodic orbit expansion (5.14) of (5.16) is exact. It is however proper to comment that, in its present form, $Z(z)$ cannot be used to determine the eigenvalues γ_μ. For this purpose a resummed formula is required. It can be obtained by expanding the infinite product over the periodic orbits and ordering the various terms in a way that leads to maximal cancelation among them. This so called cycle expansion [71] exploits the property that the dynamics of chaotic systems in phase space is coded by a skeleton of few periodic orbits. In particular, the long periodic orbits may be approximated by linear combinations of few short ones.

From (5.16) and (5.3) it follows that

$$R_P(s) = \Re \frac{1}{2\pi^2} \sum_\mu \frac{1}{(-is + \gamma_\mu)^2} \tag{5.17}$$

in complete analogy with the result of Ref. [58] for diffusive systems. The universal part of $R_P(s)$, that was obtained using H&OA sum rule, thus corresponds to the first term in the sum (5.17) ($\gamma_0 = 0$). The rest of the sum is apparently system-specific.

We turn now to the determination of the normalization constant $\mathcal{N}$ introduced in (5.12). We shall assume that the leading eigenvalue γ_0 is of unit multiplicity (this is the case when the system is ergodic). Comparison of Eqs. (5.12) and (5.4) gives the normalization factor:

$$\mathcal{N}^{-1} = \lim_{z \to 0} zZ(z). \tag{5.18}$$

It is customary to express the semiclassical density of states as the logarithmic derivative of the Selberg zeta function. The latter is defined as the spectral determinant associated with the semiclassical energy spectrum of the system under consideration:

$$\zeta_s(\epsilon) = \prod_j b_j(\epsilon - \epsilon_j) = \prod_p \prod_{k=0}^{\infty} \left(1 - \frac{e^{iS_p(\epsilon) - i\nu_p}}{|\Lambda_p|^{1/2}\Lambda_p^k}\right), \tag{5.19}$$

where b_j are regularization factors, and ϵ_j are the semiclassical energy levels of the system. The second equality above holds for two dimensional systems. One can show that the spectral determinant $\tilde{Z}(is)$, satisfies the relation

$$\tilde{Z}(is) = \exp\{\langle \ln[\zeta_s(\epsilon + s)] \ln[\zeta_s^*(\epsilon)] \rangle_{\mathrm{d}}\}. \tag{5.20}$$

where $\langle \cdots \rangle_{\mathrm{d}}$ represents an averaging which retains only the diagonal elements in the double sum. Since $\Delta\rho(\epsilon) = 1 - (\partial/\pi\partial\epsilon)\Im \ln \zeta_s(\epsilon + i0)$, the two-point correlator can be written as

$$R(s) = \frac{-\partial^2}{\pi^2 \partial s^2} \left\langle \Im \ln \zeta_s\left(\epsilon + s\right) \Im \ln \zeta_s\left(\epsilon\right) \right\rangle. \tag{5.21}$$

The diagonal approximation gives the perturbative term

$$R_P(s) = \frac{-\partial^2}{\pi^2 \partial s^2} \left\langle \Im \ln \zeta_s\left(\epsilon + s\right) \Im \ln \zeta_s\left(\epsilon\right) \right\rangle_{\mathrm{d}}. \tag{5.22}$$

The difference between Eqs. (5.21) and (5.22) can be also expressed through the diagonal average. Using Eqs. (5.20) and (5.12) it is easy to see that $R_{osc}(s)$ is given by Eq. (5.5) with

$$\mathcal{D}(s) = \mathcal{N}^2 \exp\left\{2\Re \left\langle \ln \zeta_s\left(\epsilon + s\right) \ln \zeta_s^*\left(\epsilon\right) \right\rangle_{\mathrm{d}}\right\}. \tag{5.23}$$

It is convenient to present the result in terms of the Fourier transform of the two–point level density correlator, $S(\tau) = \int ds e^{is\tau} R(s)$, known as the structure or the form factor. RM theory predicts that for the unitary ensemble $S(\tau) = min(|\tau|/2\pi, 1)$

(see the light line in Fig. 1). In the general case

$$S(\tau) = S_P(\tau) + \frac{1}{2}\left[S_{osc}(\tau + 2\pi) + S_{osc}(\tau - 2\pi)\right], \tag{5.24}$$

where S_P and S_{osc} are respectively associated with the perturbative (5.3) and the non-perturbative (5.5) parts of the two–point correlator. Assuming that the multiplicity of all the eigenvalues γ_μ is one,

$$S_P(\tau) = \frac{|\tau|}{2\pi}\sum_\mu e^{-\gamma_\mu|\tau|}. \tag{5.25}$$

Again the universal part of $S_P(\tau)$ associated with H&OA sum rule comes from the leading eigenvalue $\gamma_0 = 0$. The higher eigenvalues will contribute corrections which are in general oscillatory and decrease exponentially. For instance, the complex pair $\gamma_1' \pm i\gamma_1''$ will contribute the term $|\tau|e^{-\gamma_1'|\tau|}\cos(\gamma_1''\tau)/\pi$. The oscillatory part of the structure factor can be written as,

$$S_{osc}(\tau) = -\frac{|\tau|}{2\pi} - \sum_{\mu\neq 0}\frac{\mathcal{D}_\mu(i\gamma_\mu)}{2\pi\gamma_\mu}e^{-\gamma_\mu|\tau|}, \tag{5.26}$$

where $\mathcal{D}_\mu(s)$ is given by

$$\mathcal{D}_\mu(s) = \left(1 + \frac{s^2}{\gamma_\mu^2}\right)s^2\mathcal{D}(s). \tag{5.27}$$

For example, in the case of quasi one dimensional diffusive system, where the eigenvalues are of the form $\gamma_n = Dn^2$ one can show that $\mathcal{D}_n(iDn^2) = -4n(-1)^n/\sinh(\pi n)$, while for equally spaced eigenvalues $\gamma_n = vn$ it is $\mathcal{D}_n(ivn) = 2\pi n/\sinh(\pi n)$. In general it is expected that the contribution will come only from the lowest eigenvalues of the PF operator.

In what follows it will be assumed that the non-universal behaviour is dominated by one eigenvalue (or possibly a conjugate pair) γ_1, i.e. $\gamma_\mu' \gg \gamma_1'$ for all $\mu > 1$.

In characterizing $S(\tau)$, five domains of the parameter τ, drawn schematically in Fig 1, are identified: (I) $\tau \sim \tau_c$ where τ_c is of order of the period of the shortest periodic orbit. Here $S(\tau)$ is composed of several δ-function peaks located at the periods of the

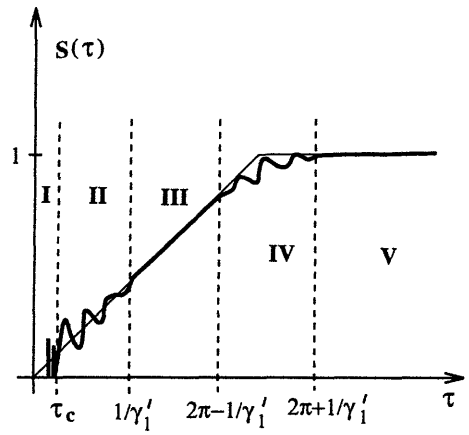

Figure 5-1: A schematic drawing of the structure factor of chaotic system belonging to the unitary ensemble. The light line represents the universal RM theory result. For the sake of clarity, the non-universal features have been exaggerated.

short orbits and weighted according to their instability. (II) $\tau_c < \tau < 1/\gamma_1'$. Deviations from universality associated with (5.25) may be noticeable also in this interval. Their period of oscillation $1/\gamma_1''$ is of the order of τ_c. (III) $1/\gamma_1' < \tau < 2\pi - 1/\gamma_1'$, the universal perturbative regime where $S(\tau) = \tau/2\pi$. This is the domain where H&OA sum rule holds. (IV) $2\pi - 1/\gamma_1' < \tau < 2\pi + 1/\gamma_1'$, the vicinity of the Heisenberg time $\tau = 2\pi$. The non-universal features here are in the form of exponentially decreasing oscillations very similar to those existing in (II). Yet their amplitude and phase may be different. In general, the RM singularity at the Heisenberg time (the light line in Fig. 1) will be smeared out by them. (V) $\tau > 2\pi + 1/\gamma_1'$. Here again the universal result $S(\tau) = 1$ holds.

These results can be generalized straightforwardly to orthogonal and symplectic chaotic systems: instead of Eq. (5.5) one should use Eq. (5) of Ref. [51] with $P(s) = s^2\mathcal{D}(s)$.

The behavior of the structure factor in the vicinity of the Heisenberg time is a manifestation of a striking property of the periodic orbit sum (5.6). Namely, that the tail of the Gutzwiller's series (the long periodic orbit) encodes its head (short periodic orbit). As a result, $S(\tau)$ in the vicinity of the Heisenberg time is determined by the same short periodic orbits as at small τ. The argument [56] is that the long

periodic orbits determine the position of the energy levels. Therefore through the long range correlation of these levels they encode the information about the short periodic orbits. In fact, Berry and Keating resummation method [72] of the periodic orbit sum associated with the quantum spectral determinant of chaotic systems is based on the bootstrapping of long periodic orbits with periods near the Heisenberg time $\tau \sim 2\pi$ to the short ones near τ_c. The behavior of $S(\tau)$ near the Heisenberg time reflect this sort of symmetry in the sense that it is determined by the short time dynamics of the classical system.

To summarize, we identified the diffusion operator in disordered grains with the Perron-Frobenius operator in the general case. This relates the spectral determinant associated with the diffusion equation in the grain to the dynamical zeta function which can be expressed in terms of the classical periodic orbits. We used these relations to extend the theory of the structure factor of disordered grains to general chaotic systems. It would be interesting and important to derive these relations for generic chaotic systems. In this respect the recently proposed σ-model-like approach for ballistic systems [73] looks promising.

Chapter 6

A Semiclassical Field Theory Approach to Quantum Chaos

6.1 Introduction

The quantum behavior of systems which are chaotic in their classical limit has no simple correspondence to the underlying classical evolution. More than a decade ago, it was conjectured that, over a certain energy interval, chaotic systems exhibit universal behavior [87, 56, 65] that is described by the theory of random matrices [7]. Extensive numerical simulations performed since support this conjecture, but the origin of the success of the universal description based on random matrix theory (RMT) remained obscure. The intuition behind this phenomena comes from the following argument: A classical probability density of a particle in a chaotic system relaxes, after some time τ_c, to the ergodic distribution which is uniform over the energy shell in phase space. At this ergodic stationary state, the classical behavior of all chaotic systems is similar and can be characterized statistically. The volume of the energy shell is, however, specific for a given system. It fixes the mean level spacing Δ of the corresponding quantum system. This picture suggests that when energy is rescaled by Δ, the spectral statistics of quantum chaotic systems is universal over energy scales smaller than $\hbar/\tau_c$. However, on time scales smaller than τ_c chaos is still not fully developed. Therefore, over the *large scale* structure of the quantum

spectrum, one expects to observe "clean features" associated with deviations from universality. These should be related to the specific manner in which the system relaxes into its ergodic state.

The study of the universal and non-universal statistical properties of quantum chaotic systems has largely been conducted along two lines: The first has relied on a semiclassical approach based on Gutzwiller's trace formula in which the semiclassical density of states (DoS) is expressed as a sum over the classical periodic orbits [8]. This approach focuses on the behavior of *individual* systems, and statistical properties are based implicitly on energy averaging. This approach has proved to be a powerful tool in describing non-universal properties associated with short time behavior. However, its success in reproducing universal long time properties associated with RMT has been limited [56]. In particular, it fails to account for the correct behavior at times in excess of the inverse level spacing or Heisenberg time $\tau_H = \hbar/\Delta$.

The second approach involves the study of an *ensemble* of similar systems, such as weakly disordered metallic grains in which electrons experience scattering by a random potential. Here, ensemble averaging is a crucial step exploited at an early stage of calculation. In this framework, the relation between disordered systems and RMT is firmly established. In fact, the supersymmetric nonlinear sigma model provides a microscopic justification for the RMT description of universal long-time properties of such systems. However, such an approach suffers from two disadvantages: Firstly, it relies on the very existence of an ensemble. Very often we are concerned with non-stochastic chaotic systems, such as a Rydberg atom in a magnetic field, where the notion of an ensemble is absent. Secondly, this type of averaging tends to erase information about individual features of a system.

The goal of this chapter is to develop a common framework in which both disordered and general chaotic systems can be treated on the same footing. For this purpose we shall derive a general semiclassical field theoretic description of quantum chaotic systems which relies *only on energy averaging.* This method offers a novel semiclassical approach for the study of the statistical quantum properties of individual chaotic systems, which accounts both for universal as well as non-universal

features. The basic ingredients of the underlying classical dynamics are *no longer* the individual periodic orbits, but general properties of the classical flow in phase space.

Before introducing the main conclusions of this study, we begin by identifying the questions which will be of most concern. To do so it is convenient to draw on the insight offered by the study of the dynamics of a particle moving in a background of weak randomly distributed scattering impurities. Amenable to the method of ensemble averaging, properties of this chaotic quantum mechanical problem are well understood.

What is the quantum evolution of a wave packet in a background of impurities? According to the time of evolution, the dynamics of the wave packet is characterized by several quite distinct regimes. On time scales t in excess of the mean free scattering time τ, the initial *ballistic* evolution of the wave packet becomes diffusive. At longer times the interference of different semiclassical paths induces a *quantum renormalization* of the bare diffusion constant $D = v_F^2\tau/d$. This leads to the phenomena of "weak localization" and is responsible for the quantum coherence effects observed in transport properties of mesoscopic metallic conductors. If the impurity potential is not strong enough to localize the wave packet altogether, the wave packet continues to spread. After a time $\tau_c = L^2/D$, the typical transport or diffusion time, the wave packet is spread approximately uniformly throughout the system. Further evolution of the wave packet is therefore said to be *ergodic*. Beyond the ergodic time $t \gg \tau_c$ the evolution of the wave packet becomes *universal*, independent of the individual features of the system. Finally, the spectral rigidity characteristic of quantum chaotic systems, leads to an approximately coherent superposition or "echo" of the wave packet at $t = \tau_H$ before the wave packet relaxes to a uniform distribution.

One can now ask about the quantum evolution of a wave packet introduced into, say, an irregular cavity (quantum billiard) without impurities. In such systems it is widely believed that there too exists some ergodic time τ_c after which properties of the system become universal. However, at shorter time scales, how is the unstable nature of the classical dynamics reflected in the quantum evolution? Is there an analogue of quantum renormalization? What, in general, plays the role of the diffusion operator

in describing the low energy degrees of freedom?

In a recent study by three of us [62], a comparison of results taken from the leading order of diagrammatic perturbation theory for disordered metals within the diagonal approximation in periodic orbit theory led to the conjecture that, for general chaotic systems, the role of the diffusion operator is, in general, played by the classical evolution (or Perron-Frobenius) operator, $e^{-\hat{\mathcal{L}}t}$. If $\rho(\mathbf{x},0)$ is an initial smooth probability density distribution defined as a function of phase space variables $\mathbf{x} \equiv (\mathbf{q},\mathbf{p})$, where $\mathbf{q}$ and $\mathbf{p}$ are the vectors of coordinates and momenta respectively, the density at a later time t is given by

$$\rho(\mathbf{x},t) = e^{-\hat{\mathcal{L}}t}\rho(\mathbf{x},0) \equiv \int_{\Gamma} d\mathbf{y}\ \delta[\mathbf{x}-\mathbf{u}^t(\mathbf{y})]\ \rho(\mathbf{y},0), \tag{6.1}$$

where $\mathbf{u}^t(\mathbf{y})$ is the solution of classical equations of motion with initial conditions $\mathbf{y}$ at $t=0$, and Γ covers the region of available phase space. The eigenvalues of the Perron-Frobenius operator are of the form $e^{-\gamma_\mu t}$. They are associated with the decaying modes in which a smooth distribution relaxes into the ergodic state and are analogous to the diffusion modes of disordered system.

The conjecture of Ref. [62] will be confirmed in this chapter. Furthermore, the efficiency of the field theory approach will be demonstrated by showing that all the statistical spectral properties of the quantum system, in the semiclassical limit, depend only on the analytic properties of the corresponding classical zeta function $1/Z(z) = \det(z-\hat{\mathcal{L}})$ [9]. In particular, by considering only its zero at the origin one exactly reproduces RMT, while by taking into account its analytic structure one is able to characterize the deviations from the universality. The point of contact of the conventional periodic orbit theory with our approach appears in the exact periodic orbit representation of $1/Z(z)$. Yet, the field theoretic approach provides, in principle, a systematic way to study the quantum corrections which lie beyond the diagonal approximations typically employed in the periodic orbit theory. This will be demonstrated by considering the example of the two-point correlation function.

The range of energy scales in which the approach presented here is valid is set out

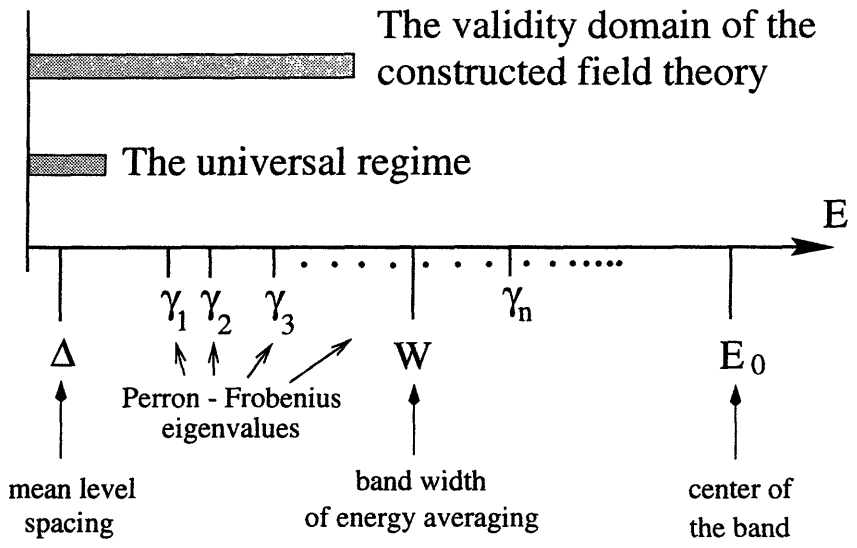

Figure 6-1: A schematic picture of the various energy scales, and the domains of validity of RMT and the field theory which is constructed in this chapter.

in Fig. 6-1. Energy averaging will be performed over a wide energy band of width W centered at E_0, where $E_0 \gg W$. Furthermore, we will require that W is much larger than the energy scale set by the first nonzero eigenvalue of $\hat{\mathcal{L}}$, γ_1. This is to ensure that the time scale $\hbar/W$ is fine enough to resolve the behavior of the classical dynamics over a time interval smaller than the ergodic time $\tau_c \sim \hbar/\gamma_1$. It will be also assumed that the finest energy scale, the mean level spacing Δ, is much smaller than γ_1. Thus, in this work we will focus on a range of energy scales where $\Delta \ll \gamma_1 \ll W \ll E_0$. The universal regime described by RMT corresponds to energies of order of Δ or smaller. The field theory which will be developed here also properly describes energies which are much larger than Δ. However, to avoid non-universal features associated with the finite band width W, it will be always assumed that all correlators involve energy differences much smaller than W. In this way only the non-universal features that emerge from the underlying classical dynamics (and not the finite band width) will be described.

6.2 The Nonlinear σ-Model

To present the derivation of the effective field theory describing spectral correlations of quantum chaotic systems, we will focus on the problem of a single particle confined by an irregular potential described by the Hamiltonian

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{q}). \tag{6.2}$$

The classical counterpart of the quantum Hamiltonian is assumed to be chaotic and to have no discrete symmetries. We confine attention to closed systems so that classical motion inhabits a finite region of the $2d$-dimensional phase space. We will assume that all classical orbits are unstable and, in particular, exclude (KAM) systems where the phase space contains islands of regular motion. Later in the discussion we will return to consider whether this restriction is crucial.

We will concentrate on statistical properties defined on an energy band of width W centered at an energy E_0. To discuss meaningful averages it is necessary to assume that the average DoS, specified by the Weyl formula

$$\nu(E) = \frac{1}{h^d} \int d\mathbf{x}\; \delta\left[E - H(\mathbf{x})\right], \tag{6.3}$$

is approximately constant within this interval. Taking as an example a particle in a random impurity potential, the accuracy of this approximation is of order W/E_0, and can be made arbitrarily small by going into the semiclassical limit $E_0 \to \infty$. On the other hand, the bandwidth is assumed to be sufficiently large that the number of levels, $N = \nu(E_0)W \gg 1$ can be employed as an expansion parameter — final expressions will be expressed in the zeroth order approximation in $1/N$. Henceforth we will express energy in units of the mean level spacing, $\Delta = 1/\nu(E_0)$ and denote such energies by $\epsilon = E/\Delta$. For simplicity, it is convenient to employ Gaussian averaging

$$\langle \cdots \rangle_{\epsilon_0} = \int \frac{d\epsilon}{(2\pi N^2)^{1/2}} \exp\left[-\frac{(\epsilon - \epsilon_0)^2}{2N^2}\right] (\cdots). \tag{6.4}$$

A general n-point correlator of physical operators, such as the local or global DoS

or current densities, can be obtained from a generating function which depends on appropriate external sources. Here we focus on two-point correlators. Expressed as a field integral, the generating function for two-point correlators takes the form

$$\mathcal{Z}(\hat{J}) \;=\; z\int D\Psi\, \exp\left[-i\int d\mathbf{q}\Psi^{\dagger}(\mathbf{q})L\left(\hat{G}^{-1}(\epsilon)-\hat{J}k\Lambda\right)\Psi(\mathbf{q})\right], \qquad (6.5)$$

where $\hat{G}^{-1}(\epsilon) = \epsilon - s^{+}\Lambda/2 - \hat{H}$ denotes the matrix Green function with energy difference s between retarded (R) and advanced (A) blocks, $\hat{J}k\Lambda$ represents the source, and the constant z is included to enforce the correct normalization.[1] By expressing the functional integral in terms of 8-component superfields

$$\Psi^{d}_{gp}(\mathbf{q}) = \begin{pmatrix}\Psi^{A}_{gp}\\ \Psi^{R}_{gp}\end{pmatrix}_{d}, \quad \Psi^{d}_{g=1,p} = \begin{pmatrix}\chi^{d}\\ \chi^{d*}\end{pmatrix}_{p}, \quad \Psi^{d}_{g=2,p} = \begin{pmatrix}S^{d}\\ S^{d*}\end{pmatrix}_{p}; \qquad (6.7)$$

where $\chi(\mathbf{q})$ and $S(\mathbf{q})$ respectively denote fermionic (F) and bosonic (B) components, the generating function is normalized, $\mathcal{Z}(0) = 1$. (The introduction of equal numbers of bosonic and fermionic fields is a standard trick which obviates the need to introduce replicas.) Matrices

$$\Lambda = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}_{d}\otimes\mathbf{1}_{g}\otimes\mathbf{1}_{p}, \qquad k = \mathbf{1}_{d}\otimes\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}_{g}\otimes\mathbf{1}_{p}, \qquad (6.8)$$

break the symmetry between the advanced/retarded and graded components respectively, and we have chosen a convention which introduces the supermatrix

$$L = \begin{pmatrix}\mathbf{1} & 0\\ 0 & k\end{pmatrix}_{d}\otimes\mathbf{1}_{p}. \qquad (6.9)$$

[1] Energy averaging of the generating functional (6.5) induces a quartic interaction of the form $(\Psi^{\dagger}L\Psi)^2$ among the supervector fields. The matrix L serves as a metric tensor. In Ref. [21] it was shown that the appropriate group of transformations preserving the interactions in the fermionic sector is compact while in the bosonic sector it must be chosen non-compact. This fixes the definition of L (see Eq. (6.9). With this definition, the constant

$$z = \exp\left[\frac{1}{2}\mathrm{STr}_{\mathbf{q}}\ln(\Lambda L)\right], \qquad (6.6)$$

accounts for the correct normalization.

The operations of complex conjugation, and transposition of supervectors are defined following Efetov [20], and $\mathrm{STr}_{\mathbf{q}}$ denotes the trace operation for supermatrices, $\mathrm{STr}M = \mathrm{Tr}M_{FF} - \mathrm{Tr}M_{BB}$, with a subscript $\mathbf{q}$ used to denote a further extension of the trace to include the coordinate integration.

The inclusion of complex conjugated fields effectively doubles the number of fields and implies the relation

$$\Psi^{\dagger}(\mathbf{q}) = \Psi(\mathbf{q})^{T} C^{T}, \tag{6.10}$$

where

$$C = \mathbf{1}_{d} \otimes \begin{pmatrix} -i\tau_2 & 0 \\ 0 & \tau_1 \end{pmatrix}_{g}, \tag{6.11}$$

denotes the "charge conjugation" matrix and

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.12}$$

represent Pauli matrices which act inside time-reversal blocks. This convention differs slightly from that discussed by Ref. [20].

As an application, Eq. (6.5) can be used to represent the two-point correlator of DoS fluctuations

$$R_2(s) = \langle \hat{\nu}(\epsilon + s/2)\hat{\nu}(\epsilon - s/2) \rangle - 1, \tag{6.13}$$

where $\hat{\nu}(\epsilon) = \mathrm{Tr}\ \delta(\epsilon - \hat{H})$ defines the DoS. If we choose the source $J(\mathbf{q}, \mathbf{q}') = J\delta(\mathbf{q} - \mathbf{q}')$, then

$$R_2(s) = -\frac{1}{16\pi^2} \frac{\partial^2}{\partial J^2} \langle \mathcal{Z}(\hat{J}) \rangle_{\epsilon_0} \Big|_{J=0}. \tag{6.14}$$

If the energy difference s is chosen to be much narrower than the width of the energy band N, correlators become independent of N and of the particular shape of the band (whether it is Gaussian or Lorentzian, etc). Performing the Gaussian integral in Eq. (6.4) we obtain

$$\langle \mathcal{Z} \rangle_{\epsilon_0} \;=\; z \int D\Psi \left[-i \int d\mathbf{q} \Psi^{\dagger}(\mathbf{q}) L \left(\hat{G}^{-1}(\epsilon_0) - \hat{J} k \Lambda \right) \Psi(\mathbf{q}) - S_{\mathrm{int}}[\Psi] \right], \tag{6.15}$$

$$S_{\rm int} = \frac{N^2}{2}\left(\int d\mathbf{q}\Psi^\dagger(\mathbf{q})L\Psi(\mathbf{q})\right)^2. \tag{6.16}$$

So, in contrast to an impurity averaging, energy averaging induces a *nonlocal* interaction of Ψ, and represents a departure from the usual consideration of random Hamiltonians.

The next step involves the decoupling of the interaction induced by the averaging by means of a Hubbard-Stratonovich transformation involving the introduction of 8×8 component *nonlocal* supermatrices $\hat{Q}(\mathbf{q}_1, \mathbf{q}_2)$

$$e^{-S_{\rm int}[\Psi]} = \int DQ \exp\left[-\,{\rm STr}_{\mathbf{q}}\left(\frac{\hat{Q}^2}{2} - iN\Psi^\dagger L\hat{Q}\Psi\right)\right], \tag{6.17}$$

Eq. 6.10 implies that the dyadic product $A(\mathbf{q},\mathbf{q}') = \Psi(\mathbf{q})\otimes\Psi^\dagger(\mathbf{q}')L$ obeys the symmetry property $A(\mathbf{q},\mathbf{q}') = C^T L A^T(\mathbf{q}',\mathbf{q})LC$. This induces the corresponding symmetry

$$\hat{Q} = C^T L\hat{Q}^T LC, \tag{6.18}$$

where the transposition should be understood in the operator sense.

Substituting Eq. (6.17) into Eq. (6.15) and integrating over Ψ we obtain the following expression for the averaged generating functional

$$\begin{aligned}\langle\mathcal{Z}(\hat{J})\rangle_{\epsilon_0} &= \int DQ\ \exp\left[-\frac{1}{2}{\rm STr}_{\mathbf{q}}\hat{Q}^2\right. \\ &- \left.\frac{1}{2}{\rm STr}_{\mathbf{q}}\ln\left(\hat{\mathcal{G}}^{-1}(\hat{Q}) - \left(\frac{s^+}{2} + \hat{J}k\right)\Lambda\right)\right], \qquad (6.19)\\ \hat{\mathcal{G}}^{-1}(\hat{Q}) &= \epsilon_0 - \hat{H} - N\hat{Q}. \qquad (6.20)\end{aligned}$$

Thus far no approximations have been made. The next step is to identify the low energy degrees of freedom and obtain an effective action. To do so, we will employ a saddle-point approximation and find the matrix $\hat{Q}_0$ which minimizes the action in Eq. (6.19). The effective field theory is described by the expansion of the action in fluctuations of $\hat{Q}$ around the saddle-point. These fluctuations are strongly anisotropic and can be classified into massive and massless modes. The integral over the former

can be safely evaluated within the saddle-point approximation to leading order in $1/N$ (see Appendix A.6). The integral over the remaining massless modes, which arise from the underlying symmetry of the action (6.15) must be evaluated exactly. The resulting field theory has the form of a nonlinear σ-model.

6.2.1 Saddle-point approximation and the σ-model

Varying the action in Eq. (6.19) with respect to $\hat{Q}$, and neglecting the terms of order s and $\hat{J}$, we find minima at $\hat{Q}_0$ which satisfy the equation

$$\hat{Q}_0 \, \hat{\mathcal{G}}^{-1}(\hat{Q}_0) = N, \tag{6.21}$$

where $\hat{Q}_0$ must be thought of as an operator. The saddle-point solution which is diagonal in superspace is given by

$$\hat{Q}_0 \;=\; \frac{\epsilon_0 - \hat{H}}{2N} + i\left[1 - \left(\frac{\epsilon_0 - \hat{H}}{2N}\right)^2\right]^{1/2} \Lambda. \tag{6.22}$$

Note that $N\hat{Q}_0$ plays the role of the self-energy in the average Green function $\mathcal{G}(\hat{Q}_0)$.

The saddle-point solution in Eq. (6.22) is not unique but is in fact one member of a degenerate manifold of solutions. Their existence follows from the underlying symmetry of the action of Eq. (6.15). The interaction term $S_{\rm int}[\Psi]$ is invariant under the group of transformations $\Psi \to \hat{U}\Psi$ such that

$$\hat{U}^\dagger L \hat{U} = L, \tag{6.23}$$

where $\hat{U}$ is an operator in Hilbert space. Terms that break the symmetry of the total action in Eq. (6.15) are $s\Lambda$, $\hat{J}k\Lambda$ and the commutator $[\hat{H}, \hat{U}]$. The property $\Psi^\dagger = \Psi^T C^T$ should be invariant under $\Psi \to \hat{U}\Psi$, which induces an additional constraint on $\hat{U}$

$$\hat{U}^\dagger = C\hat{U}^T C^T. \tag{6.24}$$

¿From Eq. (6.17) it follows that these transformations induce the following transfor-

mations on the Hubbard-Stratonovich field $\hat{Q}$: $\hat{Q} \to \hat{U}^{-1}\hat{Q}\hat{U}$.

The saddle-point solution in Eq. (6.22) is not invariant under this group of transformations. Therefore the low energy modes of the action are of the form $\hat{Q} = \hat{U}^{-1}\hat{Q}_0\hat{U}$. However, not all of these transformations should be taken into account. The group of transformations (6.23) contains a subgroup of matrices which commute with the Hamiltonian. The matrix $\hat{Q}$ remains diagonal in Hilbert space in the basis of eigenstates of the Hamiltonian. In Appendix A.6 we show that the massive mode integration gives rise to the fact that such fluctuations of $\hat{Q}$ are suppressed by large N. The only matrix $\hat{U}$ commuting with $\hat{H}$ which "survives" the $N \to \infty$ limit is the one proportional to the unit matrix in Hilbert space. Admitting matrices $\hat{Q}$ of such form into Eq. (6.19) we obtain

$$\langle Z(\hat{J})\rangle_{\epsilon_0} = \int DQ \ \exp\left(-S_{\mathit{eff}}[Q]\right), \tag{6.25}$$

where

$$\begin{aligned} S_{\mathit{eff}}[\hat{Q}] &= \frac{1}{2}\mathrm{STr}_{\mathbf{q}} \ln\left[\hat{\mathcal{G}}^{-1}(\hat{Q}) - \left(\frac{s^+}{2} + \hat{J}k\right)\Lambda\right] \\ &= \frac{1}{2}\mathrm{STr}_{\mathbf{q}} \ln\left[\hat{\mathcal{G}}^{-1}(\hat{Q}_0) - \hat{U}\left(\frac{s^+}{2} + \hat{J}k\right)\Lambda\hat{U}^{-1} - \hat{U}[\hat{H}, \hat{U}^{-1}]\right]. \end{aligned} \tag{6.26}$$

The last three terms under the logarithm in Eq. (6.26) are small as compared to the first, and we can expand in them. Each order in this expansion brings an additional power of $1/N$, and suggests the inclusion of just the leading order term:

$$S_{\mathit{eff}}[\hat{Q}] = -\frac{1}{2N}\,\mathrm{STr}_{\mathbf{q}}\left[\hat{Q}\left(\frac{s^+}{2}\Lambda + \hat{J}k\Lambda - \hat{U}^{-1}[\hat{H}, \hat{U}]\right)\right]. \tag{6.27}$$

This approximation is justified only if $s \ll N$ and the commutator $[\hat{H}, \hat{U}]$ is not anomalously large. The validity of this approximation must be considered individually for each system. In section 6.5 we will discuss an example where this is not the case, and one has to keep the second order expansion of the logarithm in Eq. (6.26).

6.2.2 Semi-classical approximation

In the limit $\epsilon_0 \to \infty$, the configurations of the Q-matrix that contribute substantially to the functional integral in Eq. (6.15) can be described within the semi-classical approximation. It is therefore convenient to re-express all operators in the Wigner representation. Given an operator $\hat{\mathcal{O}}$ as a set of matrix elements $\mathcal{O}(\mathbf{q}_1, \mathbf{q}_2)$ between two position states at $\mathbf{q}_1$ and $\mathbf{q}_2$, its Wigner representation is a function of the phase space variables $\mathbf{x} = (\mathbf{q}, \mathbf{p})$ defined by

$$\mathcal{O}(\mathbf{x}) = \int d^d q' \ \exp(i\mathbf{p}\mathbf{q}'/\hbar) \ \mathcal{O}(\mathbf{q} + \mathbf{q}'/2, \mathbf{q} - \mathbf{q}'/2). \tag{6.28}$$

We will use the fact that, in the semiclassical limit, the Wigner transform of a product of operators is equal the product of the Wigner transformed operators, $(\mathcal{O}_1\mathcal{O}_2)(\mathbf{x}) \to \mathcal{O}_1(\mathbf{x})\mathcal{O}_2(\mathbf{x})$, where $\mathcal{O}_{1,2}(\mathbf{x})$ are smooth slowly varying functions on quantum scale [75]. In this approximation Eq. (6.23) becomes

$$U^\dagger(\mathbf{x}) L U(\mathbf{x}) = L, \tag{6.29}$$

and implies that the matrices $U(\mathbf{x})$ belong to the pseudounitary supergroup $U(2, 2/4)$. Expressed in the Wigner representation, the constraint in Eq. (6.24)

$$U^*(\mathbf{q}, \mathbf{p}) = C U(\mathbf{q}, -\mathbf{p}) C^T \tag{6.30}$$

shows that the matrices $U(\mathbf{x})$ at different $\mathbf{x}$ are not independent.

The massless modes in the Wigner representation are generated by those matrices $U(\mathbf{x})$ that do not commute with Λ. Such matrices, denoted by $T(\mathbf{x})$, belong to the coset space $\mathbf{H} = \mathbf{G}/\mathbf{K} = U(2, 2/4)/\left[U(2/2) \otimes U(2/2)\right]$ and, as follows from Eq. (6.30), satisfy the following symmetry relation

$$T^*(\mathbf{q}, \mathbf{p}) = C T(\mathbf{q}, -\mathbf{p}) C^T. \tag{6.31}$$

As was mentioned below Eq. (6.24) the matrices $\hat{U}$ commuting with the Hamiltonian

are suppressed by massive modes (see Appendix A.6). In the semiclassical limit it amounts to only considering those matrices $T(\mathbf{x})$ which are independent of the energy. The massless modes are then given by

$$Q(\mathbf{x}) = T^{-1}(\mathbf{x}_{\|})Q_0(H)T(\mathbf{x}_{\|}), \tag{6.32}$$

where $\mathbf{x}_{\|}$ denotes a phase space coordinate on the energy shell $\epsilon_0 = H(\mathbf{x})$.

Substituting $\hat{T}$ for $\hat{U}$ in Eq. (6.27), and applying the semiclassical approximation, in which the commutator with the Hamiltonian becomes the Liouville operator $\hat{\mathcal{L}}$,

$$[\hat{H}, \hat{T}] \to -i\hbar\hat{\mathcal{L}}T(\mathbf{x}_{\|}) = -i\hbar\left\{T(\mathbf{x}_{\|}), H\right\}, \tag{6.33}$$

where

$$\{A, B\} = \sum_i \left[\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q_i}\frac{\partial A}{\partial p_i}\right] \tag{6.34}$$

denotes the Poisson bracket of A and B, we obtain

$$S_{\mathit{eff}}[Q] \;=\; -\frac{1}{2N}\int \frac{d\mathbf{x}}{h^d}\,\mathrm{STr}\left[Q(\mathbf{x})\left(\frac{s^+}{2}\Lambda + \hat{J}k\Lambda + i\hbar T^{-1}\hat{\mathcal{L}}T\right)\right]. \tag{6.35}$$

Since the only dependence on the coordinate $x_{\perp} \equiv H(\mathbf{x})$ normal to the energy shell enters through $Q_0(H)$ the integral over this variable can be performed. Introducing the notation

$$\mathcal{Q}(\mathbf{x}_{\|}) = \frac{-i}{\pi N}\int dH\; T^{-1}(\mathbf{x}_{\|})Q_0(H)T(\mathbf{x}_{\|}) = T^{-1}(\mathbf{x}_{\|})\Lambda T(\mathbf{x}_{\|}) \tag{6.36}$$

we obtain the final expression

$$S_{\mathit{eff}}[\mathcal{Q}] \;=\; \frac{i\pi}{2}\int \frac{d\mathbf{x}_{\|}}{h^d}\,\mathrm{STr}\left[\mathcal{Q}\left(s^+\Lambda + \hat{J}k\Lambda + i\hbar T^{-1}\hat{\mathcal{L}}T\right)\right]. \tag{6.37}$$

Here and henceforth when the arguments of $\mathcal{Q}$ and T are omitted they should be understood as functions of $\mathbf{x}_{\|}$.

The matrix $\mathcal{Q}(\mathbf{x}_{\|})$ introduced in Eq. (6.36) satisfies the constraints

$$\mathcal{Q}(\mathbf{x}_{\|})^2 = 1, \qquad \mathcal{Q}(\mathbf{q},\mathbf{p}) = C^T L \mathcal{Q}^T(\mathbf{q},-\mathbf{p}) L C. \tag{6.38}$$

It seems that the expression (6.37) does not correspond to a σ-model since it expressed through the matrices T rather than through $\mathcal{Q} = T^{-1}\Lambda T$. A general property of σ-models is the invariance of the action under gauge transformations $T \to RT$, where R commutes with Λ. Using the fact that $\hat{\mathcal{L}} = \dot{\vec{x}}_{\|}\vec{\partial}_{x_{\|}}$ (where $\dot{\vec{x}}_{\|}$ is the phase space velocity) is a first order differential operator it is easy to show that under a gauge transformation the change of the action (6.37) is given by

$$\delta S_{\mathit{eff}}[\mathcal{Q}] = -\frac{\pi\hbar}{2}\int \frac{d\mathbf{x}_{\|}}{h^d}\,\mathrm{STr}\big[\Lambda R^{-1}\hat{\mathcal{L}}R\big] = -\frac{\pi\hbar}{2}\int \frac{d\mathbf{x}_{\|}}{h^d}\,\mathrm{STr}\big[\Lambda\hat{\mathcal{L}}\ln R\big] = 0. \tag{6.39}$$

To arrive at the last equality we used the fact that the flow in phase space is incompressible: $\vec{\partial}_{x_{\|}}\dot{\vec{x}}_{\|} = 0$.

The kinetic part of the action in Eq. (6.37) is equivalent to that introduced by Muzykantskii and Khmel'nitskii [73], where it is written in the form of the Wess-Zumino-Witten action

$$S_{\mathrm{WZW}}[\mathcal{Q}] = \frac{\pi\hbar}{2}\int \frac{d\mathbf{x}_{\|}}{h^d}\int_0^1 du\mathrm{STr}\left(\tilde{\mathcal{Q}}\big[\frac{\partial\tilde{\mathcal{Q}}}{\partial u},\hat{\mathcal{L}}\mathcal{Q}\big]\right). \tag{6.40}$$

where $\tilde{\mathcal{Q}}$ is a smooth function of u and $\mathbf{x}_{\|}$, such that $\tilde{\mathcal{Q}}(\mathbf{x}_{\|},1) = \mathcal{Q}(\mathbf{x}_{\|})$ and $\tilde{\mathcal{Q}}(\mathbf{x}_{\|},0) = \Lambda$.

6.2.3 Range of validity of the σ-model

To clarify the domain of applicability of the non-linear σ-model in Eq. (6.37) let us review the main steps involved in its derivation. The construction of the effective generating functional in Eq. (6.19) involved purely formal manipulations which involved no approximation. To proceed beyond this expression we invoked a saddle-point approximation in which the fluctuations of the massive modes were neglected. The

parameter which controlled this approximation was the inverse bandwidth $1/N$.

The second approximation involved the replacement of quantum mechanical commutators by the semi-classical Poisson bracket. Such an approximation is justified at high energies where the shortest length scale is set by the wavelength of the particle. Finally, in treating fluctuations of the massless modes around the saddle-point, we treat only the leading order term in the expansion. Formally, if the commutator $[\hat{H}, \hat{T}]$ is not anomalously large, this approximation is also justified by large N. Since characteristic configurations of T are assumed semi-classical this assumption can be violated only if $\hat{H}$ contains some non-semiclassical contributions.

The validity of this semi-classical approximation is discussed in more detail later in section 6.5 when we return to consider scattering from quantum impurities and the relation of the ballistic σ-model to the conventional diffusive non-linear σ-model.

The symmetry breaking terms in the action place additional constraints on the range of validity. The expansion around the saddle-point relies on characteristic frequencies (or energy scales arising from the Poisson bracket in the action (6.37)) being much smaller than the bandwidth N.

The derivation of the σ-model of Eq. (6.37) relies solely on the presence of energy averaging which allows us to neglect the contribution of massive modes in the functional integral. Indeed, energy averaging is crucial in the ballistic limit even in the presence of disorder. This was emphasized in the study of Altland and Gefen [85] of spectral statistics of ballistic metallic grains. There it was pointed out that ensemble averages of spectral correlators differ from averages performed over both ensemble and energy. In the semiclassical language of periodic orbit theory this difference emerges from trajectories which are not scattered by impurities [86] which give rise to "clean features" in the quantum spectrum. The neglect of interference terms among different trajectories (namely the diagonal approximation) is allowed only upon energy averaging over a wide band. Otherwise, the interference among these trajectories is substantial. In section 6.5 we show that without energy averaging only the diffusive σ-model of Efetov [20] can strictly be justified. In this case, the large parameter which suppresses the fluctuations of the massive modes is $(\tau\Delta)^{-1}$.

6.3 Regularization of the Functional Integral

The functional integral in Eq. (6.25) with the action (6.37) suffers from ultraviolet divergences and needs to be regularized. This ultraviolet divergence is not an artifact of the approximations made to derive Eq. (6.37) and is present even in the original expression (6.5). Indeed, Eq. (6.5) represents a ratio of quantum spectral determinants $\mathrm{Det}(\epsilon - \hat{H})$. Supersymmetry in Eq. (6.5) improves the ultraviolet properties if the functional integral but in higher dimensions is not sufficient to make it converge. Therefore an ultraviolet regulator needs to be introduced. This regularization induces the corresponding regularization on the functional integral in Eq. (6.25) with the action given by Eq. (6.37). The Lagrangian in (6.37) can be presented as $\mathrm{STr}[\mathcal{Q}T^{-1}\dot{\vec{x}}_{\|}\vec{\partial}_{x_{\|}}T]$, where $\dot{\vec{x}}_{\|}$ is the classical the phase space velocity. This action is only sensitive to the variations of the $\mathcal{Q}$-matrix along the classical trajectories, therefore nothing prevents the $\mathcal{Q}$-field from fluctuating in the directions transverse to $\dot{\vec{x}}_{\|}$. These short scale fluctuations lead the to divergence of the functional integral. The ultraviolet divergences are independent of the classical dynamics and of the shape of the constant energy surface and are unphysical. The diverging contribution to the functional integral needs to be extracted. This is achieved by a regularization procedure. The problem of ultraviolet regularization of functional integrals is a long studied problem in field theory (see for example [83]). One can introduce the following regulator into the action (6.37)

$$\delta S_R = m \int dx_{\|} \mathrm{STr}(\vec{\partial}_{x_{\|}} \mathcal{Q})^2. \tag{6.41}$$

This term suppresses strong fluctuations of $\mathcal{Q}$ in the directions transversal to $\dot{\vec{x}}_{\|}$ and favors the smooth physical functions $\mathcal{Q}(x_{\|})$. Depending on the dimensionality of the phase space this may not be sufficient to make the integral convergent and additional regularization procedures should be invoked. To discuss this question we will consider the functional integral which arises from the lowest order perturbative expansion of the action (6.37).

In this case we can represent $T = 1 + \delta T$ and expand the action (6.37) with

the regulator (regulator) to second order in δT. The resulting functional integral becomes Gaussian and amounts to evaluation of the determinant of the operator $is-\hat{\mathcal{L}}_R = is-\hat{\mathcal{L}}-m\vec{\partial}^2_{x_\parallel}$ which is elliptic. Elliptic operators are operators whose highest rank in derivatives is positive definite. The problem of regularization the determinants of such operators is discussed in [83]. One method involves the construction of a zeta function of the operator defined as

$$\zeta(is-\hat{\mathcal{L}}_R|z) = \sum_i (is-\lambda_i)^{-z} = \frac{1}{\Gamma(z)}\int_0^\infty t^{-z-1}dt \mathrm{Tr}\exp[-(is-\hat{\mathcal{L}}_R)t]. \qquad (6.42)$$

Here λ_i denote the eigenvalues of $\hat{\mathcal{L}}_R$, and we assume that s is such that the operator $is-\hat{\mathcal{L}}_R$ has no zero modes. Then the integral in the right hand side converges at the upper limit. At the lower limit $t \to 0$ it can diverge depending on the value of z. This divergence has ultraviolet nature and can be removed by taking the integral at sufficiently large positive z. The expression can then be analytically continued to the rest of the complex plane. A regularized spectral determinant is expressed through the derivative of the zeta function (6.42)

$$\ln \mathrm{Det}(is-\hat{\mathcal{L}}_R) = -\zeta'(is-\hat{\mathcal{L}}_R|z)\Big|_{z=0}. \qquad (6.43)$$

This regularized spectral determinant $\mathrm{Det}(is-\hat{\mathcal{L}}_R)$ is a function of s. It has zeroes at the positions of eigenvalues of $\hat{\mathcal{L}}_R$ and nowhere else on the complex plane. One then has to take the limit $m \to 0$ to obtain the result which is independent of the regulator (6.41)

$$1/Z(is) = \mathrm{Det}(is-\hat{\mathcal{L}}) = \lim_{m\to 0}\mathrm{Det}(is-\hat{\mathcal{L}}_R) \qquad (6.44)$$

This limit is very different for integrable and chaotic systems. One might expect that due to antihermiticity of $\hat{\mathcal{L}}$ in the $m \to 0$ limit the zeroes of the regularized determinant $\mathrm{Det}(is-\hat{\mathcal{L}})$ will lie on the imaginary axis of is. This however is incorrect for chaotic systems [76, 77].

To understand the subtleties of this limit which arise for nonintegrable systems consider the classical evolution. Let us form an initially nonuniform probability den-

sity distribution $\rho(\mathbf{x}_{\|})$ in the phase space. The classical dynamics involves stretching along the unstable manifold and contraction along the stable one. Thus, any nonuniform initial distribution will evolve into a highly singular function along the stable manifold. The regularization term (6.41) in the classical evolution can be ignored for short times but eventually, when contractions along the stable manifold make the phase space gradients sufficiently big, it becomes relevant[2]. Therefore the limits time-to-infinity and $m \to 0$ do not commute. To find the spectrum one has to take the time-to-infinity limit first and then set the regulator to zero. In this limit the eigenvalues γ_μ of $\hat{\mathcal{L}}$ have finite real parts corresponding to relaxation rates into the equilibrium distribution. These physical eigenvalues which reflect intrinsic irreversible properties of the *purely classical dynamics* are known as Ruelle resonances or the Perron–Frobenius spectrum [76, 77].

There are several ways of calculating the Perron-Frobenius spectrum other than diagonalizing $\mathcal{L}_R$ and taking the "zero noise limit" $m \to 0$ [78]. These employ, for instance, symbolic dynamics [9], course graining of the flow dynamics in phase space [79], and analytic continuation [80].

An exact formal expression for the dynamical zeta function $1/Z(z) = \prod_\mu (z - \gamma_\mu)$ is given in terms of the classical periodic orbits of the system. For two dimensional systems it is of the form [70]

$$1/Z(z) = \prod_p \prod_{k=0}^{\infty} \left(1 - \frac{e^{zT_p}}{|\Lambda_p|\Lambda_p^k}\right)^{k+1} \tag{6.45}$$

where T_p is the period of the p-th primitive orbit and Λ is the eigenvalue of the monodromy matrix (which is the linearized map on the Poincare surface of section in the vicinity of the orbit) with absolute value larger than one. In its present form,

[2] An analogous situation arises in the theory of turbulence [84]. In the inertial range viscosity can be ignored and turbulence can be considered as dissipationless. However for sufficiently small scales velocity gradients become large and viscosity becomes relevant. In this picture the energy which is pumped into the system at large spatial scales is transfered without dissipation in the inertial range to smaller spatial scales and is eventually absorbed at microscopic scales determined by viscosity. The latter can be viewed as an ultraviolet regulator, which is eventually set to zero but has a finite effect on the velocity correlators since it is necessary to produce a stationary solution.

$1/Z(z)$ cannot be used to determine the eigenvalues γ_μ. For this purpose a re-summed formula is required. It can be obtained by expanding the infinite product over the periodic orbits and ordering the various terms in a way that leads to maximal cancelation among them. This method known as the cycle expansion [71] exploits the property that the dynamics of chaotic systems in phase space is coded by a skeleton of a small number of periodic orbit called fundamental orbits. Namely, the long periodic orbits may be viewed as linear combinations of the fundamental orbits.

6.4 Applications

To interpret the findings of the previous sections we will apply the generalized nonlinear σ-model to the region of long-time or low energy scales. This will establish a firm connection of level statistics with RMT. Corrections to RMT will be studied within the framework of a perturbation theory involving the modes of the Perron-Frobenius operator. These results indicate a close correspondence between spectral correlations of the classical and quantum operators which we discuss.

6.4.1 Random Matrix Theory

It is widely believed that the statistical quantum properties of systems with a *small number* of degrees of freedom can be described, at least over some range of energy scales, by RMT. To interpret this, various approaches has been developed largely along two parallel lines. The first approach concerned the study of ensembles of random systems such as disordered metallic grains [82, 58, 20]. Randomness in this case is introduced on the level of the Hamiltonian itself usually as a consequence of some unknown impurity configuration. The second approach involves the study of non-stochastic systems which are chaotic in their classical limit such as the Sinai or the stadium billiards [87]. In this case "randomness" is generated by the underlying deterministic classical dynamics itself. Nevertheless, it has been conjectured [87] that spectral fluctuations of strongly chaotic quantum systems are described by level statistics of random matrix ensembles.

Despite being supported by extensive numerical studies, the origin of the success of RMT as well as its domain of validity are still not completely resolved. Below we will show that, in the semiclassical limit, this conjecture is indeed valid for chaotic systems without any discrete symmetries, and which are characterized by an exponential decay of classical correlation functions in time.

If we define by $\{\gamma_n\}$ the set of the eigenvalues of the Perron-Frobenius operator, $\hat{\mathcal{L}}$, the lowest eigenvalue in ergodic systems $\gamma_0 = 0$. This eigenvalue is non-degenerate, and manifests the conservation of probability density. Any initial density eventually relaxes to the state associated with γ_0. If, in addition, this relaxation is exponential in time, then the Perron-Frobenius spectrum has a gap associated with the slowest decay rate. Thus, for the first nonzero eigenvalue γ_1 we have $\gamma_1' \equiv \mathrm{Re}(\gamma_1) > 0$. This gap sets the ergodic time scale, $\tau_c = 1/\gamma_1'$ over which the classical dynamics relaxes to equilibrium. In the case of disordered metallic grains, it coincides with the Thouless time, while in ballistic systems or billiards it is of order of the time of flight across the system.

In the limit $s \ll \gamma_1'$, or equivalently at times which are much longer than τ_c, the dominant contribution to Eq. (6.25) with the effective action of Eq. (6.37) comes from the ergodic classical distribution, the zero-mode $\hat{\mathcal{L}}T_0 = 0$. With this contribution alone the functional integral (6.25) becomes definite,

$$\langle \mathcal{Z}(J) \rangle_{\epsilon_0} = \int dQ_0 \exp\left[-i\frac{\pi}{4}\mathrm{STr}\left(\left[s^+ + 2Jk\right]\Lambda Q_0\right)\right], \tag{6.46}$$

where $Q_0 = T_0^{-1}\Lambda T_0$. This expression coincides with that obtained from RMT and implies Wigner-Dyson level statistics [20, 21]. A straightforward generalization to include other statistical correlators confirms the correspondence between the zero-mode an RMT (see for example Ref. [21]). The quantum statistics of chaotic systems with exponential classical relaxation are therefore described by RMT at energies smaller than γ_1'.

The RMT description is expected to hold even for certain chaotic systems where the Perron-Frobenius spectrum is gapless [87] such as the stadium or the Sinai billiards

where classical correlation functions decay algebraically in time [81]. The resolvent $(z-\hat{\mathcal{L}})^{-1}$, in this case, is expected to have cuts which reach the $\Im z$ axis. Nevertheless, we expect the RMT description to hold whenever the spectral weight of the resolvent inside the strip $0 \leq \Re z \leq 1$ (which excludes the pole at the origin, however) is much smaller than unity.

6.4.2 Beyond universality

To proceed, let us make use of the σ-model to examine how corrections to RMT appear at larger energy scales. Again, focusing of the two-point correlator of DoS fluctuations, the generating function leads to the expression

$$R_2(s) = \lim_{u\to 0} \frac{1}{64} \Re \int \mathcal{D}Q \left(\int d\mathbf{x}_{\|} \mathrm{STr}[\Lambda k Q(\mathbf{x}_{\|})] \right)^2 \exp\left[-S_{\mathit{eff}}(s)\right] \tag{6.47}$$

where

$$S_{\mathit{eff}}(s) \;=\; \frac{\pi}{2} \int (d\mathbf{x}_{\|}) \mathrm{STr} \left[i\frac{s^+}{2} \Lambda Q - u^2 (\Lambda Q)^2 + Q T^{-1} \hat{\mathcal{L}} T \right]. \tag{6.48}$$

Here we have introduced into the action an additional term $-u^2(\Lambda Q)^2$. It is introduced as a regulator to control the stationary point evaluation of the functional integral and should ultimately be sent to zero.

In the limit of high frequencies s, the two-point correlator takes the asymptotic form

$$R_2(s) = R_P(s) + R_{NP}(s), \tag{6.49}$$

where both the non-perturbative term $R_{NP}(s)$ and the perturbative one $R_P(s)$ are expressed through the classical spectral determinant $\mathcal{D}(s)$ as

$$R_{NP}(s) = \frac{\cos(2\pi s)}{2\pi^4} \mathcal{D}(s), \quad R_P(s) = -\frac{1}{\pi^2} \frac{\partial^2}{\partial s^2} \ln[\mathcal{D}(s)]. \tag{6.50}$$

$\mathcal{D}(s)$ can itself be expressed in terms of determinants of the Perron-Frobenius operator

$$\mathcal{D}(s) = \Re \frac{\mathrm{Det}'(\hat{\mathcal{L}})^4}{\mathrm{Det}^2\left[(is-\hat{\mathcal{L}})(-is-\hat{\mathcal{L}})\right]}, \tag{6.51}$$

where Det should be understood as a regularized determinant as explained in section 6.3, and the prime on the determinant denotes the exclusion of the zero eigenvalue.

We can express $\mathcal{D}(s)$ in terms eigenvalues γ_μ of $\hat{\mathcal{L}}$, the Ruelle resonances

$$\mathcal{D}(s) = \prod_\mu A^2(\gamma_\mu)\left(\gamma_\mu^2 + s^2\right)^{-2}. \tag{6.52}$$

Here $A(\gamma_0 = 0) = 1$ (there is always such an eigenvalue zeta function, it corresponds to the state which s uniform on the constant energy shell), and $A(\gamma_\mu) = \gamma_\mu^2$ for $\gamma_\mu \neq 0$. Eq. (6.52) is written assuming that the product converges. If the product diverges then $\mathcal{D}(s)$ should be understood as the regularized determinant (6.44).

The derivation of the results in Eqs. (6.50) and (6.51) closely parallels that of Ref. [51] [3].

At high frequency the integrand in Eq. (6.48) becomes highly oscillatory, and we can use the stationary phase method to evaluate the integral. There are two such points: $Q = \Lambda$ and $Q = -\Lambda k$. The presence of the term $u^2(\Lambda Q)^2$ in the action makes both stationary points stable. We can expand the integrand in small fluctuations of Q-matrix around Λ and $-\Lambda k$ to obtain the leading high-frequency asymptotics of $R_2(s)$.

We first consider the expansion around $Q = \Lambda$. This corresponds to the usual perturbation expansion in the theory of disordered conductors [20, 58]. We begin with the parametrization

$$T = \mathbf{1} + iP, \qquad P = \begin{pmatrix} 0 & B \\ \bar{B} & 0 \end{pmatrix}, \tag{6.53}$$

[3] Note that the sign in the right hand side of Eq. (5b) in this paper is incorrect.

where, from Eq. (6.31), it follows that P satisfies the condition

$$P(\mathbf{q},\mathbf{p})^* = -CP(\mathbf{q},-\mathbf{p})C^T. \tag{6.54}$$

Next we substitute Eq. (6.53) into Eq. (6.47) and expand the integrals in the pre-exponential factor and the free energy (6.48) to second order in P. Due to the presence of the infinitesimal imaginary part in s^+, the stationary point $Q = \Lambda$ is stable and we can safely set $u = 0$ in the free energy (6.48). To second order in B and $\bar{B}$ we have

$$\begin{aligned} \mathrm{STr}(\Lambda k Q) &\approx 8 - 2\mathrm{STr}(kB\bar{B} + k\bar{B}B), \\ \mathrm{STr}(\Lambda k Q)^2 &\approx -8\mathrm{STr}(k\bar{B}kB + \bar{B}B), \qquad (6.55) \\ \mathrm{STr}(\Lambda Q) &\approx -4\mathrm{STr}(\bar{B}B). \qquad (6.56) \end{aligned}$$

Using these relations we obtain the following expression for the perturbative part of $R_2(s)$

$$R_P(s) = \Re \int \mathcal{D}[B,\bar{B}] \left(\int d\mathbf{x}_{\|} [1 - \frac{1}{2}\mathrm{STr}(kB\bar{B} + k\bar{B}B)] \right)^2 \exp\left[-S_{\mathit{eff}}(s)\right] \tag{6.57}$$

where

$$S_{\mathit{eff}}(s) = i\pi \int (d\mathbf{x}_{\|})\ \mathrm{STr}\left[-s\bar{B}B - i\bar{B}\hat{\mathcal{L}}B\right] + O(B^4) \tag{6.58}$$

In order to perform the integration over B and $\bar{B}$ it is convenient to represent these matrices as

$$B = \sum_{i=0}^{3} B_i \tau_i, \qquad \bar{B} = \sum_{i=0}^{3} \bar{B}_i \tau_i. \tag{6.59}$$

As follows from Eq. (6.29) the matrices B and $\bar{B}$ in Eq. (6.53) obey the relation $\bar{B} = kB^\dagger$, which implies

$$\bar{B}_i = kB_i^\dagger, \quad i = 0, \ldots, 3. \tag{6.60}$$

In this notation, Eq. (6.58) becomes

$$S_{eff,P}(s) = -i\pi \int (d\mathbf{x}_{\|})\ \mathrm{STr}\left[\sum_{i=0}^{3}\left(\bar{B}_i(s+i\hat{\mathcal{L}})B_i\right)\right]. \tag{6.61}$$

Each matrix B_i can be parametrized as

$$B_i = \begin{pmatrix} a_i & i\sigma_i \\ \eta_i^* & ib_i \end{pmatrix}. \tag{6.62}$$

The parametrization for $\bar{B}_i$ can be obtained from Eq. (6.60). To evaluate the integral (6.57) over the variables (6.62) one can use Wick's theorem. It is necessary to take into account Eq. (6.54) which reduces the number of independent integration variables by a factor of two. As a result we obtain the second part in Eq. (6.50).

Now let us turn to the other stationary point $Q = -k\Lambda$. Consider the functional integral (6.47). The Q-integration goes over the symmetric space $\mathbf{H} = \mathbf{G}/\mathbf{K}$. If we perform a global coordinate transformation $Q \to U_0^{-1}QU_0$, where $U_0 \in \mathbf{H}$, the integral will remain invariant. The integrand, however, will change because it contains matrices Λ and $-k\Lambda$ that break the symmetry in the coset space. Such a coordinate transformation maps $\Lambda \to U_0\Lambda U_0^{-1}$ and $-k\Lambda \to -U_0 k\Lambda U_0^{-1}$. There exists a transformation U_0 such that $\Lambda \to U_0\Lambda U_0^{-1} = -k\Lambda$ and $-k\Lambda \to -U_0 k\Lambda U_0^{-1} = \Lambda$. Therefore Eq. (6.47) can be rewritten as

$$R_2(s) = \lim_{u\to 0}\frac{-1}{64}\Re\int \mathcal{D}Q\left(\int d\mathbf{x}_{\|}\mathrm{STr}[\Lambda Q(\mathbf{x}_{\|})]\right)^2 \exp\left[-\tilde{S}_{eff}(s)\right], \tag{6.63}$$

$$\tilde{S}_{eff}(s,u) = \frac{\pi}{4}\int (d\mathbf{x}_{\|})\mathrm{STr}\left[-is^+k\Lambda Q - u^2(\Lambda Q)^2 + QT^{-1}\hat{\mathcal{L}}T\right]. \tag{6.64}$$

We now expand Eq. (6.63) in powers of P using Eq. (6.53). This expansion is equivalent to expanding the Q-matrix around $-k\Lambda$ in Eq. (6.47). Expanding the free energy (6.64) to second order in P we use Eq. (6.55). Note that with the parametrization of Eq. (6.62)

$$\mathrm{STr}\left(k\bar{B}B + kB\bar{B}\right) = -4\sum_{i=0}^{3}(|a_i|^2 - |b_i|^2), \tag{6.65}$$

$$\mathrm{STr}\left(k\bar{B}kB + \bar{B}B\right) = -4\sum_{i=0}^{3}(|a_i|^2 + |b_i|^2). \tag{6.66}$$

Therefore the Grassmann variables in the parametrization (6.62) do not couple to s and u^2. As follows from (6.65) the ordinary variables a_i and b_i couple to s with opposite signs. Due to the presence of the infinitesimal imaginary part in s the integral over a_i^0 (the zero mode variable) would diverge at $u = 0$. Equation (6.66) shows that the term $\mathrm{STr}(k\Lambda Q)^2$ makes the integration over a_i^0 convergent. We therefore have to keep u finite during the evaluation of the functional integral and take the limit $u \to 0$ only in the final expressions. The quadratic approximation to the free energy (6.64) becomes

$$\begin{aligned}\tilde{S}_{\mathit{eff}}(s) &= -2\pi i s + 2\pi \int (d\mathbf{x}_{\|})\mathrm{STr}\left[\sum_{i=0}^{3}\left(a_i^*(-is^+ - u^2 + \hat{\mathcal{L}})a_i\right.\right. \\ &+ \left.\left. b_i^*(is^+ - u^2 + \hat{\mathcal{L}})b_i + \sigma_i^*\hat{\mathcal{L}}\sigma_i + \eta_i^*\hat{\mathcal{L}}\eta_i\right)\right] \end{aligned} \tag{6.67}$$

The zero mode Grassmann variables η_i^0 and σ_i^0 do not appear in the quadratic expansion of the free energy (6.67). For the integral (6.63) not to vanish they have to come from the pre-exponential factor. While evaluating the integral we have to take into account the symmetry (6.54) which reduces the number of independent integration variables by factor of two. Therefore there are eight independent Grassmann variables in the zero mode. Thus, in order to obtain a non-zero result we should expand the pre-exponential factor to eighth order in P. Then in the eighth order expansion of the prefactor we should keep only the zero mode terms. This renders the integration over the zero mode variables nonvanishing, whereas the integration over the ordinary zero mode variables yields a factor $(s^2 + u^4)^{-2}$. The integral over the nonzero modes yields the superdeterminant of the operator (6.67). After we perform the integration we take the $u \to 0$ limit to obtain equation (6.51).

6.5 Beyond the Semi-Classical Approximation

The derivation of the non-linear σ-model in Eq. (6.37) relied on the use of the semi-classical approximation. However, often we are concerned with quantum chaotic systems which can not be treated straightforwardly within the framework of semi-classics. A familiar example involves the quantum mechanical scattering of particles from a weak random impurity potential. In such cases, a formal justification of the ballistic non-linear σ-model in Eq. (6.37) does not seem possible. However, if the quantum Hamiltonian can be resolved into a part that can be treated within semi-classics and a part which can not, when the latter is small, a perturbation treatment may still be possible.

Consider a general Hamiltonian $\hat{H}$ with matrix elements

$$\hat{H} = \hat{H}_{\rm cl} + \hat{H}_{\rm qu}, \tag{6.68}$$

where $\hat{H}_{\rm cl}$ represents the contribution which can be treated within a semi-classical approximation, and $\hat{H}_{\rm qu}$ determines the part which can not.

If the matrix elements of $\hat{H}_{\rm qu}$ are small as compared to the band width N (a more precise criterion can be formulated for a specific operator $\hat{H}_{\rm qu}$, see below) their effect can be treated within the σ-model approach. In this case the saddle-point is governed by $\hat{H}_{\rm cl}$, and we can use Eq. (6.22) with $\hat{H}$ replaced by $\hat{H}_{\rm cl}$, The contribution of $\hat{H}_{\rm qu}$ to the effective action can be found by expanding Eq. (6.26),

$$S_{\mathit{eff}}[\hat{Q}] = \frac{1}{2}{\rm STr}_{\mathbf{q}} \ln \left[\hat{\mathcal{G}}^{-1}(\hat{Q}_0) - \hat{H}_{\rm qu} - \hat{U}\left(\frac{s^+}{2} + \hat{J}k\right)\Lambda\hat{U}^{-1} - \hat{U}[\hat{H}, \hat{U}^{-1}] \right], \tag{6.69}$$

where the supermatrix Green function involves only $\hat{H}_{\rm cl}$. Expanding to second order in $\hat{H}_{\rm qu}$ we obtain

$$\begin{aligned} S_{\mathit{eff}}[\hat{Q}] &= -\frac{1}{2N}\,{\rm STr}_{\mathbf{q}} \left[\hat{Q}\left(\frac{s^+}{2}\Lambda + \hat{J}k\Lambda + \hat{H}_{\rm qu} - \hat{U}^{-1}[\hat{H}_{\rm cl}, \hat{U}]\right) \right. \\ &+ \left. \frac{1}{2N}\left(\hat{Q}\hat{H}_{\rm qu}\right)^2 \right]. \end{aligned} \tag{6.70}$$

Finally, representing the Q-matrices in the Wigner representation the second order correction to the action takes the form

$$-\frac{1}{4N^2}\mathrm{STr}_{\mathbf{q}}\left(\hat{Q}\hat{H}_{\mathrm{qu}}\right)^2 = -\frac{1}{2N^2}\int\prod_{i=1}^{4}dq_i\prod_{i=1}^{2}\frac{dp_i}{h^d}e^{-i\mathbf{p}_1(\mathbf{q}_1-\mathbf{q}_2)/2\hbar - i\mathbf{p}_2(\mathbf{q}_3-\mathbf{q}_4)/2\hbar}$$
$$\times H_{\mathrm{qu}}(\mathbf{q}_2,\mathbf{q}_3)H_{\mathrm{qu}}(\mathbf{q}_4,\mathbf{q}_1)\mathrm{STr}\left[Q\left(\mathbf{p}_1,(\mathbf{q}_1+\mathbf{q}_2)/2\right)Q\left(\mathbf{p}_2,(\mathbf{q}_3+\mathbf{q}_4)/2\right)\right], \quad (6.71)$$

where $H_{\mathrm{qu}}(\mathbf{q},\mathbf{q}') = \langle\mathbf{q}|\hat{H}_{\mathrm{qu}}|\mathbf{q}'\rangle$.

Random impurities and the restoration of the diffusive non-linear σ-model

A physical example concerns the case of a background of weakly scattering impurities. If the $\mathcal{Q}$ matrices vary on a scale that is long as compared to the scattering length $\ell = v\tau$, particle dynamics becomes diffusive and we should recover the supersymmetric non-linear σ-model introduced by Efetov [20]. In the opposite limit, the impurities generate a new term in the action which takes the form of a collision integral.

The problem of dilute scattering impurities in an otherwise ballistic system has been discussed previously. A description within the framework of diagrammatic perturbation theory was investigated by Altland and Gefen [85]. More recently, in a substantial development Muzykantskii and Khmel'nitskii [73] introduced an effective field theory to extend the diffusive σ-model into the ballistic regime.

For simplicity, let us consider a δ-correlated white noise impurity potential

$$H_{\mathrm{qu}}(\mathbf{q},\mathbf{q}') = V(\mathbf{q})\delta^d(\mathbf{q}-\mathbf{q}'), \quad (6.72)$$

where the dimensionless mean free time τ is defined by the second moment

$$\langle\delta V(\mathbf{q})\rangle_V = 0, \qquad \langle\delta V(\mathbf{q})\delta V(\mathbf{q}')\rangle_V = \frac{\Omega}{2\pi\tau}\delta^d(\mathbf{q}-\mathbf{q}'), \quad (6.73)$$

where Ω is the volume of the system.

In this case, the expansion of the action around the saddle-point of the Hamiltonian $\hat{H}_{\mathrm{cl}}$ is justified in the limit $1/\tau \ll N$. The same condition allows the truncation of the perturbation series at second order. Once again, performing the energy shell

integration we obtain the effective action

$$S_{eff}[\mathcal{Q}] = \frac{i\pi}{2}\int \frac{d\mathbf{x}_{\|}}{h^d}\,\mathrm{Str}\Big[\left(\frac{s^+}{2}\Lambda + \hat{J}k\Lambda - i\hbar T^{-1}\{H_{\mathrm{cl}},T\}\right)\mathcal{Q}\Big] - \frac{\pi\Omega}{2\tau}\int \frac{d\mathbf{x}_{\|}}{h^d}\frac{d\mathbf{x}'_{\|}}{h^d}\delta^d(\mathbf{q}-\mathbf{q}')\mathrm{STr}\left[\mathcal{Q}(\mathbf{x}_{\|})\mathcal{Q}(\mathbf{x}'_{\|})\right]. \quad (6.74)$$

Although this action is precisely of the form of that introduced in Ref. [73], its derivation and the domain of validity seems far removed from that proposed in this earlier work. The sigma model description of the ballistic regime holds only if the frequencies of interest (or, equivalently the characteristic gradient energies) are small as compared with the width of the band $N\Delta$. In the absence of energy averaging the validity range of such description is restricted to the diffusive regime, where it coincides with the diffusive σ-model [73]. At higher energies the massive modes have to be taken into account, and the σ-model description fails. The distinction drawn by energy averaging has been emphasized by Altland and Gefen [85]. Physically the difference comes from those orbits whose period is shorter than τ but longer than the inverse band width $(N\Delta)^{-1}$. Technically the energy averaging suppresses the massive mode fluctuations and facilitates the σ-model description.

From Eq. (6.74) it is straightforward to establish the relation between the ballistic σ-model and the conventional diffusive counterpart. This is achieved by separating the supermatrix fields into fast degrees of freedom corresponding to momentum relaxation and slow ones corresponding to density relaxation [73]. Upon the integration over the fast degrees of freedom one recovers the diffusive σ-model [20].

Appendix A

Appendix

A.1 Invariant measure for the average DOS

In this appendix we will determine the invariant measure used in calculation of average DOS in section 2.2. To do so we use parameters of $T_0[12]$ and $R^{-1}Q_D R$ as independent variables. Following the method of Ref. [21] it is straightforward to show that the invariant measure at the saddle-point is again given by $d\mu(Q) = \mathrm{detg}(\delta T_0'[12]/\delta[\hat{\rho}_1, \xi])d\rho_{1B}id\rho_{1F}d\xi_1 d\xi_3$, where $\delta T_0' = \delta T_0 T_0^{-1}$, and the variables ρ and ξ were introduced in Eq. (2.50).

We want to express the invariant measure through $d\theta_{1B}$, $d\theta_{1F}$, $d\xi_1$, $d\xi_3$. We evaluate the Berezinian of this transformation using the chain rule:

$$\mathrm{detg}\left(\frac{\delta(T_0')[12]}{\delta(\hat{\rho}_1,\xi)}\right) = \mathrm{detg}\left(\frac{\delta(T_0')[12]}{\delta\tau}\right) \times \mathrm{detg}\left(\frac{\delta\tau}{\delta(\hat{\vartheta},\xi)}\right) \times \mathrm{detg}\left(\frac{\delta(\hat{\vartheta},\xi)}{\delta(\hat{\rho}_1,\xi)}\right). \quad \text{(A.1)}$$

As in the usual case we have to evaluate $d\mu((\delta T_0 T_0^{-1})[12])$. Following Ref. [21], to calculate the invariant measure we introduce a "rational" parametrization; $w_1 = 2(1-\tau^2)^{-1}\tau$, when

$$T_0 = \begin{pmatrix} (1+\tau^2)(1-\tau^2)^{-1} & -2i(1-\tau^2)^{-1}\tau \\ 2i(1-\tau^2)^{-1}\tau & (1+\tau^2)(1-\tau^2)^{-1} \end{pmatrix}. \quad \text{(A.2)}$$

For $(\delta T_0')[12] = (\delta T_0 T_0^{-1})[12]$ we obtain

$$(\delta T_0')[12] = -2i(1-\tau^2)^{-1}(\delta\tau - \tau\delta\tau\tau)(1-\tau^2)^{-1}. \tag{A.3}$$

It is clear that τ is diagonalized by the same matrix u_1 as w_1. We use the following eigenvalue parametrization

$$\tau = u_1\hat{\vartheta}u_1^{-1} = u_1 \begin{pmatrix} \vartheta_1 & 0 \\ 0 & i\vartheta_2 \end{pmatrix} u_1^{-1}. \tag{A.4}$$

The eigenvalues of w_1 and of τ are related by

$$\rho_{1B} = \tfrac{2\vartheta_1}{1-\vartheta_1^2} \quad , \quad \rho_{1F} = \tfrac{2\vartheta_2}{1+\vartheta_2^2}. \tag{A.5}$$

Using this we obtain the following relation

$$u_1^{-1}(\delta T_0')[12]u_1 = -2i(1-\hat{\vartheta}^2)^{-1}(u_1^{-1}\delta\tau u_1 - \hat{\vartheta}u_1^{-1}\delta\tau u_1\hat{\vartheta})(1-\hat{\vartheta}^2)^{-1}. \tag{A.6}$$

Now we observe that $\mathrm{detg}\left(\frac{(\delta T_0')[12]}{\delta\tau}\right) = \mathrm{detg}\left(\frac{u_1^{-1}(\delta T_0')[12]u_1}{u_1^{-1}\delta\tau u_1}\right)$. Explicitly writing down the components of $u_1^{-1}\delta\tau u_1$, $u_1^{-1}(\delta T_0')[12]u_1$ and relations between them that follow from Eq. (A.6) we obtain

$$\mathrm{detg}\left(\frac{(\delta T_0')[12]}{\delta\tau}\right) = \frac{(1-\vartheta_1^2)(1+\vartheta_2^2)}{(1-i\vartheta_1\vartheta_2)^2}. \tag{A.7}$$

We use Eq. (A.4) to calculate the second Berezinian in the product Eq. (A.1). A straightforward calculation gives

$$\mathrm{detg}\left(\frac{\delta\tau}{\delta(\vartheta,\xi)}\right) = \frac{1}{(\vartheta_1 - i\vartheta_2)^2}. \tag{A.8}$$

In the last Berezinian in Eq. (A.1) ξ's are the same since τ and w are diagonalized by the same matrix u, and the relation between the eigenvalues is given by Eq. (A.5), which yields

$$\mathrm{detg}\left(\frac{\delta(\hat{\vartheta},\xi)}{\delta(\hat{\rho}_1,\xi)}\right) = \frac{(1-\vartheta_1^2)(1+\vartheta_2^2)}{4(1+\vartheta_1^2)(1-\vartheta_2^2)}. \tag{A.9}$$

Combining all these expressions we obtain the invariant measure

$$d\mu(Q) = \frac{(1-\vartheta_1^2)^3(1+\vartheta_2^2)^3 d\rho_{1B} d\rho_{1F} d\xi_1 d\xi_3}{4(1-i\vartheta_1\vartheta_2)(\vartheta_1 - i\vartheta_2)^2(1+\vartheta_1^2)^2(1-\vartheta_2^2)^2}. \tag{A.10}$$

Expressing the differentials of ρ's through θ's we find

$$d\mu(Q) = \frac{\left[(1-\vartheta_1^2\vartheta_2^2)(\vartheta_1^2-\vartheta_2^2) - 4\vartheta_1^2\vartheta_2^2 + 2i\vartheta_1\vartheta_2(1+\vartheta_1^2)(1-\vartheta_2^2)\right] d\theta_{1B} d\theta_{1F} d\xi_1 d\xi_3}{(\cosh(\theta_{1B}/2)+1)^2(\cos(\theta_{1F}/2)+1)^2(1+\vartheta_1^2\vartheta_2^2)^2(\vartheta_1^2+\vartheta_2^2)^2} \tag{A.11}$$

Since the effective Lagrangian and the invariant one-point function of Eq. (2.52) are functions only of $\cosh(\hat{\theta})$ and thus even functions of $\hat{\theta}$, the last term in the numerator gives no contribution to the integrals and can be neglected. After some algebra the last expression can be reduced to the more compact one shown in Eq. (2.53).

A.2 Integration over the Saddle-Point Manifold for the Average DOS

In this appendix we calculate the integrals over the saddle-point manifold to determine the average DOS. It is straightforward to show that the integrals over the variables in $R^{-1}Q_D R$ are equal to unity, and we only have to integrate over the Goldstone modes. The Berezinian of Eq. (2.53) is singular at $\hat{\theta} = 0$, and the integral contains two contributions: the first from the ordinary part of the invariant one-point function and the second from the part containing Grassmann variables. The former is equal to the the value of the one-point function at the origin (Ref. [21, 31]). Using Eqs. (2.36), (2.52) and (2.53) and performing the integration over the Grassmann variables we can obtain the following expression for the average DOS

$$\langle\rho(E)\rangle = \frac{1}{\Delta} + \frac{i}{8\pi\Delta}\int_{-\infty}^{+\infty} d\theta_{1B} \int_{-\pi}^{\pi} d\theta_{1F} \frac{\cosh\theta_{1B}\cos\theta_{1F} - 1}{\cosh\theta_{1B} - \cos\theta_{1F}}$$

$$\times \left\{ \exp\left[\frac{i\pi E^+}{\Delta}(\cosh\theta_{1B} - \cos\theta_{1F})\right] \right.$$
$$\left. + \exp\left[\frac{-i\pi E^-}{\Delta}(\cosh\theta_{1B} - \cos\theta_{1F})\right] \right\}. \tag{A.12}$$

Next we assume that E is positive and deform the integration contour for the variable θ_{1B} in the first exponential as shown in Fig. 2-3 a), and that for the second exponential as in Fig. 2-3 b). Such a deformation ensures the convergence of the integrals for positive E and we do not need to keep the imaginary part of E any more. The integrals over legs 1 and 1′ cancel each other, as do the integrals over legs 3 and 3′. The integrals over legs 2 and 2′ yield for $\langle\rho(E)\rangle$

$$\langle\rho(E)\rangle = \frac{1}{\Delta} + \frac{1}{8\pi\Delta}\mathrm{Im}\int_{-\pi}^{\pi} d\theta_{1B}\int_{-\pi}^{\pi} d\theta_{1F}\frac{\cos\theta_{1B}\cos\theta_{1F} - 1}{\cos\theta_{1B} - \cos\theta_{1F}}\exp\left[\frac{i\pi E}{\Delta}(\cos\theta_{1B} - \cos\theta_{1F})\right], \tag{A.13}$$

Taking the derivative of the last expression with respect to energy we obtain,

$$\frac{d\langle\rho(E)\rangle}{dE} = \frac{\pi^2}{2\Delta^2}\left[J_0^2(\pi E/\Delta) - J_1^2(\pi E/\Delta)\right]. \tag{A.14}$$

By integrating the last expression we find

$$\langle\rho(E)\rangle = \frac{\pi^2 E}{2\Delta^2}\left(J_0^2(\pi E/\Delta) + J_1^2(\pi E/\Delta)\right) \tag{A.15}$$

where the integration constant is zero, since at $E = 0$ the DOS should vanish. Thus we obtain Eq. (2.54), which coincides with the result of Ref. [14, 16, 17].

A.3 Invariant Measure for the Two-Point Function

In this appendix we calculate the invariant measure for the two point function. We follow closely the method of Ref. [32]. The invariant measure is again given by $\delta T_0'$[12], where $\delta T_0' = \delta T_0 T_0^{-1} = \delta T_u T_u^{-1} + T_u \delta T_{ch} T_{ch}^{-1} T_u^{-1}$. Equations(2.73) and (2.74)

suggest that the matrix of partial derivatives $\frac{\delta T_0'[12]}{\delta(T_u'[12],T_{ch}'[12])}$ has the structure

$$\begin{array}{lcc} & \delta T_u'[12] & \delta T_{ch}'[12] \\ \delta T_0'^{RR}[12] & 0 & \star \\ \delta T_0'^{AA}[12] & 0 & \star \\ \delta T_0'^{RA}[12] & 1 & \star \\ \delta T_0'^{AR}[12] & 1 & \star. \end{array} \tag{A.16}$$

where $\star$ denotes nonvanishing terms. Thus, our Berezinian is simply equal to

$$J = \mathrm{detg}\left(\frac{\delta T_0'[12]}{\delta(T_u'[12], T_{ch}'[12])}\right) = \mathrm{detg}\left(\frac{\delta[(T_0')^{RR}[12], (T_0')^{AA}[12]]}{\delta[(T_{ch}')^{RR}[12], (T_{ch}')^{AA}[12]]}\right). \tag{A.17}$$

Using Eq. (2.76) we obtain

$$\begin{aligned} \delta(T_0')^{RR}[12] &= T_u^{RR}[11]\delta(T_{ch}')^{RR}[12](T_u^{-1})^{RR}[22] + T_u^{RA}[12]\delta(T_{ch}')^{AA}[21](T_u^{-1})^{AR}[12] \\ &= u[\cosh(\hat{\Omega}/2)u^{-1}\delta(T_{ch}')^{RR}[12]u\cosh(\hat{\Omega}/2) - \\ &\sinh(\hat{\Omega}/2)\exp(i\hat{\phi})v^{-1}\delta(T_{ch}')^{AA}[12]v\sinh(\hat{\Omega}/2)\exp(-i\hat{\phi})]u^{-1}. \end{aligned} \tag{A.18}$$

The matrices u and v do not affect the value of the superdeterminant, and after a simple calculation we find

$$J = \frac{4\cosh\Omega_B\cos\Omega_F}{(\cosh\Omega_B + \cos\Omega_F)^2} \tag{A.19}$$

The invariant measure $d\mu(T_u')$ is just the usual measure for the unitary ensemble equal to $\frac{(2\pi)^2 d(\cosh\Omega_B)d(\cos\Omega_F)d\alpha d\alpha^* d\beta d\beta^*}{(\cosh\Omega_B - \cos\Omega_F)^2}$, where the variables ϕ_B and ϕ_F have been integrated over since nothing depends on them. The invariant measure for T_{ch}' is given by Eq. (2.53). Combining everything and introducing the variables $s_1 = \cosh\Omega_B$ and $s_2 = \cos\Omega_F$ we arrive at Eq. (2.78) for the invariant measure.

A.4 Integration limits

In this appendix we establish the integration limits for the variables for the two-point function. To do this it is sufficient to consider only the transformations not containing the Grassmann variables. In the BB block all the variables range from $-\infty$ to $+\infty$, and we have to establish the integration limits for the variables in the FF block only. The FF block can be parametrized as $Q[FF] = -i\Sigma_z \cos(\hat{a}) + i\Sigma_x \sin(\hat{a})$, where $\hat{a}$ is a hermitean matrix. It suffices to consider the $Q[FF]_{12}$ block only. Then $Q[FF]_{12} = i\sin(\hat{a})$, but $i\sin(\hat{a})$ is just an antihermitean part of a $U(2)$ matrix $\exp(i\hat{a})$, which can be parametrized as

$$\begin{pmatrix} \cos\phi \exp(i(\gamma_0+\gamma_1)) & \sin\phi \exp(i(\gamma_0+\gamma_2)) \\ -\sin\phi \exp(i(\gamma_0-\gamma_2)) & \cos\phi \exp(i(\gamma_0-\gamma_1)) \end{pmatrix}, \tag{A.20}$$

where $\phi \in [0, \pi/2]$, $\gamma_0 \in [0, 2\pi]$, $\gamma_1 \in [0, 2\pi]$, $\gamma_2 \in [0, 2\pi]$. Its antihermitean part is then

$$\begin{pmatrix} i\cos\phi \sin(\gamma_0+\gamma_1) & \sin\phi \cos\gamma_0 \exp(i\gamma_2) \\ -\sin\phi \cos\gamma_0 \exp(-i\gamma_2) & i\cos\phi \sin(\gamma_0-\gamma_1) \end{pmatrix}. \tag{A.21}$$

In our notation, in the absence of Grassmann variables, this block is given by

$$\begin{pmatrix} i\cos\Omega_F \sin(\theta_{1F}) & i\sin\Omega_F \cos(\frac{\theta_{1F}+\theta_{2F}}{2}) \exp(i\phi_F) \\ i\sin\Omega_F \cos(\frac{\theta_{1F}+\theta_{2F}}{2}) \exp(-i\phi_F) & i\cos\Omega_F \sin(\theta_{2F}) \end{pmatrix}. \tag{A.22}$$

Comparing the last two expressions we identify the variables as follows:

$$\Omega_F = \phi \in [0, \pi/2], \quad \theta_{1F} = \gamma_0 + \gamma_1, \quad \theta_{2F} = \gamma_0 - \gamma_1, \quad \phi_F = (\gamma_2 - \pi/2) \in [0, 2\pi]. \tag{A.23}$$

Q is periodic with respect to γ_0 and γ_1 with period 2π and, thus forms a $2D$ lattice. The integration region is the lattice unit cell whose orientation we are free to choose, provided we preserve the total area. One possibility is $\theta_{1F} \in [0, 4\pi]$, $\theta_{2F} \in [0, 2\pi]$.

This choice preserves the area of the unit cell. Although the expression for Q contains $\theta_{1F}/2$ and $\theta_{2F}/2$ it is still periodic in θ's. The elementary translation periods, however, are directed at an angle to the θ axes.

A.5 SPM integration for the two-point function

In this section we obtain the explicit expressions for the effective Lagrangian and the invariant two-point function in Eqs. (2.66), (2.67) and perform the integration over the saddle-point manifold. Using the parametrizations of section 2.3 we obtain,

$$\begin{aligned}
\mathrm{trg}(Q\Lambda)^2 &= \mathrm{trg}(\hat{\Upsilon}_u^{-1} Q_D \hat{\Upsilon}_u \Lambda)^2 && \text{(A.24)}\\
\mathrm{trg}(Q\hat{E}\Sigma_z) &= -2i[E_1 \mathrm{trg}(u_1^{-1} u \cosh(\hat{\Omega}) u^{-1} u_1 \cosh(\hat{\theta}_1)) \\
&\quad + E_2 \mathrm{trg}(u_2^{-1} v \cosh(\hat{\Omega}) v^{-1} u_2 \cosh(\hat{\theta}_2))] && \text{(A.25)}\\
\mathrm{trg}(Q\Sigma_z k(1+\Lambda)) &= -4i\mathrm{trg}(u_1^{-1} u \cosh(\hat{\Omega}) u^{-1} u_1 [\cosh(\hat{\theta}_1/2) u_1^{-1} k u_1 \cosh(\hat{\theta}_1/2)] \\
&\quad + \sinh(\hat{\theta}_1/2) u_1^{-1} k u_1 \sinh(\hat{\theta}_1/2)]) && \text{(A.26)}\\
\mathrm{trg}(Q\Sigma_z k(1-\Lambda)) &= -4i\mathrm{trg}(u_2^{-1} v \cosh(\hat{\Omega}) v^{-1} u_2 [\cosh(\hat{\theta}_2/2) u_2^{-1} k u_2 \cosh(\hat{\theta}_2/2) \\
&\quad + \sinh(\hat{\theta}_2/2) u_2^{-1} k u_2 \sinh(\hat{\theta}_2/2)]) && \text{(A.27)}
\end{aligned}$$

which after expansion gives

$$\mathrm{trg}(Q\Lambda)^2 = -8(s_1^2 - s_2^2), \qquad \text{(A.28)}$$

$$\begin{aligned}
\mathrm{trg}(Q\hat{E}\Sigma_z) &= -2iE_1[s_1 \cosh\theta_{1B} - s_2 \cos\theta_{1F} + (s_1 - s_2)\delta_1(\xi_1\xi_3 - \alpha^*\alpha - \xi_1\alpha^* - \alpha\xi_3)] \\
&\quad -2iE_2[s_1 \cosh\theta_{2B} - s_2 \cos\theta_{2F} + (s_1 - s_2)\delta_2(\xi_2\xi_4 + \beta^*\beta - i\xi_2\beta^* - i\beta\xi_4)] \qquad \text{(A.29)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{trg}(Q\Sigma_z k(1+\Lambda)) &= -4i[s_1 \cosh\theta_{1B} + s_2 \cos\theta_{1F} + \xi_1\xi_3[(s_1 + s_2)\delta_1 + 2(s_1 - s_2)\varpi_1] \\
&\quad -\alpha^*\alpha(s_1 - s_2)(\cosh\theta_{1B} + \cos\theta_{1F}) - 2\alpha^*\alpha\xi_1\xi_3(s_1 - s_2)\delta_1 \\
&\quad -(\alpha\xi_3 + \xi_1\alpha^*)(s_1 - s_2)\varpi_1 \qquad \text{(A.30)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{trg}(Q\Sigma_z k(1-\Lambda)) &= -4i[s_1 \cosh\theta_{2B} + s_2 \cos\theta_{2F} + \xi_2\xi_4[(s_1 + s_2)\delta_2 + 2(s_1 - s_2)\varpi_2] \\
&\quad +\beta^*\beta(s_1 - s_2)(\cosh\theta_{2B} + \cos\theta_{2F}) + 2\beta^*\beta\xi_2\xi_4(s_1 - s_2)\delta_2 \\
&\quad -(i\beta\xi_3 + i\xi_1\beta^*)(s_1 - s_2)\varpi_2, \qquad \text{(A.31)}
\end{aligned}$$

here we have used the notation $s_1 = \cosh\Omega_B$, $s_2 = \cos\Omega_F$, $\delta_1 = \cosh\theta_{1B} - \cos\theta_{1F}$, $\delta_2 = \cosh\theta_{2B} - \cos\theta_{2F}$, $\varpi_1 = \cosh\theta_{1B} + \cos\theta_{1F} - 2\cosh([\theta_{1B} + i\theta_{1F}]/2)$ and $\varpi_2 = \cosh\theta_{2B} + \cos\theta_{2F} - 2\cosh([\theta_{2B} + i\theta_{2F}]/2)$.

We note that the integrand is a product of a function of E_1 and a function of E_2. We proceed by expanding the effective Lagrangian in the Grassmann variables. Since $(\xi_1\xi_3 - \alpha^*\alpha - \xi_1\alpha^* - \alpha\xi_3)^2 = (\xi_2\xi_4 + \beta^*\beta - i\xi_2\beta^* - i\beta\xi_4)^2 = 0$ it is necessary to expand the exponential in each factor of the integrand only to first order in Grassmann variables. We classify all terms in the integrand according to the number of Grassmann variables they contain as $F^k_{i,j}$, where i is the number of Grassmann variables from $\hat{w}_1$ (ξ_1, ξ_3), j the number of ξ_2 or ξ_4 and k is the number of Grassmann variables from the "unitary" blocks. Applying "Zirnbauer's theorem" [31] it is clear that the only nonvanishing contributions to the integrals come from $F^0_{0,0}$, $F^0_{2,0}$, $F^0_{0,2}$, $F^0_{2,2}$, $F^4_{0,0}$, $F^4_{2,0}$, $F^4_{0,2}$ and $F^4_{2,2}$. The calculations are further simplified due to the factorization of the integral into a product of variables in blocks RR and AA. Using the invariant measure of Eq. (2.78) we arrive at the following expression for the two-point function

$$\begin{aligned} W(E_1, E_2, X) &= \left(\frac{\pi}{2\Delta}\right)^2 \int_1^\infty ds_1 \int_0^1 ds_2 \frac{(4\pi)^2 s_1 s_2}{(s_1+s_2)^2(s_1-s_2)^2} \\ &\times \exp\left[-\left(\frac{\pi X}{2N\Delta}\right)^2 \mathrm{tr}(\Theta^2)(s_1^2 - s_2^2)\right] I_1 I_2, \end{aligned} \tag{A.32}$$

where

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} d\theta_{1B} \int_{-\pi}^{\pi} d\theta_{1F} \exp\left(\frac{i\pi E_1}{\Delta}[s_1\cosh\theta_{1B} - s_2\cos\theta_{1F}]\right) \frac{(\cosh\theta_{1B}\cos\theta_{1F} - 1)}{2(\cosh\theta_{1B} - \cos\theta_{1F})^2} \\ &\times \Big\{1 + \xi_1\xi_3\delta_1 - \alpha\alpha^*(s_1 - s_2)\Big[1 \\ &+ \xi_1\xi_3\delta_1\Big(1 + \frac{i\pi E_1}{\Delta}[s_1\cosh\theta_{1B} - s_2\cos\theta_{1F}]\Big)\Big]\Big\} d\xi_1 d\xi_3 d\alpha d\alpha^*, \end{aligned} \tag{A.33}$$

$$\begin{aligned} I_2 &= \int_{-\infty}^{+\infty} d\theta_{2B} \int_{-\pi}^{\pi} d\theta_{2F} \exp\left(\frac{i\pi E_2}{\Delta}[s_1\cosh\theta_{2B} - s_2\cos\theta_{2F}]\right) \frac{(\cosh\theta_{2B}\cos\theta_{2F} - 1)}{2(\cosh\theta_{2B} - \cos\theta_{2F})^2} \\ &\times \Big\{1 + \xi_2\xi_4\delta_2 + \beta\beta^*(s_1 - s_2)\Big[1 \\ &+ \xi_2\xi_4\delta_2\Big(1 + \frac{i\pi E_2}{\Delta}[s_1\cosh\theta_{2B} - s_2\cos\theta_{2F}]\Big)\Big]\Big\} d\xi_2 d\xi_4 d\beta d\beta^*, \end{aligned} \tag{A.34}$$

Just as with the one-point function of DOS we require only the part even in both E_1 and E_2. We use Eq. (A.32) to obtain the expression for $K(E_1, E_2, X)$. In these expressions terms not containing the "unitary" Grassmann variables should be taken at $s_1 = s_2 = 1$, and from Eq. (A.12) they can be recognized as the disconnected part of the DOS correlator. The terms containing $\alpha\alpha^*\beta\beta^*$ generate the connected part.

Performing the Grassmann integrations and deforming the integration contours for θ_B's as in appendix A.2 we obtain the following expression for the connected part of DOS correlation function

$$\begin{aligned} K(E_1, E_2, X) &= \frac{-1}{\Delta^2}\int_1^\infty ds_1 \int_0^1 ds_2 \frac{s_1 s_2 L(E_1, s_1, s_2) L(E_2, s_1, s_2)}{(s_1 + s_2)^2} \\ &\times \exp\left[-\frac{\pi^2 X^2 \mathrm{tr}(\Theta^2)}{4N^2\Delta^2}(s_1^2 - s_2^2)\right], \end{aligned} \tag{A.35}$$

where

$$\begin{aligned} L(E_1, s_1, s_2) &= \cos[\frac{\pi E_1}{\Delta}(s_1 - s_2)] + \frac{1}{4\pi}\int_{-\pi}^{\pi} d\theta_{1B} \int_{-\pi}^{\pi} d\theta_{1F} e^{[\frac{i\pi E_1}{\Delta}(s_1 \cos\theta_{1B} - s_2 \cos\theta_{1F})]} \\ &\times \frac{\cos\theta_{1B}\cos\theta_{1F} - 1}{\cos\theta_{1B} - \cos\theta_{1F}}\left[1 + \frac{i\pi E_1}{\Delta}(s_1\cos\theta_{1B} - s_2\cos\theta_{1F})\right]. \end{aligned} \tag{A.36}$$

The integrand in the last equation factorizes into a product of two terms, one depending only on E_1 while the other depends only on E_2. If we differentiate $L(E_1, s_1, s_2)$ in Eq. (A.36) with respect to $\frac{\Delta}{\pi E_1}(\frac{\partial}{\partial s_1} + \frac{\partial}{\partial s_2})$ (and analogously for $L(E_2, s_1, s_2)$) we remove the denominators which come from the measure. Then the integrals for the derivative of Eq. (A.36) can be taken and give

$$\begin{aligned} &-i\pi\left(2J_0(\pi E_1 s_1/\Delta)J_0(\pi E_1 s_2/\Delta) - \tfrac{\pi E_1(s_1+s_2)}{\Delta}\right. \\ &\times[J_0(\pi E_1 s_1/\Delta)J_1(\pi E_1 s_2/\Delta) + J_1(\pi E_1 s_1/\Delta)J_0(\pi E_1 s_1/\Delta)]) = \\ &-i\pi\tfrac{\Delta}{\pi E_1}(\tfrac{\partial}{\partial s_1} + \tfrac{\partial}{\partial s_2})\left(\tfrac{\pi E_1(s_1+s_2)}{\Delta}J_0(\pi E_1 s_1/\Delta)J_0(\pi E_1 s_2/\Delta)\right) \end{aligned} \tag{A.37}$$

together with an analogous expression for $L(E_2, s_1, s_2)$. Since this expression is a derivative of $\frac{\pi E_1(s_1+s_2)}{\Delta}J_0(\pi E_1 s_1/\Delta)J_0(\pi E_1 s_2/\Delta)$ with respect to $\frac{\pi E_1(s_1+s_2)}{\Delta}$ we know $L(E_1, s_1, s_2)$ up to an additive function of $(\frac{\pi E_1(s_1-s_2)}{\Delta})^2$ (the integrand of Eq. (A.36)

is obviously a symmetric function of s_1 and s_2). Let us proceed by assuming that this function vanishes and use the consistency of the result as justification. Then we obtain

$$K(E_1,E_2,X)_c = \left(\frac{\pi}{\Delta}\right)^4 \int_1^\infty ds_1 \int_0^1 ds_2 s_1 s_2 E_1 E_2 \exp\left[-\left(\frac{\pi X}{2N\Delta}\right)^2 \mathrm{tr}(\Theta^2)(s_1^2-s_2^2)\right] \times J_0(\pi E_1 s_1/\Delta) J_0(\pi E_1 s_2/\Delta) J_0(\pi E_2 s_1/\Delta) J_0(\pi E_2 s_2/\Delta). \quad \text{(A.38)}$$

which coincides with the result found by Macédo [24].

A.6 Saddle-Point Approximation: Identifying the Massive Modes

In this section we examine the fluctuations around the solution (6.22) of the saddle-point equations Eq. (6.21) to identify the massive modes in the effective theory of Eq. (6.20). To be specific we consider the orthogonal case studied in the main text, but the general conclusions of this appendix hold for all ensembles.

To identify the massive modes it is convenient to work in the eigenbasis $\{\varphi_n\}$ of the quantum Hamiltonian, where Eq. (6.22) takes the form

$$[Q_0]_{\mu\nu} = \delta_{\mu\nu} Q_\nu = \delta_{\mu\nu} \left(\frac{\epsilon_0 - \epsilon_\mu}{2N} + \left[1 - \left(\frac{\epsilon_0 - \epsilon_\mu}{2N}\right)^2\right]^{1/2} \Lambda \right). \quad \text{(A.39)}$$

Massive modes appear as fluctuations δQ that commute with Q_0 in superspace. Expanding the action in Eq. (6.20) around the saddle-point to second order in δQ and neglecting s (and using the fact that Q_0 is diagonal in the Hilbert space indices) we obtain

$$\delta S_2 = -\frac{1}{2} \sum_{\mu\nu} \mathrm{Str}\ \left(Q_\mu \delta Q_{\mu\nu} Q_\nu \delta Q_{\nu\mu} + \delta Q_{\mu\nu} \delta Q_{\nu\mu}\right). \quad \text{(A.40)}$$

The mass of these modes is not apparently large but is of order one. However, if we

consider their contribution to a local observable such as the local DoS $\nu(\mathbf{q})$ we find

$$\begin{aligned}\langle\delta\nu(\mathbf{q}_1)\delta\nu(\mathbf{q}_2)\rangle_{\text{massive}} &= -\frac{1}{(4\pi N)^2}\sum_{\mu\nu}\varphi_\mu(\mathbf{q}_1)\varphi_\nu^*(\mathbf{q}_1)\varphi_\mu^*(\mathbf{q}_2)\varphi_\nu(\mathbf{q}_2) \\ &\times \langle \mathrm{STr}(\Lambda k\delta Q_{\mu\nu})\ \mathrm{STr}(\Lambda k\delta Q_{\nu\mu})\rangle_Q, \end{aligned} \tag{A.41}$$

where $\langle\cdots\rangle_Q$ denotes the average over supermatrices Q with respect to the action in Eq. (6.19). The contribution from both the diagonal ($\mu = \nu$) and off-diagonal terms is small: The former is of order N^{-1}, while the latter involves N^2 terms each of which is of order N^{-2}. However, since the the phases of wave functions at different point are almost uncorrelated so the off-diagonal terms arise with random phases. This implies a contribution of the off-diagonal terms which is also of order N^{-1}. This consideration enables us to neglect the massive modes.

The integration measure in Eq. (6.19) is invariant under the group of transformations $\hat{Q} \to \hat{U}^{-1}\hat{Q}\hat{U}$, where $\hat{U}$ is an operator satisfying Eq. (6.23) with indices both in the Hilbert space $U_{\mu\nu}$ and in superspace. The action Eq. (6.20) is also invariant under such transformations, provided that $\hat{U}$ commutes with $\hat{H}$. We will denote such transformations by $\hat{U}_0$ This symmetry leads to the existence of a degenerate manifold of saddle-point solutions (at $s = 0$). All matrices of the form

$$\hat{Q} = \hat{U}_0^{-1}\hat{Q}_0\hat{U}_0, \tag{A.42}$$

where $[\hat{U}_0, \hat{H}] = 0$ satisfy Eq. (6.21). In the basis of the eigenstates of the Hamiltonian such matrices are of the form $U_{0,\mu\nu} = \delta_{\mu\nu}U_{0,\mu}$ with $U_{0,\mu} \in UOSP(2,2/4)$[21]. We assume the absence of degeneracies due to non-integrability. All such matrices generate zero-modes.

It is shown below that the integration over the massive modes strongly favors the ground state configurations of Q which correspond to identical Q_μ's. This happens because the grond state in which Q_μ's are different breaks supersymmetry of the action for the massive modes. This leads to a rapid decay (as a function of inhomogeneity of Q_μ) of the superdeterminant which arises from the integration over the massive modes.

Therefore the integration over the massive modes gives a nonvanishing contribution into the effective action which depends on Q_μ's. This contribution can be interpreted as an effective interaction between Q_μ's which favors configurations with identical Q_μ's. Hence, it can be thought of as "ferromagnetic" interaction of "spins" Q_μ which reside on the nonlinear manifold $UOSP(2,2/4)/[UOSP(2/2)\otimes UOSP(2/2)]$. This interaction is long range (all "spins" within the band interact with approximately equal strength) and therefore in the thermodynamic limit $N \to \infty$ leads to a ferromagnetic ground state. The fluctuations of "spins" from the the ground state configurations are small as $1/N$ and can be neglected.

To see how the supersymmetry breaking for the massive modes arises let us consider one term in the sum (A.40) corresponding to particular μ and ν. The matrix Q_μ has the same symmetries as the Q-matrix in Efetov's non-linear σ-model and can be parametrized as

$$\begin{aligned} Q_\mu &= \begin{pmatrix} u_\mu & 0 \\ 0 & v_\nu \end{pmatrix} Q_E(\hat{\Theta}_\mu) \begin{pmatrix} \bar{u}_\mu & 0 \\ 0 & \bar{v}_\mu \end{pmatrix}, \\ Q_E(\hat{\Theta}_\mu) &= \begin{pmatrix} \cos(\hat{\Theta}_\mu) & i\sin(\hat{\Theta}_\mu) \\ -i\sin(\hat{\Theta}_\mu)) & -\cos(\hat{\Theta}_\mu) \end{pmatrix}, \hat{\Theta}_\mu = \mathrm{diag}(\Theta, \Theta, i\Theta^+, i\Theta^-). \end{aligned} \quad \text{(A.43)}$$

Here we deviate from Efetov's original parametrization by introducing the angles Θ^+ and Θ^- which can be expressed through the angles appearing in Ref. [20], Θ_1 and Θ_2 as $\Theta^+ = \Theta_1 + \Theta_2$ and $\Theta^- = \Theta_1 - \Theta_2$. The particular form of the matrices u and v is not important for what follows and will be left unspecified.

If the angles $\hat{\Theta}_\mu$ and $\hat{\Theta}_\nu$ coincide then the massive modes line up with $\hat{\Theta}_\nu$. In other words we can make a global rotation to bring $\hat{\Theta}_\nu$ to zero, and in this coordinate frame the massive fluctuations correspond to $\delta Q^{RR}_{\mu\nu}$ and $\delta Q^{AA}_{\mu\nu}$. If the angles $\hat{\Theta}_\mu$ and hat $\hat{\Theta}_\nu$ differ by a small amount, we can go to the "center of mass" coordinate where $\hat{\Theta}_\mu = -\hat{\Theta}_\nu$ In this frame the massive modes will still correspond to $\delta Q^{RR}_{\mu\nu}$ and $\delta Q^{AA}_{\mu\nu}$.

The contribution of $\delta Q_{\mu\nu}^{AA}$ to the effective action is

$$\mathrm{STr}\left(Q_E(\hat{\Theta}_\mu)\bar{u}_\mu\delta Q_{\mu\nu}^{AA}u_\nu Q_E(\hat{\Theta}_\nu)\bar{u}_\nu\delta Q_{\nu\mu}^{AA}u_\mu+\delta Q_{\mu\nu}^{AA}\delta Q_{\nu\mu}^{AA}\right) \tag{A.44}$$

Instead integration variables $\delta Q_{\mu\nu}^{AA}$ and $\delta Q_{\nu\mu}^{AA}$ it is more convenient to use $\delta\tilde{Q}_{\mu\nu}^{AA}=\bar{u}_\mu\delta Q_{\mu\nu}^{AA}u_\nu$ and $\delta\tilde{Q}_{\nu\mu}^{AA}=\bar{u}_\nu\delta Q_{\nu\mu}^{AA}u_\mu$. Since the super Jacobian of such transformation is equal to unity,

$$\mathrm{Sdet}\left(\frac{\partial(\delta Q_{\mu\nu}^{AA},\delta Q_{\nu\mu}^{AA})}{\partial(\delta\tilde{Q}_{\mu\nu}^{AA},\delta\tilde{Q}_{\nu\mu}^{AA})}\right)=1. \tag{A.45}$$

the invariant measure is preserved.

With the parametrization involving ordinary variables a_i, b_i, and Grassmann variables σ_i, σ_i^*,

$$\delta\tilde{Q}_{\mu\nu}^{AA}=\begin{pmatrix} a_1 & a_2 & i\sigma_1 & i\sigma_2\\ -a_2^* & a_1^* & -i\sigma_2^* & -i\sigma_1^*\\ \sigma_3^* & \sigma_4 & ib_1 & ib_2\\ \sigma_4^* & \sigma_3 & ib_2^* & ib_1^*\end{pmatrix},\delta\tilde{Q}_{\nu\mu}^{AA}=\begin{pmatrix} a_1^* & -a_2 & -\sigma_3 & -\sigma_4\\ a_2^* & a_1 & \sigma_4^* & \sigma_3^*\\ -i\sigma_1^* & -i\sigma_2 & ib_1^* & ib_2\\ -i\sigma_2^* & -i\sigma_1 & ib_2^* & ib_1\end{pmatrix}, \tag{A.46}$$

which obey the symmetry relations

$$\delta\tilde{Q}_{\mu\nu}^{AA}=C^T(\delta\tilde{Q}_{\nu\mu}^{AA})^TC,\qquad \delta\tilde{Q}_{\mu\nu}^{AA}=k(\delta\tilde{Q}_{\nu\mu}^{AA})^\dagger \tag{A.47}$$

integration over massive modes can be performed and yields

$$I_{\mu\nu}=\frac{[2+\cos\Theta(\cosh\Theta^++\cosh\Theta^-)]^4}{(2+\cosh^2\Theta^++\cosh^2\Theta^-)(2+2\cosh\Theta^+\cosh\Theta^-)(2+2\cos^2\Theta)^2}. \tag{A.48}$$

For small $\hat{\Theta}_\mu$, writing $\cos\Theta=1-\alpha$, $\cosh\Theta^+=1+\beta^+$, and $\cosh\Theta^-=1+\beta^-$, $I_{\mu\nu}$ can be expanded to second order,

$$I_{\mu\nu}\approx 1-(2\alpha+\beta^++\beta^-)^2/8\approx\exp(-(2\alpha+\beta^++\beta^-)^2/8). \tag{A.49}$$

If all $\hat{\Theta}_\mu$ are small, then we obtain a model equivalent to spins with infinite range

interactions. In the thermodynamic limit of such a model the mean field approximation becomes exact. The fluctuations of α, β^+ and β^- become small as $1/N$ and can be neglected. This forces us to consider the matrices $\hat{Q}_0$ which are of the form of Eq. (A.39). Then the relevant (massless) fluctuations of the Q-matrix are those that anticommute with Λ in superspace.

A.7 The Inverted Harmonic Oscillator

In this appendix problems associated with the spectral decomposition of $\hat{\mathcal{L}}$ are illustrated. Consider the problem of the inverted harmonic oscillator described by the Hamiltonian $\mathcal{H} = (p^2 - q^2)/2$. Although this system is not chaotic, it displays an exponential dependence on the initial conditions. Expressed in terms of the canonical variables $\xi_0 = (p+q)/2$ and $\xi_1 = q - p$, the equations of motion

$$\frac{\partial \xi_j}{\partial t} = (-1)^j \xi_j \qquad\qquad j = 0, 1 \tag{A.50}$$

generate the solutions $\xi_j(t) = \xi_j(0) \exp\{(-1)^j t\}$. Thus the evolution of the classical distribution in phase space involves stretching along the unstable manifold $p = q$ and contraction along the stable manifold $p = -q$. The evolution of a density in phase space satisfies the equation

$$\begin{aligned} \rho(\eta_0, \eta_1; t) &= \int d\xi_0 d\xi_1 \delta(\eta_0 - \xi_0 e^t) \delta(\eta_1 - \xi_1 e^{-t}) \rho_0(\xi_0, \xi_1) \\ &= \int d\xi_1 e^{-t} \rho_0(\eta_0 e^{-t}, \xi_1) \delta(\eta_1 - \xi_1 e^{-t}), \end{aligned} \tag{A.51}$$

where $\rho_0(\xi_0, \xi_1)$ is the initial particle distribution which evolves into $\rho(\eta_0, \eta_1; t)$ after time t. In order to identify the spectral decomposition of $\hat{\mathcal{L}}$ one would like to express $\rho(\eta_0, \eta_1; t)$ in the form

$$\rho(\eta_0, \eta_1; t) = \sum_{\vec{n}} c_{\vec{n}} e^{\gamma_{\vec{n}} t} \phi_{\vec{n}}(\eta_0, \eta_1) \tag{A.52}$$

where $\gamma_{\vec{n}}$ and $\phi_{\vec{n}}$ are, respectively the eigenvalues and the (right) eigenfunctions of $\hat{\mathcal{L}}$, namely

$$\hat{\mathcal{L}}\phi_{\vec{n}}(\eta_0,\eta_1) = \gamma_{\vec{n}}\phi_{\vec{n}}(\eta_0,\eta_1) \tag{A.53}$$

Expanding Eq. (A.51) in powers of $\eta_0 e^{-t}$ and $\xi_1 e^{-t}$ one obtains the expansion

$$\rho(\eta_0,\eta_1;t) = \sum_{n_0=0}^{\infty}\sum_{n_1=0}^{\infty} e^{-(n_0+n_1+1)t} C_{n_0,n_1}\frac{\eta_0^{n_0}}{n_0!}(-1)^{n_1}\delta^{(n_1)}(\eta_1), \tag{A.54}$$

where, using the notation $\delta^{(n)}(\xi) = d^n\delta(\xi)/d\xi^n$, the expansion coefficients C_{n_0,n_1} are given by

$$C_{n_0,n_1} = \int d\xi_0 d\xi_1 \rho_0(\xi_0,\xi_1)(-1)^{n_0}\delta^{(n_0)}(\xi_0)\xi_1^{n_1}/n_1!. \tag{A.55}$$

¿From this we can draw the following conclusions: (i) the spectrum of $\hat{\mathcal{L}}$,

$$\gamma_{n_0,n_1} = -(1+n_0+n_1), \tag{A.56}$$

is discrete and negative; (ii) its right eigenfunctions,

$$\phi_{n_0,n_1}(\eta_0,\eta_1) = \frac{\eta_0^{n_0}}{n_0!}(-1)^{n_1}\delta^{(n_1)}(\eta_1) \tag{A.57}$$

are analytic monomials along the unstable manifold and derivatives of Dirac distributions along the stable manifold; (iii) the left eigenfunctions are

$$\tilde{\phi}_{n_0,n_1}(\eta_0,\eta_1) = \frac{\eta_1^{n_1}}{n_1!}(-1)^{n_0}\delta^{(n_0)}(\eta_0), \tag{A.58}$$

Notice that the largest eigenvalue (corresponding to $n_0 = n_1 = 0$) is finite and negative. This reflects the scattering nature of the system in which the particle eventually escapes to infinity. Another important point is that the left (and right) eigenfunctions are not orthogonal. Nevertheless they form a complete biorthonormal set,

$$\langle\tilde{\phi}_{n_0,n_1}|\phi_{m_0,m_1}\rangle = \delta_{n_0,m_0}\delta_{n_1,m_1} \tag{A.59}$$

$$\sum_{n_0,n_1} \phi_{n_0,n_1}(\xi)\tilde{\phi}^*_{n_0,n_1}(\xi') = \delta(\xi - \xi'). \tag{A.60}$$

It is instructive to consider the solution for the $\hat{\mathcal{L}}$ eigenmodes in the presence of noise. In this case, the equations of motion (A.50) are modified by the addition of a noise term $W_j(t)$ at the right hand side. We shall assume that $W_j(t)$ describes δ-correlated white Gaussian noise with zero mean, $\langle W_j(t)\rangle = 0$ and $\langle W_i(t)W_j(t')\rangle = 2D\delta_{ij}\delta(t-t')$. ¿From these properties one can show that the evolution of the averaged classical probability density, $\rho(\xi_0, \xi_1; t)$ satisfies the equation

$$\frac{\partial \rho}{\partial t} = \sum_j \left[D\frac{\partial^2 \rho}{\partial \xi_j^2} - (-1)^j \frac{\partial(\xi_j \rho)}{\partial \xi_j} \right]. \tag{A.61}$$

In Ref. [?] it is shown that the equation above can be mapped onto the Shcrödinger equation of harmonic oscillator. The resulting spectrum is identical to that obtained without the noise (A.56), and therefore is independent of the noise. However, it is a special case, and in generic problems the spectrum does depend on the noise via D_W. The results for the right eigenfunctions are:

$$\phi^D_{n_0,n_1}(\eta_0, \eta_1) = N_{n_0,n_1} H_{n_0}(\alpha\eta_0) H_{n_1}(\alpha\eta_1) \exp\left(-\frac{\eta_1^2}{2D}\right) \tag{A.62}$$

where N_{n_0,n_1} is a normalization constant, $H_n(\eta)$ is the Hermite polynomial of n-th order, and $\alpha = 1/\sqrt{2D}$. The left eigenfunctions can be obtained from the formula above by exchanging the roles of η_0 and η_1. Notice that the noise smoothes out the singular behavior of (A.58) and (A.60) and replace it by oscillations along the stable manifold. One can easily show that in the limit $D \to 0$ (or $\alpha \to \infty$), $\phi^D_{n_0,n_1} \to \phi_{n_0,n_1}$.

Finally, in contrast to the example above, in generic chaotic systems the stable and unstable manifolds of some periodic orbits intersect at an infinite number of points thereby forming the extremely complicated pattern known as the homoclinic tangle. These manifold are dense in phase space, thus, the eigenmodes of ergodic chaotic systems are, in general, much more complicated than those considered here. An exception is the zeroth eigenmode. It is simply the invariant density corresponding

to a uniform distribution on the energy shell.

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