MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044 Statistical Physics I — Spring Term 2003

Problem Set #6

Problem 1: Classical Magnetic Moments

Consider a system made up of N independent classical magnitic dipole moments located on fixed lattice sites. Each moment $\vec{\mu}_i$ has the same length μ, but is free to rotate in 3 dimensions. When a magnetic field of strength H is applied in the positive z direction, the energy of the i^{th} moment is given by $\epsilon_i = -m_i H$ where m_i is the z component of $\vec{\mu}_i$ (that is, $\vec{\mu}_i \cdot \hat{z} = m_i$).

The magentization M and the total energy E are given by

$$M = \sum_{i=1}^{N} m_i \qquad\qquad E = \sum_{i=1}^{N} \epsilon_i = -MH$$

a) What are the physically allowed ranges of values associated with m_i, M, and E?

b) How many microscopic variables are necessary to completely specify the state of the system?

In a certain limiting case, the accessible volume in phase space for the microcanonical ensemble is given by

$$\Omega \approx (2\mu)^N \exp[-\frac{M^2}{\frac{2}{3}N\mu^2}].$$

c) Use the microcanonical ensemble to find the equation of state, M as a function of H and T.

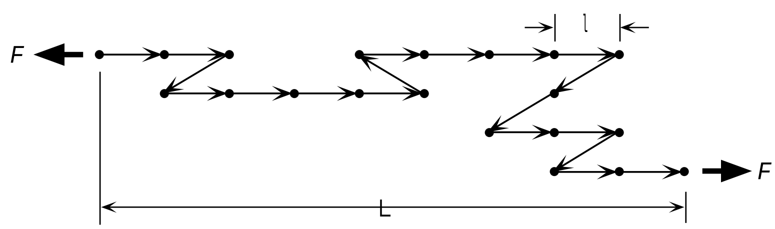
d) Is there some condition under which the solution to c) is unphysical for the system under consideraton? Explain your answer. For what values of T is the expression for Ω a good approximation?

e) The probability density $p(M)$ for the z component of a single magnetic moment can be written as $p(m) = \Omega'/\Omega$ where Ω is given above. What is Ω'?

f) Find $p(m)$. [Note: For the limit which applies here, an expression for $p(m)$ including powers of m no higher than the first is adequate.] Sketch $p(m)$ and check its normalization.

g) Use $p(m)$ to computed $< m >$. Compare the result with that which one would expect.

Problem 2: A Strange Chain



A one dimensional chain is made up of N identical elements, each of length l. The angle between successive elements can be either 0^0 or 180^0, but there is no difference in internal energy between these two possibilities. For the sake of counting, one can think of each element as either pointing to the right (+) or to the left (-). Then one has

$$\begin{aligned} N &= n_+ + n_- \\ L &= l(n_+ - n_-) = l(2n_+ - N) \end{aligned}$$

a) Use the microcanonical ensemble to find the entropy as a function of N and n_+, $S(N, n_+)$.

b) Find an expression for the tension in the chain as a function of T, N, and n_+, $\mathcal{F}(T, N, n_+)$. Notice the strange fact that there is tension in the chain even though there is no energy required to reorient two neighboring elements! The "restoring force" in this problem is generated by entropy considerations alone. This is not simply an academic oddity, however. This system is used as a model for elastic polymers such as rubber.

c) Rearrange the expression from b) to give the length as a function of N, T, and $\mathcal{F}$.

d) Use the result for the high temperature behavior from c) to find an expression for the thermal expansion coefficient $\alpha \equiv L^{-1}(\partial L/\partial T)_{\mathcal{F}}$. Note the sign. Find a stout rubber band. Hang a weight from it so that its length is extended by about a factor of two. Now heat the rubber band (a hair drier works well here) and see if the weight goes up or down.

Problem 3: Classical Harmonic Oscillators

Consider a collection of N indentical harmonic oscillators with negligable (but non-zero) interactions. In a microcanonical ensemble with energy E, the system is on a surface in phase space given by

$$\sum_{i=1}^{N}\left(\frac{p_i^2}{2m}+\frac{m\omega^2 q_i^2}{2}\right)=E.$$

a) Find the volume of phase space enclosed, $\Phi(E)$, as follows. Transform to new variables

$$\begin{aligned} x_i &= \frac{1}{\sqrt{2m}}\,p_i && 1 \le i \le N \\ x_i &= \sqrt{\frac{m\omega^2}{2}}\,q_{i-N} && N+1 \le i \le 2N \end{aligned}$$

Note that in terms of these variables the constant energy surface is a $2N$ dimensional sphere. Find its volume. Find the corresponding volume in p-q space.

b) Find the entropy S in terms of N and E.

c) Find T and express E in terms of N and T.

d) Find the joint probability density for the position coordinate q_i and the momentum coordinate p_i of *one* of the oscillators. Sketch $p(p_i, q_i)$.

Problem 4: Quantum Harmonic Oscillators

Consider a system of N almost independent harmonic oscillators whose energy in a microcanonical ensemble is given by

$$E=\frac{1}{2}\hbar\omega N+\hbar\omega M.$$

a) Find the number of ways, $\Omega(E)$, that this energy can be obtained. Note that

$$M = \sum_{i=1}^{N} n_i$$

where n_i is the occupation number $(0, 1, 2, \ldots)$ of a given harmonic oscillator. $\Omega(E)$ can be looked upon as the number of ways of putting M indistinguishable balls in N labelled boxes. It is also the number of ways of arranging $N - 1$ partitions and M indistinguishable balls along a line.

b) Find the entropy S in terms of N and M.

c) Find T and express E in terms of N and T.

d) Find the probability that a given oscillator is in its n^{th} energy eigenstate. Sketch $p(n)$.