Stiffness and Bending

Introduction

One very common problem that students have in 2.007 is not making their arm or structures stiff enough. This is a problem because the arms and structures usually need to move or support things. A lack of stiffness is very common cause of machine unreliability.

Remember from 2.001 that the following factors need to be known to calculate the stiffness of something.

**The Young's Modulus [E]:**
- This is a material property that measures the stress/strain.

**The Cross-Sectional Inertia [I]:**
- This is determined by the cross sectional geometry of the arm.

**The Loading Configuration:**
- This gives your equation to calculate the the deflection. Typical configurations and their equations are listed below.

### Young's Modulus

<table>
<thead>
<tr>
<th>Material</th>
<th>Young Modulus E (ksi)</th>
<th>Young Modulus E (GPa)</th>
<th>Shear Modulus G (ksi)</th>
<th>Shear Modulus G (GPa)</th>
<th>Poisson's Ratio v</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum (pure)</td>
<td>10,000</td>
<td>70</td>
<td>3,800</td>
<td>26</td>
<td>0.33</td>
</tr>
<tr>
<td>Aluminum alloys</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6061-T6</td>
<td>10,000</td>
<td>70</td>
<td>3,800</td>
<td>26</td>
<td>0.33</td>
</tr>
<tr>
<td>7075-T6</td>
<td>10,400</td>
<td>72</td>
<td>3,900</td>
<td>27</td>
<td>0.33</td>
</tr>
<tr>
<td>Steel</td>
<td>29,000</td>
<td>190-210</td>
<td>11,300</td>
<td>75-80</td>
<td>0.27-0.30</td>
</tr>
<tr>
<td>Delrin</td>
<td></td>
<td>3.1</td>
<td></td>
<td></td>
<td>0.35</td>
</tr>
</tbody>
</table>

### Moments of Inertia

<table>
<thead>
<tr>
<th>Cross Section</th>
<th>Inertia</th>
</tr>
</thead>
</table>
This is an approximation of a simple truss, ignoring the cross members. Both upper and lower members have the same area (A).

\[
I_x = \frac{b_o \cdot h_o^3 - b_i \cdot h_i^3}{12}
\]

\[
I_x = \frac{b \cdot h^3}{12}
\]

\[
I_x = \frac{\pi \cdot (d_o^4 - d_i^4)}{64}
\]

\[
I_x = \frac{\pi \cdot d^4}{64}
\]

\[
I_x = 2A\left(\frac{h}{2}\right)^2
\]
v = deflection in y direction

v' = dv/dx = slope of deflection curve

db = v(L) = deflection at right end of beam

Theta b = v'(L) = angle at right end of beam

\[
0 < X < A \quad A < X < L
\]
\[
v = \frac{Px^2}{6EI} (3a - x) \quad \delta_b = \frac{Pa^2}{6EI} (3L - a)
\]
\[
v' = \frac{Px}{2EI} (2a - x) \quad \theta_b = \frac{Pa}{2EI}
\]
\[
v = \frac{Mo x^2}{2EI} \quad \delta_b = \frac{Mo a}{2EI} (2L - a)
\]
\[
v' = \frac{Mo x}{EI} \quad \theta_b = \frac{Mo a}{EI}
\]
\[
v = \frac{Pb x}{6LEI} (L^2 - b^2 - x^2) \quad \theta_a = \frac{Pab(L + b)}{6LEI}
\]
\[
v' = \frac{Pb}{6LEI} (L^2 - b^2 - 3x^2) \quad \theta_b = \frac{Pab(L + a)}{6LEI}
\]

Substitute a for b and (L-x) for x, to calc v and v' for a < x < L.

**Tools for Evaluating Deflection**

There are some tools to help you calculation the amount of deflection in your structures. If you forget how to
recinert.m* and cirinert.m* calculate the cross sectional inertia for rectangular and circular cross sections respectively. All units must be consistent.

RECIERT(A,B,C,D) and CIRINERT(A,B)

»help recinert

RECIERT(A,B,C,D) calculates the cross sectional moment of interia for
a rectangular cross section
A = outer width, B = outer height
C = inner width, and D = inner height

To evaluate a filled rectangle set C and D = 0

```
»help cirinert
```

CIRINERT(A,B) calculates the cross sectional moment of inertia for
a pipe cross section
A = outer diameter, B = inner diameter

To evaluate a filled circle set B = 0

```
cantbeam.m* shows the deflection of a cantilevered beam loaded from 1 or more points.
```

CANTBEAM(P,a,L,E,I,incr)

```
»help cantbeam
```

[def] = CANTBEAM(P,a,L,E,I,incr) returns arrays with
the deflection [def]. [def] is also plotted.

P is the force applied to the beam in NEWTONS
a is the distance from the left end that the force is applied in METERS
L is the length of the beam in METERS
E is the young's modulus of the material in Pa
I is the cross sectional inertia in METER^4
incr is the number of increments to sample along the beam

Multiple forces can be entered in P, however, a must be the same length
to give a position for each force. P can be positive or negative.

all values of a may not be greater then L
E, I, a, incr, and L must be greater then 0

def is in meters (plotted in mm)

Below is a sample plot from cantbeam.m for the following data:

```
L = 0.10 m
I = 2.1333e-11 m^4  (4mm by 4mm shaft)
E = 200E+9 Pa  (steel)
incr = 100
P = [-400 600 -200] N
a = [0.03 0.06 0.09] m
```
On the deflection plot the red lines are at the point of force application and they are "pushing" the beam (e.i. a positive force will have its red line below the beam "pushing" up). The values of each force is displayed at the end of its force line.

The deflection is plotted in **mm** but the array returned for [def] is in **meters**!

**simpbeam.m**

shows the deflection of a simple beam supported

at either

end, loaded from 1 or more points.

**SIMPBEAM(P,a,L,E,I,incr)**

»help simpbeam

[def] = SIMPBEAM(P,a,L,E,I,incr) returns an array with the deflection [def]. [def] is also plotted.
Multiple forces can be entered in \( P \), however, \( a \) must be the same length to give a position for each force. \( P \) can be positive or negative.

- All values of \( a \) may not be greater then \( L \)
- \( E, I, a, \text{incr, and } L \) must be greater then 0
- def array is in meters (plotted in mm)

Below is a sample plot from simpbeam.m for the following data:

\[
\begin{align*}
L &= 0.10 \text{ m} \\
I &= 2.1333 \times 10^{-11} \text{ m}^4 \quad (4\text{mm by 4mm shaft}) \\
E &= 200 \times 10^9 \text{ Pa} \quad (\text{steel}) \\
\text{incr} &= 100 \\
P &= [-100 \ 100 \ -100] \text{ N} \\
a &= [0.02 \ 0.05 \ 0.08] \text{ m}
\end{align*}
\]

On the deflection plot the red lines are at the point of force application and they are "pushing" the beam (e.i. a positive force will have its red line below the beam "pushing" up). The values of each force is displayed at the end of its force line.

The plot may look like the beam is bending a lot, but compare at the scale on the \( x \) and \( y \) axis. In this example the maximum deflection is 0.075 mm for a 10 cm long beam!