Control of a Dual Inverted Pendulum System Using Linear-Quadratic and H-Infinity Methods

by

Lara C. Phillips

B.S., University of Missouri - Rolla (1991)

Submitted to the Department of Electrical Engineering and Computer Science in Partial Fulfillment of the Requirements for the Degree of Master of Science in Electrical Engineering at the Massachusetts Institute of Technology September 1994

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June 10, 1994

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Linear-Quadratic Regulator (LQR) and H-Infinity ($H_\infty$) with full state feedback design methods are applied to a cart with two inverted pendulums attached, named a dual inverted pendulum system. A linearized model of the system is obtained, and its open-loop properties are examined. The LQR and $H_\infty$ design methods are applied to the system with the objective of stabilizing it and approximately maintaining the open-loop bandwidth. The methods are applied to the dual inverted pendulum system for two different ratios of rod lengths to see the differences in the closed-loop responses. With the $H_\infty$ design method, frequency weighting the control is needed to shape the frequency response to meet the bandwidth constraint. The LQR and frequency weighted $H_\infty$ designs are compared in both the time and frequency domains. These comparisons indicate that the closed-loop systems have satisfactory transient responses (not always the case for the $H_\infty$ designs), stability robustness, and bandwidth, but amplify the disturbances in some frequency ranges.

Thesis Supervisor: Dr. Michael Athans
Title: Professor of Electrical Engineering
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Chapter 1

Introduction

In this chapter, the background and outline of this thesis are presented.

1.1 Background

The single inverted pendulum system is often used in classical controls classes to demonstrate the stabilization of an open-loop unstable system, and has also been used in demonstrating the design of intelligent integral control [1], the dynamic principle of parametric excitation, and the considerations and problems involved in actually constructing such a system [2], [3]. A more interesting and complicated problem in terms of controller design is the dual inverted pendulum system, in which there are two inverted pendulums on one cart. This system has an additional unstable pole and nonminimum phase zero in the open-loop plant than the single inverted pendulum system and is therefore more difficult to stabilize.

Systems with open-loop plants containing unstable poles and nonminimum phase zeros can be stabilized using constant gain feedback controllers, such as those that result from the Linear-Quadratic Regulator (LQR) design method, or using unstable
compensators that might result from Linear-Quadratic Gaussian (LQG) or $H_\infty$-Optimization design methodologies. However, even though the unstable, nonminimum phase systems are stabilized using the above design methodologies, the closed-loop systems generally exhibit poor performance characteristics such as sensitivity to parameter value variations and disturbance amplification in some frequency ranges [4].

Stabilization of the dual inverted pendulum system is demonstrated using the LQR and $H_\infty$ with full state feedback design methodologies. Each design is realized for two different ratios of rod lengths. System 1 represents the system with a ratio of 16:1 for the rod lengths, and System 2 represents the system with a ratio of 2:1. The controllers are designed such that the bandwidths of the open- and closed-loop systems are comparable. The closed-loop systems for each design methodology are compared in both the time domain and frequency domain, comparing such design characteristics as transient response, stability robustness, and disturbance rejection properties, and in the sensitivity of the closed-loop system to parameter variations.

1.2 Thesis Overview

Chapter 2 introduces the dual inverted pendulum system and the state space representation of its dynamics. Details of the development of the state space equations are included in Appendix A. The open-loop characteristics of the system, such as pole and zero
locations, controllability and observability, and frequency responses, are also investigated. The open-loop pole and zero locations for specific transfer functions are given, and the derivations of the locations are presented in Appendix B. The controllability and observability properties of the system are discussed, and the equations providing the results are included in Appendix C. Finally, specific transfer functions are examined in the frequency domain.

Chapter 3 presents the design methodology for the linear-quadratic regulator (LQR). The effect of the control weighting coefficient on the closed-loop poles is discussed, and a value is determined to satisfy the bandwidth constraint. Time and frequency responses of the closed-loop system are shown.

Chapter 4 presents the $H_{\infty}$ with full state feedback design methodology. It is found that the $H_{\infty}$ with full state feedback controller does not meet the bandwidth constraint. To reduce the bandwidth, the control is frequency weighted. The method of frequency weighting is presented, and a frequency weighting is determined for which the bandwidth constraint is better satisfied. Time and frequency responses of this closed-loop system are also shown.

In Chapter 5, the two design methodologies are compared for both System 1 and System 2. The designs are compared in the time domain in terms of the amount of the perturbations from equilibrium, the magnitude of the necessary control, and the transient times. Time responses for several initial conditions are shown in Appendix D. In the frequency domain, the stability robustness and disturbance rejection properties are compared. The
final design comparison involves a parameter variation, in which the length of the longer rod is varied until the closed-loop systems again become unstable.

Chapter 6 briefly summarizes the conclusions, and recommendations for future research are given. Complete appendices and selected references are included at the end of the thesis.
Chapter 2

The Dual Inverted Pendulum System

In this chapter, the state space representation of the dual inverted pendulum system is presented, and properties of the open-loop system are examined.

2.1 State Space Representation

The first step in analyzing any system is to obtain a representation of its dynamics. For many design purposes, a linear representation about an equilibrium point is sufficient to obtain a good controller. A linear model of the system can be obtained from the nonlinear dynamic equations which describe the system by establishing the steady-state equilibrium conditions, defining the perturbations about this equilibrium point, introducing the perturbations into the nonlinear equations, and then retaining only the linear terms. The state and output perturbation dynamics can then be expressed in the standard state space representation form

\[ \dot{x}(t) = A \delta x(t) + B \delta u(t) + L \delta d(t), \quad y(t) = C \delta x(t) + D \delta u(t) \] [5].

The derivation of the state space representation for the dual inverted pendulum system of Figure 2.1 is included in Appendix A. The assumptions for finding the nonlinear equations describing the system are: the cart is mounted on frictionless wheels; the rods are
attached to the cart with frictionless pivots; the gravitational field is uniform; the disturbances can be modeled as forces acting on the tips of the rods; and the control for the system is a translational force acting on the cart. We also assume that all of the states can be measured without sensor noise (i.e., we have full-state feedback), and that the position of the cart from some reference point is the state variable we want to control.

In Figure 2.1, $M$ is the mass of the cart, $2l_1$ and $2l_2$ are the lengths of the shorter and longer rods, respectively, and $m_1$ and $m_2$ are the masses per unit length of each rod.
In Appendix A, the state, control, and disturbance vectors are defined to be:

\[
\delta x(t) = \begin{bmatrix}
\delta z(t) \\
\delta \dot{z}(t) \\
\delta \theta_1(t) \\
\delta \dot{\theta}_1(t) \\
\delta \theta_2(t) \\
\delta \dot{\theta}_2(t)
\end{bmatrix}
\]

- perturbed cart position (m)
- perturbed cart velocity (m/s)
- perturbed angular position of the shorter rod (rad)
- perturbed angular velocity of the shorter rod (rad/s)
- perturbed angular position of the longer rod (rad)
- perturbed angular velocity of the longer rod (rad/s)

\[
\delta u(t) = \begin{bmatrix}
\delta f(t)
\end{bmatrix}
\]

perturbed control force (N)

\[
\delta d(t) = \begin{bmatrix}
\delta f_1(t) \\
\delta f_2(t)
\end{bmatrix}
\]

perturbed disturbance force on the shorter rod (N)
perturbed disturbance force on the longer rod (N)

and the state space representation for the dual inverted pendulum system is shown to be

\[
\delta \dot{x}(t) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -3M_1g/Z & 0 & -3M_2g/Z & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 3g(Z + 3M_1)/4Zl_1 & 0 & 9M_1g/4Zl_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 9M_2g/4Zl_2 & 0 & 3g(Z + 3M_2)/4Zl_2 & 0
\end{bmatrix}
\delta x(t) + \begin{bmatrix}
0 \\
4/Z \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\delta u(t) + \begin{bmatrix}
0 \\
-2 \\
0 \\
0 \\
0 \\
3/(2Zl_2)
\end{bmatrix}
\delta d(t)
\]

\[
\delta y(t) = [1 \ 0 \ 0 \ 0 \ 0 \ 0] \delta x(t)
\]

where \( M_1 = 2m_1l_1, \ M_2 = 2m_2l_2, \) and \( Z = 4M + M_1 + M_2. \) \hspace{1cm} (2.1)
In subsequent chapters, the $\delta$'s are omitted from the equations and the state space representation is written as

$$\dot{x}(t) = Ax(t) + Bu(t) + Ld(t)$$
$$y(t) = Cx(t)$$

(2.2)

2.2 Open-Loop Analysis

The open-loop analysis of the dual inverted pendulum system involves finding open-loop pole and zero locations in terms of the system parameters, investigating the controllability and observability of the system, and examining such transfer functions as the control to the angles of the rods, and the disturbances to the cart position.

2.2.1 Open-Loop Poles and Zeros

The open-loop pole and zero locations of the dual inverted pendulum system from the control to the cart position (from $u$ to $y$) can be found from the open-loop transfer function $T_{yo}(s) = C(sI - A)^{-1}B$. Because the system is SISO (single input, single output), the poles are the roots of the denominator of the transfer function and the zeros are the roots of the numerator. From the transfer function, presented in Appendix B, the open-loop pole locations are found to be

$$0, 0, \pm \sqrt{X}, \pm \sqrt{Y}, \text{ where}$$
\[
x, y = \frac{3}{8} \left( \frac{g}{Z} \right) \left[ \frac{Z + 3M_1}{l_1} + \frac{Z + 3M_2}{l_2} \right] \pm \frac{3}{8} \left( \frac{g}{Z} \right) \sqrt{ \left[ \frac{Z + 3M_1}{l_1} - \frac{Z + 3M_2}{l_2} \right]^2 + \frac{3}{4} \left( \frac{M_1 M_2}{l_1 l_2} \right) } \tag{2.3}
\]

and the open-loop zero locations are:

\[
\pm \sqrt{ \frac{3}{4} \left( \frac{g}{l_1} \right) }, \pm \sqrt{ \frac{3}{4} \left( \frac{g}{l_2} \right) } \tag{2.4}
\]

There are two open-loop poles at the origin and the other four poles are symmetric about the \( j\omega \)-axis. Because of the poles in the right-half plane, the system is unstable. The four open-loop zeros are also symmetric about the \( j\omega \)-axis. The zeros in the right-half plane cause the system to be nonminimum phase. The magnitudes of the zeros are the natural frequencies of the individual rods.

Note that the poles are dependent on the masses of the cart and the rods. Increasing the mass of the cart or the masses of the rods pushes the non-zero poles farther away from the \( j\omega \)-axis, causing the system to become harder to control. Physically, the heavier a cart is, the harder it is to move back and forth, and the heavier an inverted rod is, the more difficult it is to keep vertical.

The open-loop pole and zero locations from the disturbances to the cart position (\( f_1 \) and \( f_2 \) to \( y \)) can be found from the transfer functions \( T_{f_1}(s) = C(sI - A)^{-1} L_1 \) and \( T_{f_2}(s) = C(sI - A)^{-1} L_2 \), respectively, with the disturbance matrix \( L \) rewritten as \( L = [L_1 \ L_2] \). Since the poles depend only on the \( A \) matrix, these transfer functions have the same pole locations as given in (2.3). The transfer functions are included in Appendix B, and the zeros are found to be:

\[
\text{zeros of } T_{f_1}(s): \pm \sqrt{ \frac{3}{4} \left( \frac{g}{l_1} \right) }, \pm j \sqrt{ \frac{3}{2} \left( \frac{g}{l_1} \right) } \tag{2.5}
\]
zeros of \( T_{y}(s) : \pm \sqrt{\frac{3}{4} \frac{g}{l}} \), \( \pm j \sqrt{\frac{3}{2} \frac{g}{l}} \) (2.6)

Note that these transfer functions have real and complex zeros. If a disturbance force is of a frequency equal in magnitude to its respective complex zero, the rod oscillates at the same frequency in such a way that the cart does not "see" the disturbance, and therefore does not move [5], [6].

The open-loop pole and zero locations for the three transfer functions mentioned above are given below for System 1 and System 2, and are also shown in Figures 2.2, 2.3, and 2.4.

**SYSTEM 1**: \( l_1 = 1 \text{ m}, l_2 = 16 \text{ m} \):

- open-loop poles: \( 0, 0, \pm 2.7485, \pm 0.8022 \)
- zeros of \( T_y(s) : \pm 2.7130, \pm 0.6783 \)
- zeros of \( T_{f1y}(s) : \pm 0.6783, \pm j3.8368 \)
- zeros of \( T_{f2y}(s) : \pm 2.7130, \pm j0.9592 \)

**SYSTEM 2**: \( l_1 = 1 \text{ m}, l_2 = 2 \text{ m} \):

- open-loop poles: \( 0, 0, \pm 2.7545, \pm 1.9703 \)
- zeros of \( T_y(s) : \pm 2.7130, \pm 1.9184 \)
- zeros of \( T_{f1y}(s) : \pm 1.9184, \pm j3.8368 \)
- zeros of \( T_{f2y}(s) : \pm 2.7130, \pm j2.7130 \)

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Figure 2.2 Open-Loop Pole and Zero Locations, $T_{y_0}(s)$

Figure 2.3 Open-Loop Pole and Zero Locations, $T_{y_1}(s)$
Figure 2.4 Open-Loop Pole and Zero Locations, $T_{ij}(s)$

2.2.2 Controllability and Observability

The controllability and observability of the dual inverted pendulum system are also considered because if a system is not controllable and observable, it is meaningless to search for an optimal controller. Because the system contains unstable modes, we are actually looking at the stabilizability and detectability properties of the system.

For a system to be controllable, for any initial state $\xi$ and any final state $\vartheta$ at $t=T$, one can find a control $u(t)$ such that $x(T) = \vartheta$. For a system to be observable, the unique initial state $\xi$ can be calculated from the measurements of $u(t)$ and $y(t)$. A system with
unstable modes is said to be stabilizable and detectable if the unstable modes are controllable and observable, respectively [5]. The controllability and observability properties of the dual inverted pendulum system are examined in Appendix C in terms of the system parameters in the controllability and observability matrices.

The loss of controllability and/or observability can also be caused by a pole/zero cancellation. In Figure 2.5, the open-loop pole and zero locations are shown as the length of the longer rod, \( l_2 \), is decreased to equal the length of the shorter rod, \( l_1 \). The plot indicates that when the rods are of equal length there are pole/zero cancellations at \( \pm 2.7130 \) (two zeros and one pole at each). A pole/zero cancellation implies a loss of controllability and/or observability. To determine which loss the system experiences, the mode and the pole and zero "directions" must be examined. The loss of controllability and observability for this pole/zero cancellation is also discussed in Appendix C, along with an examination of other possible cases which

![Figure 2.5 Open-Loop Pole and Zero Locations as the Length of the Longer Rod, \( l_2 \), is Decreased](image)

Figure 2.5 Open-Loop Pole and Zero Locations as the Length of the Longer Rod, \( l_2 \), is Decreased
give a cancellation. It is found, however, that the system loses controllability and observability only when the lengths of the rods are equal, and that the system cannot lose only one or the other.

The fact that the dual inverted pendulum system is both uncontrollable and unobservable when the rods are equal in length makes sense physically. The uncontrollability of the system can be thought of in this way: rods of equal length oscillate at the same frequency, \( \sqrt{\frac{3g}{4l}} \), so that if they are started with the initial condition \( \theta_1(0) = -\theta_2(0) \), the final condition of \( \theta_1(T) = \theta_2(T) \) can not be reached for any control \( u(t) \), and thus the system is uncontrollable. The unobservability of the system can be reasoned as follows: with the same initial conditions as above, the rods exert equal and opposite forces on the cart when they fall, and therefore the cart does not move. However, since a net force of zero results on the cart for several initial conditions (\( \theta_1(0) = 10' = -\theta_2(0) \) or \( \theta_1(0) = -5' = -\theta_2(0) \), for example), the system is unobservable.

### 2.2.3 Open-Loop Frequency Responses

Finally, we examine some open-loop transfer functions of the dual inverted pendulum system. The transfer functions shown in Figure 2.6, the control to the angles of the rods for System 1 and System 2, indicate that we can control the angle of the shorter rod, \( \theta_1 \), better at the higher frequencies, and that the control affects \( \theta_2 \) more for System 2, where the ratio of the rod lengths is smaller, than for System 1. Both of these statements make sense physically since a
horizontal movement at the base of a shorter rod causes a larger angular deflection of the tip than if the same horizontal movement is applied to a longer rod.

The transfer functions in Figure 2.7, the disturbances to the cart position, show that disturbances greatly affect the position of the cart at the lower frequencies. The plots also reveal the fact that there are complex zeros for these transfer functions (refer to (2.5) and (2.6)). It appears from Figure 2.7 that if the disturbance is of a frequency equal in magnitude to that of a complex zero, the disturbance has less effect on the position of the cart than at many of the other frequencies. These slight dips in the responses should extend much farther down because as explained in Section 2.2.1, if
Now that the open-loop properties of the dual inverted pendulum system have been examined, the LQR and $H_\infty$ compensators can be designed.

Figure 2.7 Open-Loop Frequency Response, Disturbances to Cart Position

the disturbance is of a frequency equal in magnitude to that of a complex zero, the cart does not move.
Chapter 3

Linear-Quadratic Regulator Design

The linear-quadratic regulator design methodology is presented in this chapter, and a controller is designed for the dual inverted pendulum system. The closed-loop time and frequency responses are also examined.

3.1 LQR Methodology

The linear-quadratic regulator (LQR) method is used to design a linear controller for a system with full state feedback. The LQR design method develops controllers for systems of the form

\[ x(t) = Ax(t) + Bu(t), \ x(0) = x_0 \]
\[ x(t) \in \mathbb{R}^n, \ u(t) \in \mathbb{R}^m \]
\[ [A,B] \text{ stabilizable} \tag{3.1} \]

which minimize the cost performance

\[ J = \int_0^T [x'(t)Qx(t) + u'(t)Ru(t)]dt \tag{3.2} \]
where $Q$ is the symmetric, positive semidefinite state weighting matrix, and $R$ is the symmetric, positive definite control weighting matrix. It is also necessary that $[A, Q^{1/2}]$ is detectable.

The solution of the regulator problem is a time varying control law which in steady-state becomes the linear time-invariant, state feedback law [6], [7]:

$$u(t) = -Gx(t)$$

(3.3)

where $G$ is the $(m \times n)$ control gain matrix given by

$$G = R^{-1}B'K$$

(3.4)

and $K$ is the unique symmetric, positive semidefinite $(n \times n)$ solution of the algebraic Riccati equation

$$KA + A'K + Q - KBR^{-1}B'K = 0$$

(3.5)

The control law defined by (3.3) is unique and minimizes the cost functional, provided that the stabilizability and detectability conditions are met. If there are input disturbances to the plant, the optimal controller is determined as above, ignoring the fact that there are disturbances, and it is implemented as shown in Figure 3.1 [7].
3.2 LQR Design

As mentioned in Chapter 2, the state variable we want to control is the position of the cart from a reference point. Therefore, in the LQR cost performance of (3.2), the state weighting matrix $Q$ equals $C'C$.

The controllers are to be designed such that the open- and closed-loop bandwidths are comparable. With LQR control, if we let $R=\rho I, \rho > 0$ in the cost functional, then as $\rho \to 0$, some of the closed-loop poles go to cancel any minimum phase zeros, or go to the mirror image, about the $j\omega$-axis, of any nonminimum phase zeros of the open-loop system, and the rest go off toward infinity along stable Butterworth patterns [7]. Thus, we can vary the value of $\rho$ to obtain closed-loop poles within the same distance from the origin of the $s$-plane as the open-loop poles. From the locations of the open-loop poles for System 1 $(0,0,\pm 2.7485,\pm 0.8022)$ and System 2
(0,0,±2.7545,±1.9703), it is decided to have the closed-loop poles inside a semicircle of radius three from the origin.

The value of $\rho$ is varied until the closed-loop poles that are heading off toward infinity from the origin along stable Butterworth patterns are just inside the radius of three semicircle. The other closed-loop poles head toward minimum phase and "mirror-image" nonminimum phase zeros, which are inside the pole locations (Figure 2.2). The value of $\rho$ and the closed-loop pole locations for System 1 and System 2 are as follows:

**System 1:**

$\rho = 0.000353$

closed-loop poles: $-2.1564 \pm j2.0852, -2.7268 \pm j0.0175, -0.6785 \pm j0.0070$.

**System 2:**

$\rho = 0.000448$

closed-loop poles: $-2.1574 \pm j2.0831, -2.7285 \pm j0.0204, -1.9264 \pm j0.0193$.

Recall that the open-loop zero locations for System 1 and System 2 are ±2.7130, ±0.6783 and ±2.7130, ±1.9184, respectively. Thus we can see that four of the closed-loop poles are almost canceling the minimum phase and "mirror-image" nonminimum phase zeros of the open-loop system.
3.3 LQR Closed-Loop Simulation Results

The LQR closed-loop system has the form

\[
\dot{x}(t) = (A - BG)x(t) + Ld(t), \quad x(0) = x_0 \\
y(t) = Cx(t)
\]  

From these equations we can examine the time and frequency responses.

3.3.1 LQR Time Response

From the locations of the closed-loop poles, we would expect System 2 to have a faster transient response than System 1. Time responses for System 1 and System 2 are shown in Figure 3.2. For this simulation it is assumed that only the cart has an initial perturbation, equal to one meter, and that there are no disturbance forces acting on the rods.

First notice from Figure 3.2 that System 2 does in fact have a faster transient response. The figure also indicates that System 1 experiences the largest perturbation of the cart from equilibrium, and that the angle of the longer rod for System 2 deviates farther from equilibrium than for System 1. The perturbation of the angle makes sense physically because, as mentioned previously, a horizontal movement at the base of a shorter rod causes a larger angular deflection of the tip than if the same horizontal movement is applied to a longer rod. More time responses are discussed in
Figure 3.2 LQR Time Response, $x(0) = [1 \ 0 \ 0 \ 0 \ 0 \ 0]$
Chapter 5 when the LQR design is compared to the frequency weighted $H_{\infty}$ design.

3.3.2 LQR Frequency Response

The frequency responses examined relate to stability robustness and disturbance rejection. These are shown in Figures 3.3 and 3.4. The complementary sensitivity plot, Figure 3.3, is the frequency response of the closed-loop transfer function matrix, $C_{LQ}(s) = [I + G_{LQ}(s)]^{-1} G_{LQ}(s)$, where $G_{LQ}(s) = G(sl - A)^{-1}B$ is the loop transfer function matrix and $G$ is the control gain matrix of (3.4). With the modeling errors reflected to the plant input, Figure 3.3 indicates that the system can tolerate multiplicative errors with $\sigma(E(j\omega)) < \sqrt{2}$, for all $\omega$ (peaks are less than $6dB \equiv 20\log(2))$ [7].

![Figure 3.3 LQR Complementary Sensitivity](image)

The disturbance rejection plots of Figures 3.4 and 3.5 show the frequency responses of the open-loop and closed-loop transfer functions $T_{J\nu}(s)$ and $T_{J\nu}(s)$ for System 1 and System 2. The plots
Figure 3.4 LQR Disturbance Rejection, $T_{f_1}(s)$

Figure 3.5 LQR Disturbance Rejection, $T_{f_2}(s)$
indicate that disturbance rejection performance has been improved only for frequencies less than approximately 0.1 radians/sec. Figures 3.4 and 3.5 also indicate that disturbances of frequency less than approximately 8 radians/sec are amplified in the closed-loop system. As mentioned in Chapter 2, the disturbance amplification in some frequency ranges is a characteristic of closed-loop systems for which the open-loop system has nonminimum phase zeros [4].

In the next design methodology, $H_\infty$ with full state feedback, the disturbance is included in the cost function. By including the disturbance in the cost function, we would expect to obtain better disturbance rejection.
Chapter 4

H-Infinity with Full State Feedback Design

In this chapter, both the $H_\infty$ with full state feedback design method and the frequency weighted $H_\infty$ design method are presented. Design parameters and time and frequency responses are given for both designs.

4.1 $H_\infty$ with Full State Feedback Methodology

The $H$-Infinity ($H_\infty$) with full state feedback method can also be used to design a linear controller for a system with full state feedback. Unlike the LQR method however, the $H_\infty$ with full state feedback method accounts for how the disturbances affect the plant dynamics. The design method develops controllers for systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + Ld(t), \quad x(0) = x_0$$
$$y(t) = Cx(t)$$

$x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $d(t) \in L_2$


(4.1)

which minimize the cost performance
\[ J(u,d) = \frac{1}{2} \int_{0}^{\infty} \left[ y'(t)y(t) + \rho u'(t)u(t) - \gamma^2 d'(t)d(t) \right] dt \] (4.2)

where \( \rho \) is the control weighting parameter, as in the LQR design, and \( \gamma \) is the trade-off parameter.

The solution of the \( H_{\infty} \) with full state feedback problem in steady-state is the linear time-invariant, state feedback control law:

\[ u(t) = -\frac{1}{\rho} B'Xx(t) = -G_x x(t) \] (4.3)

where \( X \) is the symmetric, positive definite \((n \times n)\) solution of the modified algebraic Riccati equation

\[ XA + A'X + C'C - X \left( \frac{1}{\rho} BB' - \frac{1}{\gamma^2} LL' \right) X = 0 \] (4.4)

For the pure \( H_{\infty} \) with full state feedback solution, the value of \( \gamma \) is decreased to the minimum value, \( \gamma_{\text{min}} \), for which the following three conditions are satisfied: (i) the eigenvalues of the Hamiltonian matrix

\[ H = \begin{bmatrix} A & \frac{1}{\rho} BB' - \frac{1}{\gamma^2} LL' \\ -C'C & -A' \end{bmatrix} \] (4.5)

are just off of the \( j\omega \)-axis, (ii) \( X \) is symmetric, positive definite, and (iii) the real parts of the eigenvalues of \( A - \frac{1}{\rho} BB'X \) are less than zero.
[7]. Note that if $\gamma \to \infty$, the design method above is the same as the LQR design method.

4.2 $H_\infty$ with Full State Feedback Design

As with the LQR method, the design goal is to have the open- and closed-loop bandwidths be comparable. For the $H_\infty$ with full state feedback method, the design parameter is $\rho$. As a baseline design, the values of $\rho$ from the LQR design for System 1 and System 2 are used. The value of $\rho$ from the LQR design, the value of $\gamma_{\text{min}}$, and the closed-loop pole locations for System 1 and System 2 are:

**System 1:**

$\rho = 0.000353$

$\gamma_{\text{min}} = 7.48$

closed-loop poles: $-7670, -2.1601 \pm j2.0895, -2.7267, -0.8354, -0.6791$

**System 2**

$\rho = 0.000448$

$\gamma_{\text{min}} = 30.43$

closed-loop poles: $-8146, -2.1586 \pm j2.0828, -2.7274, -2.1905, -1.9288$

The open-loop zero locations for System 1 and System 2 are $\pm 2.7130, \pm 0.6783$ and $\pm 2.7130, \pm 1.9184$, respectively. We can then see that four of the closed-loop poles are again almost canceling the minimum phase and "mirror-image" nonminimum phase zeros of the
open-loop system. Each system also has a large negative closed-loop pole. This corresponds to a high bandwidth in the frequency domain.

4.3 $H_\infty$ Closed-Loop Simulation Results

The $H_\infty$ with full state feedback closed-loop system has the form

$$
\begin{align*}
\dot{x}(t) &= \left( A - \frac{1}{\rho} BB'X \right) x(t) + Ld(t), \quad x(0) = x_0 \\
y(t) &= Cx(t)
\end{align*}
$$

(4.6)

Again, we examine time and frequency responses of the closed-loop system.

4.3.1 $H_\infty$ with Full State Feedback Time Response

As predicted from the closed-loop pole locations, and as shall be seen in the following section, the $H_\infty$ with full state feedback method does not give a closed-loop system whose bandwidth is approximately that of the open-loop system. However, it is interesting to see the effects of the minimization properties of this method on the time response. Figure 4.1 indicates that the dual inverted pendulum system is brought back to equilibrium from the same initial conditions as for the LQR design, and again, System 2 has a faster transient response than System 1. Note however that the necessary control to achieve this equilibrium condition is very large.
Figure 4.1 $H_\infty$ Time Response, $x(0) = [1 \ 0 \ 0 \ 0 \ 0 \ 0]$
in magnitude. Compared to the LQR responses of Figure 3.2, the necessary control for the $H_\infty$ design is much larger in magnitude (over three thousand times larger initially), but the transients are only slightly faster.

### 4.3.2 $H_\infty$ with Full State Feedback Frequency Response

Figure 4.2 shows the frequency response of the closed-loop transfer function matrix, $C(s) = [I + T(s)]^{-1} T(s)$, where $T(s) = G_\omega (sI - A)^{-1} B$ is the loop transfer function matrix and $G_\omega$ is the control gain matrix of (4.3). As seen in Figure 4.2, the bandwidth of both System 1 and System 2 is approximately 2000 radians/sec, which is about 200 times higher than for the LQR design (Figure 3.3).

![Figure 4.2 $H_\infty$ Complementary Sensitivity](image)

Varying the value of $\rho$ in the $H_\infty$ with full state feedback design method and finding the corresponding values of $\gamma_{\text{min}}$ does not give closed-loop systems with significantly lower bandwidths. The next step is to add frequency weighting to the control.
4.4 Frequency Weighting Methodology

A simplified description of frequency weighting is that in the frequency domain, the inverse of the weighting function acts as a boundary. This is indicated in Figure 4.3, where both the weighting function and the resulting effect in the frequency domain are shown [5].

\[ J = \frac{1}{2\pi} \int \left[ y'(-j\omega)y(j\omega) + p\mu'(-j\omega)R(\omega)\mu(j\omega) - \gamma d'(-j\omega)d(j\omega) \right] d\omega \quad (4.7) \]

where \( R(\omega) = H_u(-j\omega)H_u(j\omega) \), and \( H_u(s) \) is the weighting function on the control, \( u_u(s) = H_u(s)(\sqrt{p} u(s)) \). The control weighting function is expressed in state space form as \( H_u(s) = C_u(sl-A_u)^{-1}B_u + D_u \) [7]
where \( i(t) = A_z(t) + B_u \sqrt{p u(t)} \)

\[ u_a(t) = C_z z_a(t) + D_u \sqrt{p u(t)} \] (4.8)

An augmented state space description is formed which includes the dynamics of the plant and the dynamics of the control weighting.

\[
\dot{x}_a(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{z}_a(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_a \end{bmatrix} \begin{bmatrix} x(t) \\ z_a(t) \end{bmatrix} + \begin{bmatrix} B \\ \sqrt{p B_u} \end{bmatrix} u(t) + \begin{bmatrix} L \\ 0 \end{bmatrix} d(t)
\]

\[ y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ z_a(t) \end{bmatrix} \] (4.9)

Let the subscript "a" represent the augmented state variables and state space matrices. The cost performance (4.7) then becomes, in the time domain,

\[
J = \int_0^\infty [x_a'(t)Q x_a(t) + 2x_a'(t)Su(t) + u'(t)Ru(t) - y^2 d'(t)d(t)] dt
\]

with \( Q = \begin{bmatrix} C'C & 0 \\ 0 & C_a' C_a \end{bmatrix} \), \( S = \begin{bmatrix} 0 \\ \sqrt{p C_a' D_a} \end{bmatrix} \), and \( R = [p D_a' D_a] \). (4.10)

Because of the cross-coupled term in the cost performance, the control law, the modified algebraic Riccati equation, and the Hamiltonian matrix change in form from Section 4.1. The appropriate equations for the frequency weighted \( H_\omega \) with full state feedback method are:

control law: \( u(t) = -R^{-1}[S' + B'_u X_a] x_a(t) = -G_x x_a(t) = -G_z z_a(t) - G x(t) \) (4.11)
modified algebraic Riccati equation:

\[ X_a A_a + A'_a X_a + Q - [X_a B_a + S] R^{-1} [B'_a X_a + S'] + \frac{1}{\gamma^2} X_a L_a L'_a X_a = 0 \quad (4.12) \]

Hamiltonian matrix:

\[
H = \begin{bmatrix}
A_a - B_a R^{-1} S' & \frac{1}{\gamma^2} L_a L'_a - B_a R^{-1} B'_a \\
-Q + SR^{-1} S' & -A'_a + SR^{-1} B'_a
\end{bmatrix}
\quad (4.13)
\]

For the frequency weighted $H_\infty$ design, the values of $\rho$ from the LQR designs are used, and the value of $\gamma$ is the minimum value for which the following three conditions are satisfied: (i) the eigenvalues of the Hamiltonian matrix are just off the $j\omega$-axis, (ii) the matrix $X_a$ is symmetric, positive definite, and (iii) the real parts of the eigenvalues of $A_a - B_a R^{-1} [B'_a X_a + S']$ are less than zero. The frequency weighted $H_\infty$ with full state feedback controller is implemented as shown in Figure 4.4.

![Figure 4.4 Frequency Weighted $H_\infty$ Implementation](image-url)
4.5 Frequency Weighted $H_\infty$ Design

As mentioned in the previous section, in the frequency domain, the inverse of the weighting function acts as a sort of boundary. For the first attempt to shape the frequency response of the $H_\infty$ design to decrease its bandwidth to that of the LQR design, the weighting function of Figure 4.5 is added to the control of both System 1 and System 2.

![Figure 4.5 Preliminary $H_\infty$ Control Weighting](image)

This weighting function gives a closed-loop bandwidth of approximately 200 radians/sec. The breakpoint frequencies and the initial magnitude of the control weighting function are varied until a weighting function is found that gives a lower bandwidth. For the final weighting functions for System 1 and System 2, shown in Figures 4.6 and 4.7, the closed-loop bandwidth is approximately 20 radians/sec, closer to that of the LQR design than without the weighting.

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Note that the initial magnitude of the weighting functions above are approximately equal to $20 \log(p)$. The values of $p$ and $\gamma$, and the locations of the closed-loop poles for System 1 and System 2 are:

**System 1**
\[
\begin{align*}
\rho &= 0.000353 \\
\gamma_{\min} &= 4.38 \\
\text{closed-loop poles:} & \quad -294.5, -4.9058 \pm j11.7000, -11.7104 \pm j4.7911, \\
& \quad -2.7130, -1.0337, -0.6783
\end{align*}
\]

**System 2**
\[
\begin{align*}
\rho &= 0.000448 \\
\gamma_{\min} &= 18.51 \\
\text{closed-loop poles:} & \quad -214.2, -4.3700 \pm j10.4259, -10.4565 \pm j4.2791, \\
& \quad -2.7130, -2.2157, -1.9184
\end{align*}
\]

Once again, there are closed-loop poles at locations of open-loop zeros ($-2.7130, -0.6783, -1.9184$). With the unweighted $H_{\infty}$ design, there
were large negative closed-loop poles at -7670 and -8146 for System 1 and System 2, respectively, and the bandwidth of the closed-loop system was high, approximately 2000 radians/sec. The addition of frequency weighting has brought the poles in to -295 and -214, and thus the bandwidth is smaller.

4.6 Frequency Weighted $H_\infty$ Closed-Loop Simulation Results

The frequency weighted closed-loop system has the form

$$\begin{align*}
\dot{x}_a(t) &= [A_a - B_a G_a] x_a(t) + L_a d(t) \\
y(t) &= [C \ 0] x_a(t)
\end{align*}$$

The time and frequency responses are shown in the next sections.

4.6.1 Frequency Weighted $H_\infty$ Time Response

Time responses for the frequency weighted $H_\infty$ with full state feedback design for System 1 and System 2 are shown in Figure 4.8. The figure indicates that System 1 has a slightly faster transient response than System 2, as is expected from the location of the closed-loop poles, and System 2 experiences the largest perturbation of the cart from equilibrium. These time response characteristics are opposite from what the system experienced for the LQR design (Figure 3.2). However, the responses of the rod angles are similar,
Figure 4.8 Frequency Weighted $H_\infty$ Time Response, $x(0) = [1 \ 0 \ 0 \ 0 \ 0 \ 0]$
with the angle of the longer rod deviating more for System 2 than for System 1. Comparing the time responses of Figure 4.8 to those for the unweighted design (Figure 4.1), we see that the frequency weighted $H_\infty$ design has faster transients, smaller perturbations, and requires much less control. Again, more time responses comparing the LQR and frequency weighted $H_\infty$ designs are discussed in Chapter 5.

4.6.2 Frequency Weighted $H_\infty$ Frequency Response

The frequency responses related to stability robustness and disturbance rejection for the frequency weighted $H_\infty$ design are shown in Figures 4.9, 4.10, and 4.11. The complementary sensitivity plot, Figure 4.9, is the frequency response of the closed-loop transfer function matrix, $C(s) = [I + T(s)]^{-1} T(s)$, where $T(s) = G(sI - A)^{-1} B$. Notice that the LQR complementary sensitivity plot of Figure 3.3 has a similar shape near 10 radians/sec. Figure 4.9 shows that the bandwidth has been brought down to approximately 20 radians/sec for both System 1 and System 2, and that the system can tolerate

![Figure 4.9 Frequency Weighted $H_\infty$ Complementary Sensitivity](image)

Figure 4.9 Frequency Weighted $H_\infty$ Complementary Sensitivity
multiplicative errors (modeling errors reflected to the plant input) with $\mathcal{H}[E(j\omega)] < \frac{1}{4}$, for all $\omega$ (peaks are less than $12dB \equiv 20\log(4)$).

The disturbance rejection plots of Figures 4.10 and 4.11 show the frequency responses of the open-loop and closed-loop transfer functions $T_{f_1}(s)$ and $T_{f_2}(s)$ for System 1 and System 2. As with the LQR designs (Figures 3.4 and 3.5), disturbance rejection performance has been improved only for frequencies less than approximately 0.1 radians/sec. Figures 4.10 and 4.11 also indicate that disturbances of frequency less than approximately 20 radians/sec are amplified in the closed-loop system.

![Disturbance Rejection Plots](image)

**Figure 4.10** Frequency Weighted $H_\infty$ Disturbance Rejection, $T_{f_1}(s)$
Figure 4.11 Frequency Weighted $H_\infty$ Disturbance Rejection, $T_{f2}(s)$

In the next chapter, the LQR and frequency weighted $H_\infty$ designs are compared directly in the time and frequency domains.
Chapter 5

Design Comparisons

In this chapter, the LQR and frequency weighted $H_\infty$ with full state feedback designs for the dual inverted pendulum system are compared. The comparisons are done in both the time domain and frequency domain, and a parameter variation on the length of the longer rod is also investigated.

5.1 Time Response

Figures 5.1 and 5.2 show the time responses for the two designs for System 1 and System 2, respectively. The figures show that for the LQR designs, the states are perturbed from equilibrium less than for the frequency weighted $H_\infty$ designs, and the LQR designs require a control that is smaller in magnitude. However, the transients of the frequency weighted $H_\infty$ design are faster. Time responses for the controller designs for other initial conditions are shown in Appendix D. These plots exhibit the same perturbation and transient response characteristics as mentioned above. For some of the responses of the frequency weighted $H_\infty$ closed-loop system, the angle of the shorter rod may violate the approximation for small angles used to determine the state space equations. In fact, for some initial conditions, the approximation is violated (note $\theta_i$ in Figures
Figure 5.1 Time Domain Comparison, System 1, $x(0) = [1 \ 0 \ 0 \ 0 \ 0 \ 0]$
Figure 5.2 Time Domain Comparison, System 2, $x(0) = [1 \ 0 \ 0 \ 0 \ 0 \ 0]$
D.3, D.9, and D.10).

5.2 Frequency Response

The first frequency response comparison is the complementary sensitivity of the closed-loop systems, shown in Figure 5.3. The plot indicates that the bandwidth for the LQR design is approximately 10 radians/sec, and for the frequency weighted $H_\infty$ design, 20

![Complementary Sensitivity Comparison](image)

Figure 5.3 Complementary Sensitivity Comparison
radians/sec. Both designs have roll-offs of 20dB/decade, although the roll-off does not begin for the $H_\infty$ design until about 200 radians/sec. The plot also indicates that the LQR designs can tolerate larger multiplicative errors at the plant input.

Figures 5.4 and 5.5 show the disturbance rejection properties for the closed-loop systems for both design methods. The plots indicate that the frequency weighted $H_\infty$ designs amplify the disturbances less than the LQR designs for frequencies less than approximately 3 radians/sec, but the LQR designs have better disturbance rejection properties at frequencies higher than this. For both design methods, the disturbances are better rejected when they appear on the longer rod.

![Figure 5.4 Disturbance Rejection Comparison, System 1](image)

Figure 5.4 Disturbance Rejection Comparison, System 1
5.3 Parameter Variation

For this comparison, the LQR and frequency weighted $H_{\infty}$ controllers for System 1 and System 2 are kept the same, and the length of the longer rod is varied. It is then determined by how much the length of the longer rod can vary from the value for which the controllers were designed before the system again becomes unstable.

The length of the longer rod is increased from half to twice the length for which the controllers were designed, and the closed-loop poles are recalculated for each length. Figures 5.6 and 5.7 show the locations of the closed-loop poles for the different lengths for System
1 and System 2, respectively. The closed-loop poles marked with an 'x' are the closed-loop pole locations given for the two designs in sections 3.2 and 4.5. Noting the step size in the length for each plot and the closed-loop pole locations, it can be seen that the LQR designs allow for more variation in the length of the longer rod before the system again becomes unstable. The approximate percentages that the length of longer rod can vary are:

**System 1:**

- **LQR Design:** $-10\%, +12\%$
- **Frequency Weighted $H_\infty$ Design:** $-0.4\%, +2.4\%$

**System 2**

- **LQR Design:** $-6\%, +10\%$
- **Frequency Weighted $H_\infty$ Design:** $-0.6\%, +2.7\%$

Note that while the LQR design allows for more variation in the length of the longer rod for System 1, the frequency weighted $H_\infty$ design allows for more variation for System 2.

The properties of the two design methods shown in these comparisons are discussed again in the next chapter as part of the conclusions.
Figure 5.6 Parameter Variation Comparison, System 1
Figure 5.7 Parameter Variation Comparison, System 2
Chapter 6

Conclusions and Suggestions for Future Work

This chapter briefly summarizes the conclusions based on the comparisons of Chapter 5, and gives recommendations for future work.

6.1 Conclusions

The design of controllers for a system that was unstable with nonminimum phase zeros was considered in this thesis. The two design methods used to design the controllers for the dual inverted pendulum system were the linear-quadratic regulator (LQR) method and the H-Infinity ($H_\infty$) with full state feedback method. The design goal was to approximately maintain the open-loop bandwidth. The goal was met for both design methods, with the addition of frequency weighting to the $H_\infty$ method.

The LQR and frequency weighted $H_\infty$ closed-loop systems were compared in both the time and frequency domains, and in the allowable variation of the length of the longer rod. In the time domain comparison, it was seen that for the LQR design, the state variables experienced smaller perturbations from equilibrium, and the required control was smaller in magnitude. The frequency
weighted $H_\infty$ design has faster transients, but for several initial conditions of the system, may have or did violate the assumption for small angles used to determine the state space representation.

For the frequency domain comparisons, both stability robustness and disturbance rejection properties were considered. From the stability robustness comparison, it was determined that the LQR design could tolerate larger multiplicative errors at the plant input. The disturbance rejection comparisons showed that the frequency weighted $H_\infty$ design amplified the disturbances less than the LQR design for frequencies less than approximately 3 radians/sec, but had poorer disturbance rejection properties at frequencies higher than this. The plots also indicated that the closed-loop system better rejected disturbances which occurred on the longer rod.

The final comparison involved using the controllers designed by the two methods and varying the length of the longer rod until the dual inverted pendulum system again became unstable. The greater possible variation in the length occurred for the controllers designed by the LQR method.

Both the LQR design method and the frequency weighted $H_\infty$ with full state feedback design methods were able to stabilize the system. The main tradeoffs of the two design methods are the allowed sizes of the perturbations, the speed at which equilibrium is achieved, the characteristics of the multiplicative error, the frequency of the disturbance, and the error in measuring the length of the longer rod.
6.2 Suggestions for Future Work

In this thesis, it was assumed that the state variable of interest was the position of the cart. Because the perturbations of the angles of the rods were sometimes quite large, it might prove beneficial to include these state variables (or possibly all of the state variables) in the cost performance equations, or investigate different methods of selecting the weighting matrices, such as the Bryson method [7] or high state weighting [6]. Control torques on the rods might also be introduced to the system. In this thesis, it was also assumed that all of the states could be measured accurately. Thus, another suggestion for future work is to design controllers for the dual inverted pendulum system using methods which do not assume full state feedback, such as the Linear-Quadratic Gaussian (LQG) method or the $H_{\infty}$-Optimization method. A final suggestion is to actually build and control a dual inverted pendulum system.
Appendix A

State Space Representation

This appendix shows the development of the state space equations, $\dot{x}(t) = A\delta x(t) + B\delta u(t) + L\delta d(t)$, $y(t) = C\delta x(t)$, in continuous time, for the dual inverted pendulum system of Figure 2.1, shown again in Figure A.1. The state space equations are derived from the nonlinear dynamic equations which describe the system by establishing the steady-state equilibrium conditions, defining the perturbations about this equilibrium point, introducing the perturbations into the nonlinear equations, and then retaining only the linear terms [5].

![Figure A.1 The Dual Inverted Pendulum System](image-url)
First, define the following vectors:

\[
\delta x(t) = \begin{bmatrix}
\delta z(t) \\
\delta \dot{z}(t) \\
\delta \theta_1(t) \\
\delta \dot{\theta}_1(t) \\
\delta \theta_2(t) \\
\delta \dot{\theta}_2(t)
\end{bmatrix}
\]

- perturbed cart position (m)
- perturbed cart velocity (m/s)
- perturbed angular position of the shorter rod (rad)
- perturbed angular velocity of the shorter rod (rad/s)
- perturbed angular position of the longer rod (rad)
- perturbed angular velocity of the longer rod (rad/s)

\[
\delta u(t) = [\delta f(t)]
\]
- perturbed control force (N)

\[
\delta d(t) = \begin{bmatrix}
\delta f_1(t) \\
\delta f_2(t)
\end{bmatrix}
\]
- perturbed disturbance force on the shorter rod (N)
- perturbed disturbance force on the longer rod (N)

Dynamic principles are then applied to Figures A.2 and A.3 to obtain the nonlinear equations, with \( M_1 = 2m_1l_1, M_2 = 2m_2l_2, a = \ddot{z}, \omega = \dot{\theta}, \) and \( \alpha = \ddot{\phi}. \) The pendulums are modeled as thin rods with moments of inertia about their center of mass equal to \( \frac{1}{3} (\text{mass})(\text{length})^2 \) [8].

Figure A.2 Diagram for Force Equation

Figure A.3 Diagram for Moment Equations
The nonlinear equations which describe the behavior of the system are:

\[ \sum F = ma; \]

\[ f(t) + f_1(t) + f_2(t) = \]

\[ (M + M_1 + M_2)\ddot{x} + (M_1l_1 \cos \theta_1)\ddot{\theta}_1 + (M_2l_2 \cos \theta_2)\ddot{\theta}_2 - (M_1l_1 \sin \theta_1)\dot{\theta}_1^2 - (M_2l_2 \sin \theta_2)\dot{\theta}_2^2 \]

\[ (A.1) \]

\[ + \sum M_O = I \alpha; \]

\[ + \sum M_{O_1} : \ M_1g l_1 \sin \theta_1 - (M_1l_1 \cos \theta_1)\ddot{\theta}_1 + 2l_1 \cos \theta_1 f_1(t) = \frac{4}{3} M_1l_1^2 \ddot{\theta}_1 \]

\[ + \sum M_{O_2} : \ M_2g l_2 \sin \theta_2 - (M_2l_2 \cos \theta_2)\ddot{\theta}_2 + 2l_2 \cos \theta_2 f_2(t) = \frac{4}{3} M_2l_2^2 \ddot{\theta}_2 \]

\[ (A.2) \]

In Figure A.2 and in (A.1), \( \alpha l_1 = \ddot{\theta}_1 l_1 \) and \( \alpha l_2 = \ddot{\theta}_2 l_2 \) represent the normal accelerations, and \( \omega l_1 = \dot{\theta}_1 l_1 \) and \( \omega l_2 = \dot{\theta}_2 l_2 \) represent the tangential accelerations. In Figure A.3 and in (A.2), \( M_1\ddot{x} = M_1a \) and \( M_2\ddot{x} = M_2a \) represent the inertial forces.

\[ \ddot{x}(t) = \ddot{x}_o + \delta \ddot{x}(t), \]

\[ \theta_1(t) = \theta_{1o} + \delta \theta_1(t), \theta_2(t) = \theta_{2o} + \delta \theta_2(t), \]

\[ \dot{\theta}_1(t) = \dot{\theta}_{1o} + \delta \dot{\theta}_1(t), \dot{\theta}_2(t) = \dot{\theta}_{2o} + \delta \dot{\theta}_2(t), \]

Define the perturbations to be:

\[ \ddot{\theta}_1(t) = \ddot{\theta}_{1o} + \delta \ddot{\theta}_1(t), \ddot{\theta}_2(t) = \ddot{\theta}_{2o} + \delta \ddot{\theta}_2(t), \]

\[ f(t) = f_o + \delta f(t), \]

\[ f_1(t) = f_{1o} + \delta f_1(t), f_2(t) = f_{2o} + \delta f_2(t) \]

\[ z_o = 0 \Rightarrow \ddot{z}_o = 0, \]

\[ \theta_{1o} = 0 \Rightarrow \dot{\theta}_{1o} = \ddot{\theta}_{1o} = 0, \]

and let the equilibrium conditions be:

\[ \theta_{2o} = 0 \Rightarrow \dot{\theta}_{2o} = \ddot{\theta}_{2o} = 0, \]

\[ f_o = f_{1o} = f_{2o} = 0 \]

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Insert the perturbations into the dynamic equations, and assuming \( \theta \) is small, substitute \( \sin \theta = \theta \) and \( \cos \theta = 1 \). Keep only the linear terms of the equations:

\[
\delta f(t) = (M + M_1 + M_2) \delta \ddot{x}(t) + M_1 \delta \dot{\theta}_1(t) + M_2 \delta \dot{\theta}_2(t) - \delta f_1(t) - \delta f_2(t)
\]

\[
M_1 g \delta \theta_1(t) - M_1 \delta \ddot{x}(t) + 2I_1 \delta f_1(t) = \frac{4}{3} M_1 I_1^2 \delta \dot{\theta}_1(t)
\]

\[
M_2 g \delta \theta_2(t) - M_2 \delta \ddot{x}(t) + 2I_2 \delta f_2(t) = \frac{4}{3} M_2 I_2^2 \delta \dot{\theta}_2(t)
\]

Solve for \( \delta \ddot{x}(t) \), \( \delta \dot{\theta}_1(t) \), and \( \delta \dot{\theta}_2(t) \), letting \( Z = 4M + M_1 + M_2 \):

\[
\delta \ddot{x}(t) = \frac{4}{Z} \delta f(t) - \frac{3M_1 g}{Z} \delta \theta_1(t) - \frac{3M_2 g}{Z} \delta \theta_2(t) - \frac{2}{Z} \delta f_1(t) - \frac{2}{Z} \delta f_2(t)
\]

\[
\delta \dot{\theta}_1(t) = -\frac{3}{Zl_1} \delta f(t) + \frac{3g(Z + 3M_1)}{4Zl_1} \delta \theta_1(t) + \frac{9M_1 g}{4Zl_1} \delta \theta_2(t) + \frac{3(M_1 + Z)}{2M_1 l_1 Z} \delta f_1(t) + \frac{3}{2Zl_1} \delta f_2(t)
\]

\[
\delta \dot{\theta}_2(t) = -\frac{3}{Zl_2} \delta f(t) + \frac{9M_2 g}{4Zl_2} \delta \theta_1(t) + \frac{3g(Z + 3M_2)}{4Zl_2} \delta \theta_2(t) + \frac{3}{2Zl_2} \delta f_1(t) + \frac{3(M_2 + Z)}{2M_2 l_2 Z} \delta f_2(t)
\]

From (A.4) and the definitions of \( \delta x(t) \), \( \delta u(t) \), and \( \delta d(t) \), the state space representation becomes:

\[
\delta x(t) =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{3M_1 g}{Z} & 0 & -\frac{3M_1 g}{Z} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 3g(Z + 3M_1) & 0 & \frac{9M_1 g}{4Zl_1} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & \frac{9M_2 g}{4Zl_2} & 0 & \frac{3g(Z + 3M_2)}{4Zl_2} & 0
\end{bmatrix}
\]

\[
\delta x(t) +
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\delta u(t) +
\begin{bmatrix}
0 \\
0 \\
0 \\
\frac{4}{Z} \\
\frac{-2}{Z} \\
\frac{-2}{Z}
\end{bmatrix}
\]

\[
\delta d(t) +
\begin{bmatrix}
\frac{\delta f_1(t)}{Zl_1} \\
\frac{\delta f_1(t)}{Zl_1} \\
\frac{\delta f_2(t)}{Zl_2} \\
\frac{\delta f_1(t)}{Zl_1} \\
\frac{\delta f_2(t)}{Zl_2} \\
\frac{\delta f_2(t)}{Zl_2}
\end{bmatrix}
\]

(A.5)
Because the concern is in controlling the position of the cart, the output vector is:

\[ \delta y(t) = [1 \ 0 \ 0 \ 0 \ 0] \delta x(t) \]  

(A.6)

Numerical values for the state equation matrices for the two configurations considered in this thesis, System 1 and System 2, are included in Appendix B.
Appendix B

Open-Loop Poles and Zeros

In this appendix, the open-loop poles and zeros are found in terms of the system parameters for the following transfer functions: the control, $u$, to the cart position, $y$, $T_w(s)$; the disturbance on the shorter rod, $f_1$, to the cart position, $T_{f_1}(s)$; and the disturbance on the longer rod $f_2$ to the cart position, $T_{f_2}(s)$. Their locations for both System 1 and System 2 are given, as well as the numerical values for the state space matrices of Chapter 2 and Appendix A.

Because the locations of the open-loop poles are dependent only on the $A$ matrix, all three transfer functions have the same poles. The open-loop zeros from $u$ to $y$ are found from the transfer function $C(sI-A)^{-1}B$, and the open-loop zeros from $f_1$ and $f_2$ to $y$ are found from $C(sI-A)^{-1}L_1$ and $C(sI-A)^{-1}L_2$, respectively, with the disturbance matrix $L$ rewritten as $L=[L_1 \ L_2]$. Each of these transfer functions is SISO, therefore, the poles are the roots of the denominator and the zeros are the roots of the numerator.

To simplify the transfer function equations, the state space matrices are rewritten with variables representing some of the entries. The matrices and their representations are as follows:
The transfer functions can be expressed as:

\[
T_w(s) = \frac{s^4 \Gamma + s^2 (A \Lambda + B \Pi - \Gamma \Phi - \Gamma X) + \Gamma (X \Phi - E \Delta) + \Lambda (BE - A \Phi) + \Pi (A \Delta - B \Xi)}{s^2 \left( s^4 - s^2 (X + \Phi) + (X \Phi - E \Delta) \right)}
\]

\[
T_{f_{1\nu}}(s) = \frac{s^4 \Theta + s^2 (\Sigma \Lambda + B \Xi - \Theta \Phi - \Theta X) + \Theta (X \Phi - E \Delta) + \Sigma (BE - A \Phi) + \Xi (A \Delta - B \Psi)}{s^2 \left( s^4 - s^2 (X + \Phi) + (X \Phi - E \Delta) \right)}
\]

\[
T_{f_{1\zeta}}(s) = \frac{s^4 \Theta + s^2 (A \Omega + B \Psi - \Theta \Phi - \Theta X) + \Theta (X \Phi - E \Delta) + \Omega (BE - A \Phi) + \Psi (A \Delta - B \Xi)}{s^2 \left( s^4 - s^2 (X + \Phi) + (X \Phi - E \Delta) \right)}
\]
which can be rewritten in terms of the system parameters as:

\[ T_w(s) = \frac{s^4\left(\frac{4}{Z}\right) + s^2\left(-\frac{3g}{l_1Z} + \frac{3g}{l_2Z}\right) + \frac{9}{4}\left(\frac{g^2}{l_1l_2Z}\right)}{s^2(s^2 - X)(s^2 - Y)} \]

\[ T_{f_1}(s) = \frac{s^4\left(-\frac{2}{Z}\right) + s^2\left(-\frac{3g}{l_1Z} + \frac{3g}{2l_2Z}\right) + \frac{9}{4}\left(\frac{g^2}{l_1l_2Z}\right)}{s^2(s^2 - X)(s^2 - Y)} \]

\[ T_{f_2}(s) = \frac{s^4\left(-\frac{2}{Z}\right) + s^2\left(\frac{3g}{2l_1Z} - \frac{3g}{l_2Z}\right) + \frac{9}{4}\left(\frac{g^2}{l_1l_2Z}\right)}{s^2(s^2 - X)(s^2 - Y)} \]

where

\[ X, Y = \frac{3}{8}\left(\frac{g}{Z}\right)\left[\frac{Z + 3M_1}{l_1} + \frac{Z + 3M_2}{l_2}\right] \pm \frac{3}{8}\left(\frac{g}{Z}\right)\sqrt{\left[\frac{Z + 3M_1}{l_1} - \frac{Z + 3M_2}{l_2}\right]^2 + 36\left(\frac{M_1M_2}{l_1l_2}\right)} \]

\[ M_1 = 2m_1l_1, \; M_2 = 2m_2l_2, \; \text{and} \; Z = 4M + M_1 + M_2. \quad \text{(B.3)} \]

From the equations of (B.3), the open-loop poles and zeros are found to be:

open-loop poles: \( 0, 0, \pm \sqrt{X}, \pm \sqrt{Y} \)

zeros of \( T_w(s) \): \( \pm \sqrt[3]{\frac{3}{4} \left(\frac{g}{l_1}\right)}, \pm \sqrt[3]{\frac{3}{4} \left(\frac{g}{l_2}\right)} \)

zeros of \( T_{f_1}(s) \): \( \pm \frac{3}{4} \left(\frac{g}{l_2}\right), \pm j\frac{3}{2} \left(\frac{g}{l_1}\right) \)

zeros of \( T_{f_2}(s) \):
zeros of $T_{f2}(s)$: $\pm \sqrt{\frac{3g}{4l_1}}$, $\pm j\sqrt{\frac{3g}{2l_2}}$

Notice that the transfer functions share some of the same zero locations, and that two of the zeros of the transfer functions from the disturbances to the cart position are complex.

The state space matrices and the open-loop pole and zero locations of the dual inverted pendulum system are given for the following constants and parameter values:

\[ g = 9.81 \text{ m/s}^2, \text{the acceleration of gravity} \]
\[ M = 5 \text{ kg, the mass of the cart} \]
\[ m_1 = m_2 = 0.1 \text{ kg/m, the mass per unit length of each rod} \]
\[ M_1 = 2m_1 l_1, \text{ the mass of the shorter rod} \]
\[ M_2 = 2m_2 l_2, \text{ the mass of the longer rod} \]
\[ Z = 4M + M_1 + M_2 \]

**SYSTEM 1:** $l_1 = 1\text{ m}, l_2 = 16\text{ m}$:

\[
\begin{align*}
\delta x(t) &= \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.2516 & 0 & -4.0263 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 7.5492 & 0 & 3.0197 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0.0118 & 0 & 0.6488 & 0 & 0 \\
\end{bmatrix}
\delta x(t) +
\begin{bmatrix}
0 & 0 & 0.1709 & 0 & 0 & -0.1282 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.0080 & 0 & 0.6488 & 0 & 0 \\
0 & 0 & 0.0118 & 0 & 0.6488 & 0 & 0 \\
0 & 0 & 7.5641 & 0.0641 & 0 & 0 & 0 \\
0 & 0 & 0.0040 & 0.0333 & 0 & 0 & 0 \\
\end{bmatrix}
\delta d(t)
\end{align*}
\]

open-loop poles: 0, 0, $\pm 2.7485, \pm 0.8022$

zeros of $T_{y}(s)$: $\pm 2.7130, \pm 0.6783$

zeros of $T_{f1}(s)$: $\pm 0.6783, \pm j3.8368$

zeros of $T_{f2}(s)$: $\pm 2.7130, \pm j0.9592$

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**SYSTEM 2:** $l_1 = 1 \text{ m}, l_2 = 2 \text{ m}$:

$$\delta x(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.2858 & 0 & -0.5717 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 7.5749 & 0 & 0.4288 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0.1072 & 0 & 3.8946 & 0 \end{bmatrix} \delta x(t) + \begin{bmatrix} 0 \\ 0.1942 \\ 0 \\ -0.1456 \\ 0 \\ -0.0728 \end{bmatrix} \delta u(t) + \begin{bmatrix} 0 \\ -0.0971 \\ 0 \\ 0 \\ 7.5728 \\ 0.0364 \end{bmatrix} \delta d(t)$$

open-loop poles: $0, 0, \pm 2.7545, \pm 1.9703$

zeros of $T_{xy}(s)$: $\pm 2.7130, \pm 1.9184$

zeros of $T_{f_{y},y}(s)$: $\pm 1.9184, \pm j3.8368$

zeros of $T_{f_{2y},y}(s)$: $\pm 2.7130, \pm j2.7130$
Appendix C

Controllability and Observability

In this appendix, the controllability and observability properties of the dual inverted pendulum system are examined using the controllability and observability matrices, written in terms of the system parameters. The loss of controllability or observability because of a pole/zero cancellation is also discussed. Finally, the system parameter relationships which cause a pole/zero cancellation are determined.

A constant coefficient linear system is completely controllable if and only if the matrix

\[ M_c = \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix} \]  \hspace{1cm} (C.1)

has rank \( n \), where \( x(t) \in \mathbb{R}^n \). A constant coefficient linear system is completely observable if and only if the matrix

\[ M_o = \begin{bmatrix} C' & A'C' & (A^2)'C' & \ldots & (A^{n-1})'C' \end{bmatrix} \]  \hspace{1cm} (C.2)

has rank \( n \) [9].

For the dual inverted pendulum system, the \( M_c \) and \( M_o \) matrices are:
\[
M_c = \begin{bmatrix}
0 & 4 & 0 & a & 0 & d \\
4 & 0 & a & 0 & d & 0 \\
0 & -3 & 0 & b & 0 & e \\
\frac{-3}{l_1Z} & 0 & b & 0 & e & 0 \\
0 & \frac{-3}{l_2Z} & 0 & c & 0 & f \\
\frac{-3}{l_2Z} & 0 & c & 0 & f & 0
\end{bmatrix}
\]

\[
a = \frac{9g}{2Z^2} \left( \frac{M_1 + M_2}{l_1} \right),
\]

\[
b = -\frac{9g}{4l_1Z^2} \left( \frac{Z + 3M_1 + 3M_2}{l_1} \right),
\]

\[
c = -\frac{9g}{4l_2Z^2} \left( \frac{Z + 3M_1}{l_2} \right),
\]

\[
d = \frac{27g^2}{2Z^2} \left( \frac{M_1(Z + 3M_1) + M_2(Z + 3M_2) + 6M_1M_2}{l_1^2} \right),
\]

\[
e = -\frac{27g^2}{16l_1Z^3} \left( \frac{(Z + 3M_1)^2 + 3M_1(Z + 6M_1) + 3M_1(Z + 3M_1)}{l_1} \right),
\]

\[
f = \frac{27g^2}{16l_2Z^3} \left( \frac{(Z + 3M_1)^2 + 3M_1(Z + 6M_2) + 3M_1(Z + 3M_1)}{l_2} \right)
\]

(C.3)

\[
M_o = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{3M_1g}{Z} & 0 & -\frac{9M_1g^2}{4Z^2} \left( \frac{Z + 3M_1 + 3M_2}{l_1} \right) & 0 \\
0 & 0 & 0 & -\frac{3M_1g}{Z} & 0 & -\frac{9M_1g^2}{4Z^2} \left( \frac{Z + 3M_1 + 3M_2}{l_1} \right) \\
0 & 0 & -\frac{3M_2g}{Z} & 0 & -\frac{9M_2g^2}{4Z^2} \left( \frac{Z + 3M_2 + 3M_1}{l_2} \right) & 0 \\
0 & 0 & 0 & -\frac{3M_2g}{Z} & 0 & -\frac{9M_2g^2}{4Z^2} \left( \frac{Z + 3M_2 + 3M_1}{l_2} \right)
\end{bmatrix}
\]

(C.4)

For \( l_1 \neq l_2 \), rank\((M_c) = 6 \) and rank\((M_o) = 6 \), and thus the dual inverted pendulum system is both controllable and observable (by definition, as mentioned in Chapter 2, we should actually state that the system is stabilizable and detectable).

When the rods of the system are equal in length, that is, for \( l_1 = l_2 = l \), rank\((M_c) = rank(M_o) = 4 \). The system is then both uncontrollable and unobservable. Upon careful examination of the matrix of (C.3), it is found that the 5th column of \( M_c \) is...
\[
\frac{3g}{41Z} (Z + 3M_1 + 3M_2) \text{ times greater than the } 3^{rd} \text{ column of } M_C. \]

This relationship also holds between the 6\textsuperscript{th} and 4\textsuperscript{th} columns of \(M_C\), and the 5\textsuperscript{th} and 3\textsuperscript{rd}, and 6\textsuperscript{th} and 4\textsuperscript{th}, columns of \(M_o\). This explains the loss in rank by two for the \(M_C\) and \(M_o\) matrices. Note that the multiplying factor is expressed in terms of \(M_1, M_2,\) and \(Z\). The loss of controllability and observability is found to be dependent only upon the lengths of the rods, not their masses.

A system can also lose controllability or observability because of a pole/zero cancellation. For such a loss, not only do the pole and zero have to have the same numerical value, but their "directions" have to be related. The pole and zero locations and their "directions" are defined as follows:

**definition of a pole, \(\lambda_k\):**

\[
\omega_k' (\lambda - A) = 0' \text{ or } (\lambda - A) \nu_k = 0,
\]

with \(\omega_k'\) or \(\nu_k\) representing the "direction" of the pole \(\text{(C.5)}\)

**definition of a zero, \(z_k\):**

\[
\eta_k' (z_k - A) = 0' \text{ or } (z_k - A) \xi_k = 0; \text{ with } \eta_k' \text{ and } \gamma_k' \text{ or } \xi_k \text{ and } u_k
\]

representing the "direction" of the zero \(\text{(C.6)}\)

A loss in controllability due to a pole/zero cancellation occurs if \(\lambda_k = z_k\) and \(\eta_k' = \beta \omega_k'\). Then \(\gamma_k' C = 0'\) and \(\omega_k' B = 0'\). The latter implies the \(k^{th}\) mode is uncontrollable. A loss in observability occurs if \(\lambda_k = z_k\) and \(\xi_k = \beta \nu_k\). Then \(Bu_k = 0\) and \(Cv_k = 0\), the latter implying that the \(k^{th}\) mode is unobservable \([5]\).
As shown in Figure 2.5, and again in Figure C.1, when the lengths of the rods are equal, \( l_1 = l_2 = 1 \text{m} \), there are pole/zero cancellations at \( \pm 2.7130 \).

![Figure C.1 Open-Loop Pole and Zero Locations as the Length of the Longer Rod, \( l_2 \), is Decreased](image)

Even though it was shown that the system is uncontrollable and unobservable for rods of equal length from determining the ranks of the \( M_c \) and \( M_o \) matrices, we examine these properties from the viewpoint of the pole/zero cancellations.

For \( \lambda_k = z_k = +2.7130 \),

\[
\begin{bmatrix}
0 \\
0 \\
-0.6635 \\
-0.2446 \\
0.6635 \\
0.2446
\end{bmatrix} = \eta'_k \text{ and } \begin{bmatrix}
0 \\
0 \\
0.2446 \\
0.6635 \\
-0.2446 \\
-0.6635
\end{bmatrix} = \xi_k
\]
and for \( \lambda_k = z_k = -2.7130 \),

\[
\begin{bmatrix}
0 \\
0 \\
0.2446 \\
-0.6635 \\
-0.2446 \\
0.6635
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0.2446 \\
-0.6635 \\
-0.2446 \\
0.6635
\end{bmatrix} = \xi_k
\]

Thus, these pole/zero cancellations also imply that the system is both uncontrollable and unobservable.

It has been shown that when the lengths of the rods are equal, the system loses both controllability and observability. We now determine if there are any other system parameter relationships that cause a pole/zero cancellation. Recall that the pole locations were found to be \( 0, 0, \pm \sqrt{X}, \pm \sqrt{Y} \) where

\[
X, Y = \frac{3}{8} \left( \frac{g}{Z} \right) \left[ \frac{Z + 3M_1}{l_1} + \frac{Z + 3M_2}{l_2} \right] \pm \frac{3}{8} \left( \frac{g}{Z} \right) \sqrt{\left[ \frac{Z + 3M_1}{l_1} - \frac{Z + 3M_2}{l_2} \right]^2 + 36 \left( \frac{M_1 M_2}{l_1 l_2} \right)}
\]

(C.7)

and the zero locations were \( \pm \sqrt[4]{\frac{3g}{l_1}}, \pm \sqrt[4]{\frac{3g}{l_2}} \).

(C.8)

Rewrite (C.7) as \( X, Y = \frac{3}{8} \left( \frac{g}{Z} \right) \alpha \pm \frac{3}{8} \left( \frac{g}{Z} \right) \sqrt{\beta} \) and (C.8) as \( \pm \sqrt{\gamma}, \pm \sqrt{\delta} \).

We then want to find the relationships of the system parameters such that
There are four possible combinations of these equations:

(i) Adding equations (1) and (3) gives $\alpha = \frac{2Z}{l_1}$, and adding equations (2) and (4) gives $\alpha = \frac{2Z}{l_2}$. This implies that $l_1 = l_2$ gives a pole and zero at the same location.

(ii) Adding equations (1) and (4), or (2) and (3), gives $\alpha = \frac{Z}{l_1} + \frac{Z}{l_2}$. Substituting for $\alpha$ gives the equation $\frac{3M_1}{l_1} + \frac{3M_2}{l_2} = 0$. Recalling that $M_1 = 2m_1l_1$ and $M_2 = 2m_2l_2$, the solution is $m_1 = -m_2$.

(iii) Equation (2) minus equation (3) gives $\frac{3}{4}(\frac{g}{Z})\sqrt{\beta} = \delta - \gamma$, and (1) minus (4) gives $\frac{3}{4}(\frac{g}{Z})\sqrt{\beta} = \gamma - \delta$. Setting these equations equal to each other gives $\frac{1}{l_2} - \frac{1}{l_1} = \frac{1}{l_1} - \frac{1}{l_2}$, which implies $l_1 = l_2$.

(iv) Equation (1) minus equation (3), and (2) minus (4), both give $\frac{3}{4}(\frac{g}{Z})\sqrt{\beta} = 0$. This indicates that if the system parameters are chosen such that $\beta = \left[\frac{Z + 3M_1}{l_1} - \frac{Z + 3M_2}{l_2}\right]^2 + 36\frac{M_1M_2}{l_1l_2} = 0$, there is a
pole/zero cancellation.

The only relationship of the system parameters that gives a pole/zero cancellation is $l_1 = l_2$. The solutions to both (ii) and (iv) imply that some of the system parameters would have to be negative or imaginary. Thus, the dual inverted pendulum system loses controllability and observability only when the lengths of the rods are equal, and it can not lose just controllability or just observability.
Appendix D

Additional Time Responses

This appendix shows the closed-loop time responses for various initial conditions of System 1 and System 2 (again, no disturbances are assumed to act on the system). Each plot shows the cart position, rod angles, and required controls of both the LQR design and the frequency weighted $H_\infty$ with full state feedback design.

For most of the initial conditions, the cart position and rod angles behave similarly for both the LQR and frequency weighted $H_\infty$ designs, with the $H_\infty$ design having faster transients and larger perturbations. In Figures D.5, D.7, and D.8 however, the time responses do not behave in this manner. For these initial conditions, the $H_\infty$ design still has the larger perturbations, but the transient times are almost equal. Also, the initial control is positive for the LQR design (initially pushing the cart), while for the $H_\infty$ design, the initial control is negative (initially pulling the cart), thus causing the opposite behavior in the state variable responses.

For some of the initial conditions, the assumption for small angles used in Appendix A to find the state space representation is violated for the $H_\infty$ design (note the angle of the shorter rod, $\theta_1$, in Figures D.3, D.9, and D.10).
Figure D.1 Time Response, System 1, $x(0) = [0 \ 0 \ 1' \ 0 \ 0 \ 0]$
Figure D.2 Time Response, System 2, $x(0) = [0 \ 0 \ 1^\circ \ 0 \ 0 \ 0]$
Figure D.3 Time response, System 1, $x(0)=[0\ 0\ 0\ 0\ 1\ 0]$
Figure D.4 Time Response, System 2, $x(0)=[0 \ 0 \ 0 \ 0 \ 1^\circ \ 0]$
Figure D.5 Time Response, System 1, $x(0) = \begin{bmatrix} 1 & 0 & -1' & 0 & -1' & 0 \end{bmatrix}$
Figure D.6 Time Response, System 2, $x(0)=[1 \ 0 \ -1' \ 0 \ -1' \ 0]$
Figure D.7 Time Response, System 1, $x(0) = [1 \ 0 \ 1' \ 0 \ -1' \ 0]$
Figure D.8  Time Response, System 2, $x(0)=[1 \ 0 \ 1^\circ \ 0 \ -1^\circ \ 0]$
Figure D.9 Time Response, System 1, $x(0) = [1 \ 0 \ -1' \ 0 \ 1' \ 0]$
Figure D.10 Time Response, System 2, $x(0) = [1 \ 0 \ -1' \ 0 \ 1' \ 0]$
References


