Lenses and imaging

• Huygens principle and why we need imaging instruments
• A simple imaging instrument: the pinhole camera
• Principle of image formation using lenses
• Quantifying lenses: paraxial approximation & matrix approach
• “Focusing” a lens: Imaging condition
• Magnification
• Analyzing more complicated (multi-element) optical systems:
  – Principal points/surfaces
  – Generalized imaging conditions from matrix formulae
The minimum path principle

∫ \Gamma \ n(x, y, z) \ dl \quad \text{Γ is chosen to minimize this "path" integral, compared to alternative paths}

(aka Fermat’s principle)
Consequences: law of reflection, law of refraction
The law of refraction

\[ n \sin \theta = n' \sin \theta' \]

Snell’s Law of Refraction
Ray bundles

point
source

spherical
wave
(diverging)

point
source

very-very
far away

plane
wave

\[ \infty \]
Huygens principle

Each point on the wavefront acts as a secondary light source emitting a spherical wave.

The wavefront after a short propagation distance is the result of superimposing all these spherical wavelets.

- optical wavefronts
Why imaging systems are needed

- Each point in an object scatters the incident illumination into a spherical wave, according to the Huygens principle.
- A few microns away from the object surface, the rays emanating from all object points become entangled, delocalizing object details.
- To relocalize object details, a method must be found to reassign (“focus”) all the rays that emanated from a single point object into another point in space (the “image.”)
- The latter function is the topic of the discipline of Optical Imaging.
The pinhole camera blocks all but one ray per object point from reaching the image space $\Rightarrow$ an image is formed (i.e., each point in image space corresponds to a single point from the object space).

- Unfortunately, most of the light is wasted in this instrument.
- Besides, light diffracts if it has to go through small pinholes as we will see later; diffraction introduces artifacts that we do not yet have the tools to quantify.
Lens: main instrument for image formation

The curved surface makes the rays bend proportionally to their distance from the “optical axis”, according to Snell’s law. Therefore, the divergent wavefront becomes convergent at the right-hand (output) side.
Analyzing lenses: paraxial ray-tracing

- Free-space propagation
- Refraction at air-glass interface
- Free-space propagation
- Refraction at glass-air interface
Paraxial approximation /1

• In paraxial optics, we make heavy use of the following approximate (1st order Taylor) expressions:

\[
\sin \varepsilon \approx \varepsilon \approx \tan \varepsilon \quad \cos \varepsilon \approx 1
\]

\[
\sqrt{1 + \varepsilon} \approx 1 + \frac{1}{2} \varepsilon
\]

where \( \varepsilon \) is the angle between a ray and the optical axis, and is a small number (\( \varepsilon \ll 1 \text{ rad} \)). The range of validity of this approximation typically extends up to \( \sim 10\text{-}30 \) degrees, depending on the desired degree of accuracy. This regime is also known as “Gaussian optics.”

Note the assumption of existence of an optical axis (\( i.e., \) perfect alignment!)
Paraxial approximation

Apply Snell’s law as if ray bending occurred at the intersection of the *axial* ray with the lens.

Ignore the distance between the location of the axial ray intersection and the actual off-axis ray intersection.

Valid for small curvatures & thin optical elements.
Example: one spherical surface, translation+refraction+translation

- **R**: radius of spherical surface

Paraxial rays (approximation valid)

- Non-paraxial ray (approximation gives large error)

medium 1

- index \( n \), e.g. air \( n = 1 \)

medium 2

- index \( n' \), e.g. glass \( n' = 1.5 \)

center of spherical surface
Translation+refraction+translation /1

Starting ray: location $x_0$ direction $\alpha_0$

Translation through distance $D_{01}$ (+ direction):

$$\begin{align*}
  \alpha_1 &= \alpha_0 \\
x_1 &= x_0 + D_{01} \alpha_0
\end{align*}$$

Refraction at positive spherical surface:

$$\begin{align*}
  \alpha'_1 &= \frac{n}{n'} \alpha_1 + \left[ \frac{(n-n')}{n'R} \right] x_1 \\
x'_1 &= x_1
\end{align*}$$
Translation+refraction+translation /2

Translation through distance $D_{12}$ (+ direction):

$$\begin{align*}
x_2 &= x_1 + D_{12}\alpha'_1 \\
\alpha_2 &= \alpha'_1
\end{align*}$$

Put together:
Translation+refraction+translation /3

\[ x_2 = \left[ \frac{(n-n')D_{12}}{n'R} + 1 \right] x_0 + \left[ \frac{D_{01}}{n'} + \frac{nD_{12}}{n'R} + \frac{(n-n')D_{01}D_{12}}{n'R} \right] \alpha_0 \]

\[ \alpha_2 = \left[ \frac{n-n'}{n'R} \right] x_0 + \left[ \frac{n}{n'} + \frac{(n-n')D_{01}}{n'R} \right] \alpha_0 \]
Sign conventions for refraction

- Light travels from left to right
- A radius of curvature is positive if the surface is convex towards the left
- Longitudinal distances are positive if pointing to the right
- Lateral distances are positive if pointing up
- Ray angles are positive if the ray direction is obtained by rotating the $+z$ axis counterclockwise through an acute angle
On-axis image formation

All rays emanating at $x_0$ arrive at $x_2$ irrespective of departure angle $\alpha_0$

$$\frac{n'}{D_{12}} + \frac{n}{D_{01}} = \frac{n' - n}{R}$$

“Power” of the spherical surface [units: diopters, 1D=1 m$^{-1}$]
Magnification: lateral (off-axis), angle

\[ x_0 = 0 \]

\[ \alpha_0 \]

\[ \Delta \alpha_0 \]

\[ \alpha_2 \]

\[ \Delta \alpha_2 \]

\[ D_{01} \]

\[ D_{12} \]

\[ x_2 \]

Lateral

\[ m_x = \frac{x_2}{x_0} = \frac{n - n'}{R} \frac{D_{12}}{n'} + 1 = \ldots = -\frac{n}{n'} \frac{D_{12}}{D_{01}} \]

Angle

\[ m_\alpha = \frac{\Delta \alpha_2}{\Delta \alpha_0} = -\frac{D_{01}}{D_{12}} \]
Object-image transformation

\[ x_2 = m_x x_0 \]

\[ \alpha_2 = -\frac{1}{f'} x_0 + m_\alpha \alpha_0 \]

Ray-tracing transformation (paraxial) between object and image points
Image of point object at infinity

\[ \begin{align*}
\alpha_0 &= 0 \\
x_0 &
\end{align*} \]

\[ \begin{align*}
D_{01} &= \infty \\
D_{12} &
\end{align*} \]

\[ \begin{align*}
\alpha_2 &= 0 \\
x_0 &
\end{align*} \]

\[ \begin{align*}
x_2 &= 0 \\

\frac{n'}{D_{12}} &= \frac{n' - n}{R} \\
D_{12} &= \frac{n'R}{n' - n} \equiv f' : \text{image focal length}
\end{align*} \]
Point object imaged at infinity

\[ x_0 = 0 \]

\[ D_{01} = \frac{n}{n' - n} R \quad \Rightarrow \quad D_{01} = \frac{n'R}{n' - n} \equiv f : \text{object focal length} \]
Matrix formulation /1

\[ x_1 = x_0 + D_{01} \alpha_0 \]
\[ \alpha_1 = \alpha_0 \]

translation by distance \( D_{01} \)

\[ \alpha_{out} = M_{11} \alpha_{in} + M_{12} x_{in} \]
\[ x_{out} = M_{21} \alpha_{in} + M_{22} x_{in} \]

form common to all

\[ x'_1 = x_1 \]
\[ \alpha'_1 = \frac{n}{n'} \alpha_1 + \left[ \frac{(n-n')}{n'R} \right] x_1 \]

refraction by surface with radius of curvature \( R \)

\[ x_2 = m_x x_0 \]
\[ \alpha_2 = -\frac{1}{f'} x_0 + m_{\alpha} \alpha_0 \]

ray-tracing object-image transformation
Matrix formulation /2

\[ \begin{align*}
\alpha_{\text{out}} &= M_{11} \alpha_{\text{in}} + M_{12} x_{\text{in}} \\
x_{\text{out}} &= M_{21} \alpha_{\text{in}} + M_{22} x_{\text{in}}
\end{align*} \]

\[ \begin{pmatrix} n_{\text{out}} \alpha_{\text{out}} \\ x_{\text{out}} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} n \alpha_{\text{in}} \\ x_{\text{in}} \end{pmatrix} \]

**Refraction by spherical surface**

\[ x_1' = x_1 \]
\[ \alpha_1' = \frac{n}{n'} \alpha_1 + \left[ \frac{(n - n')}{n'R} \right] x_1 \]

\[ \begin{pmatrix} n' \alpha_1' \\ x_1' \end{pmatrix} = \begin{pmatrix} 1 & \frac{n' - n}{R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n \alpha_1 \\ x_1 \end{pmatrix} \]

**Translation through uniform medium**

\[ x_1 = x_0 + D_{01} \alpha_0 \]
\[ \alpha_1 = \alpha_0 \]

\[ \begin{pmatrix} n \alpha_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{D_{01}} & 0 \\ n & 1 \end{pmatrix} \begin{pmatrix} n \alpha_0 \\ x_0 \end{pmatrix} \]
Translation+refraction+translation

\[
\begin{pmatrix}
    n' \\
    x_2
\end{pmatrix} = \begin{pmatrix}
    \text{translation by } D_{12} \\
    \text{by r.curv. } R \\
    \text{translation by } D_{01}
\end{pmatrix} \begin{pmatrix}
    n \\
    x_0
\end{pmatrix}
\]

result…

\[
n' \alpha_2 = \left[ \frac{n - n'}{R} \right] x_0 + \left[ n + \frac{(n - n')D_{01}}{R} \right] \alpha_0
\]

\[
x_2 = \left[ \frac{(n - n')D_{12}}{n'R} + 1 \right] x_0 + \left[ D_{01} + \frac{nD_{12}}{n'} + \frac{(n - n')D_{01}D_{12}}{n'R} \right] \alpha_0
\]
Thin lens

\[
\begin{pmatrix}
\alpha_{\text{out}} \\
x_{\text{out}}
\end{pmatrix} = \begin{pmatrix}
1 & -[P + P'] \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\alpha_{\text{in}} \\
x_{\text{in}}
\end{pmatrix}
\]

\[
P_{\text{thin lens}} = \frac{n-1}{R} + \frac{1-n}{R'} = (n-1) \left( \frac{1}{R} - \frac{1}{R'} \right)
\]

Lens-maker’s formula
The power of surfaces

- Positive power bends rays “inwards”

- Negative power bends rays “outwards”
The power in matrix formulation

\[
\begin{pmatrix}
  n_{\text{out}} \alpha_{\text{out}} \\
  x_{\text{out}}
\end{pmatrix} =
\begin{pmatrix}
  M_{11} & M_{12} \\
  M_{21} & M_{22}
\end{pmatrix}
\begin{pmatrix}
  n_{\text{in}} \alpha_{\text{in}} \\
  x_{\text{in}}
\end{pmatrix}
\]

\[\alpha_{\text{in}} = 0\]

\[
M_{12} = -\frac{n-1}{R}
\]

(Ray bending) = (Power) × (Lateral coordinate)

⇒ (Power) = \(-M_{12}\)
Power and focal length

\[ \alpha_{\text{in}} = 0 \]

\[ \begin{aligned}\alpha_{\text{in}} = 0 \quad &\quad \alpha_{\text{out}} \begin{cases} < 0 \\ > 0 \end{cases} \quad \Rightarrow \quad (D > 0) \\ x_{\text{in}} + D \alpha_{\text{out}} = 0 \quad \Rightarrow \quad D = -1/M_{12} \end{aligned} \]

\[ \begin{aligned} \alpha_{\text{in}} = 0 \quad &\quad \alpha_{\text{out}} \begin{cases} > 0 \\ < 0 \end{cases} \quad \Rightarrow \quad (D < 0) \\ x_{\text{in}} + D \alpha_{\text{out}} = 0 \quad \Rightarrow \quad D = -1/M_{12} \end{aligned} \]

(Focal length) = 

\[ \frac{1}{(\text{Power})} = \frac{1}{M_{12}} \]

Simple spherical refractor (positive)

Simple spherical refractor (negative)
Thick/compound elements: focal & principal points (surfaces)

Note: in the paraxial approximation, the focal & principal surfaces are flat \((i.e.,\) planar). In reality, they are curved (but not spherical!!). The exact calculation is very complicated.
Focal Lengths for thick/compound elements

**EFL**: Effective Focal Length (or simply “focal length”)

**FFL**: Front Focal Length

**BFL**: Back Focal Length
PSs and FLs for thin lenses

\[ \frac{1}{(EFL)} \equiv P = P_1 + P_2 \quad (BFL) = (EFL) = (FFL) \]

- The principal planes coincide with the (collocated) glass surfaces
- The rays bend precisely at the thin lens plane (=collocated glass surfaces & PP)
The significance of principal planes /1

[Diagram showing principal planes and optical system]

1st PS

1st FP

generalized optical system

2nd PS

2nd FP

thin lens of the same power

located at the 2nd PS for rays passing through 2nd FP
The significance of principal planes /2

generalized optical system

thin lens of the same power
located at the 1st PP for rays passing through 1st FP
Imaging condition: ray-tracing

- Image point is located at the common intersection of all rays which emanate from the corresponding object point.
- The two rays passing through the two focal points and the chief ray can be ray-traced directly.
Imaging condition: matrix form /1

\[
\begin{bmatrix}
1 & 0 \\
S'/n' & 1
\end{bmatrix}
\begin{bmatrix}
1 & -P \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
S/n & 1
\end{bmatrix}
= \begin{bmatrix}
1 - \frac{PS}{n} & -P \\
\frac{S'}{n'} & n - \frac{PSS'}{nn'}
\end{bmatrix}
\]
Imaging condition: matrix form /2

\[ n \begin{bmatrix} x \end{bmatrix} = 2^{nd} \text{FP} \]

Output coordinate \( x' \) must not depend on entrance angle \( \gamma \)

\[
\begin{pmatrix} n' \gamma' \\ x' \end{pmatrix} = \begin{pmatrix} 1 - \frac{PS}{n} & -P \\ \frac{S'}{n'} + \frac{S}{n} - \frac{PSS'}{nn'} & 1 - \frac{PS'}{n'} \end{pmatrix} \begin{pmatrix} n \gamma \\ x \end{pmatrix} = 0
\]
Imaging condition: matrix form /3

\[ \frac{S'}{n'} + \frac{S}{n} - \frac{PSS'}{nn'} = 0 \iff \frac{n}{S} + \frac{n'}{S'} = P \]

system immersed in air,
\[ n = n' = 1; \]
power \( P = 1/f \)

\[ \frac{1}{S} + \frac{1}{S'} = \frac{1}{f} \]
Lateral magnification

\[
\begin{align*}
\begin{pmatrix} n' \gamma' \\ x' \end{pmatrix} &= \begin{pmatrix}
1 - \frac{PS}{n} & -P \\
0 & 1 - \frac{PS'}{n'}
\end{pmatrix}
\begin{pmatrix} n \gamma \\ x \end{pmatrix} \\
\Rightarrow m_x & \equiv \frac{x'}{x} = 1 - \frac{PS'}{n'}
\end{align*}
\]

(assume imaging condition is satisfied)
Angular magnification

\[ (n' \gamma') = \begin{pmatrix} 1 - \frac{PS}{n} & -P \\ 0 & 1 - \frac{PS'}{n'} \end{pmatrix} \begin{pmatrix} n \gamma \\ x \end{pmatrix} \Rightarrow \]

\[ m_a \equiv \frac{\Delta \gamma'}{\Delta \gamma} = \frac{n}{n'} \left( 1 - \frac{PS}{n} \right) \]
Generalized imaging conditions

\[
\begin{pmatrix}
  n' \\
  x'
\end{pmatrix}
= \begin{pmatrix}
  M_{11} & M_{12} \\
  M_{21} & M_{22}
\end{pmatrix}
\begin{pmatrix}
  n\alpha \\
  x
\end{pmatrix}
\]

- **Power:** \( P = -M_{12} \neq 0 \)
- **Imaging condition:** \( M_{21} = 0 \)
- **Lateral magnification:** \( m_x = M_{22} \)
- **Angular magnification:** \( m_a = \frac{n}{n'} M_{11} \)