Fall 2002 13.00 Problem Set 8 — Answers

1.

$$\vec{\nabla}^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$$

$$\begin{aligned}
\text{L.H.S.} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) \\
\frac{\partial p}{\partial r} &= \frac{\partial}{\partial r}\left(\frac{A}{r}e^{-i(kr-\omega t)}\right) \\
&= \left(-\frac{A}{r^2} - \frac{iAk}{r}\right) e^{-i(kr-\omega t)} \\
r^2\frac{\partial p}{\partial r} &= (-A - iAkr)\, e^{-i(kr-\omega t)} \\
\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) &= (-iAk + (-A - iAkr)(-ik))\, e^{-i(kr-\omega t)} \\
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) &= \left(-\frac{Ak^2}{r}\right) e^{-i(kr-\omega t)}
\end{aligned}$$

$$\begin{aligned}
\text{R.H.S.} &= \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \\
&= \frac{1}{c^2}\left(-\frac{A\omega^2}{r}e^{-i(kr-\omega t)}\right)
\end{aligned}$$

Using the property, $c = \lambda f = \frac{\omega}{k}$, we can show that the L.H.S. is the same as R.H.S. (QED)

3.(a)

$$I_{ref} = \frac{p_{ref}^2}{\rho c} = \frac{(1 \times 10^{-6}\ \text{N/m}^2)^2}{(1000\ \text{kg/m}^3 \times 1500\ \text{m/s})} = 0.67 \times 10^{-18}\ \text{Watt/m}^2$$

(b)

$$0.0002\ \text{dyne/cm}^2 \times \frac{10^4\ \text{cm}^2}{1\ \text{m}^2} \times \frac{10^-5\ \text{N}}{1\ \text{dyne}} = 2 \times 10^{-5}\ \text{N/m}^2$$

$$I_{ref} = \frac{p_{ref}^2}{\rho c} = \frac{(2 \times 10^{-5})^2}{(1.21 \times 343)} = 9.63 \times 10^{-13}\ \text{Watt/m}^2$$

(c)

$$SL = 10 \log \frac{I}{I_{ref}}$$

For the whisper, $\mathcal{P} = 10^{-10}$ W and for the shout, $\mathcal{P} = 10^{-5}$ W. Assuming that both the shout and the whisper are omni-directional, we can divide by 4π to get the intensities: $I_{whisper} = 7.9^{-12}$ and $I_{shout} = 7.9^{-7}$.

Using the air reference:

$$I_{whisper} = 10 \log \left(\frac{7.9^{-12}}{9.63 \times 10^{-13}} \right) = 9.1 \text{ dB re } 0.0002 \text{ dyne/cm}^2$$

$$I_{shout} = 10 \log \left(\frac{7.9^{-7}}{9.63 \times 10^{-13}} \right) = 59.1 \text{ dB re } 0.0002 \text{ dyne/cm}^2.$$

Using the underwater reference:

$$I_{whisper} = 10 \log \left(\frac{7.9^{-12}}{0.67 \times 10^{-18}} \right) = 70.7 \text{ dB re } 1 \ \mu \text{ Pa}$$

$$I_{shout} = 10 \log \left(\frac{7.9^{-7}}{0.67 \times 10^{-18}} \right) = 120.7 \text{ dB re } 1 \ \mu \text{ Pa}$$

(d) If the whole world shouted at once (in the same place),

$$I = (6 \times 10^9) \times 7.9^{-7} W/m^2 = 4.74 \times 10^3 \quad W/m^2$$

Calculate sound pressure level in dB:

$$SL = 10 \log \left(\frac{4.74 \times 10^3}{9.63 \times 10^{-13}} \right) = 157 \text{ dB re } 0.0002 \text{ dyne/cm}^2$$

$$SL = 10 \log \left(\frac{4.74 \times 10^3}{0.67 \times 10^{-18}} \right) = 218 \text{ dB re } 1 \ \mu \text{ Pa.}$$

(e) For the rock band:

$$140 \text{ dB} = 10 \log \left(\frac{\text{I}}{9.63 \times 10^{-13}} \right)$$

$$I = (9.63 \times 10^{-13}) \times 10^{14} = 96.3 \text{ Watts/m}^2.$$

What is the corresponding sound pressure level (SPL) in water?

$$10\log\left(\frac{96.3}{0.67\times10^{-18}}\right) = 202 \text{ dB re } 1\ \mu \text{ Pa}.$$

(f) The rock band would be louder than the whale.

4. (a) From the definition, the intensity of A, I_A satisfies:

$$100 = 10\log_{10}\frac{I_A}{I_{ref}}, \quad \Rightarrow \quad \frac{I_A}{I_{ref}} = 10^{10}$$

Those of B and C satisfy respectively,

$$100 = 10\log_{10}\frac{I_B}{I_{ref}} \quad \Rightarrow \quad \frac{I_B}{I_{ref}} = 10^{10}$$
$$90 = 10\log_{10}\frac{I_C}{I_{ref}} \quad \Rightarrow \quad \frac{I_C}{I_{ref}} = 10^{9}$$

Therefore, $I_{A+B} = (I_A + I_B)$ gives us

$$\begin{aligned} 10\log_{10}\left(\frac{I_A+I_B}{I_{ref}}\right) &= 10\log_{10}\left(\frac{I_A}{I_{ref}}+\frac{I_B}{I_{ref}}\right) \\ &= 10\log_{10}(2\cdot10^{10}) = 10(\log_{10}2+10) \\ &\cong 103.0 \text{ dB re } 1\ \mu \text{ Pa} \end{aligned}$$

(b)

$$\begin{aligned} 10\log_{10}\left(\frac{I_A+I_C}{I_{ref}}\right) &= 10\log_{10}\left(\frac{I_A}{I_{ref}}+\frac{I_C}{I_{ref}}\right) \\ &= 10\log_{10}(10^{10}+10^{9}) \cong 100.4 \text{ dB re } 1\ \mu \text{ Pa} \end{aligned}$$

(c)

$$\begin{aligned} 10\log_{10}\left(\frac{I_A+I_B+I_C}{I_{ref}}\right) &= 10\log_{10}\left(\frac{I_A}{I_{ref}}+\frac{I_B}{I_{ref}}+\frac{I_C}{I_{ref}}\right) \\ &= 10\log_{10}(10^{10}+10^{10}+10^{9}) \cong 103.2 \text{ dB re } 1\ \mu \text{ Pa} \end{aligned}$$

5.

1. $f = 30$ kHz; $\lambda = c/f = 1500/15000 = 0.05$m.

   $\tan\theta_{3dB} = (25/1000) \Longrightarrow \theta_{3dB} = \pm 1.43^\circ$

   $\theta_{3dB} = \pm\frac{29.5\lambda}{D} = \pm 1.43$

   $D = \frac{29.5(0.05)}{1.43} = 1.03$ meters.

2. $\mathrm{DI} = 10\log(\frac{\pi \mathrm{D}}{\lambda})^2 = 10\log(\frac{1.03\pi}{0.05})^2 = 36 \;\; \mathrm{dB}$

3. $\mathrm{SL} = 171 + 10\log\mathcal{P} + DI = 171 + 10\log(1) + 36 = 207$ dB re 1 μPa at 1 meter.

4. From matlab, $\alpha \approx 7.6 \;\; \mathrm{dB/km}$.

   $\mathrm{TL} = 20\log r + \alpha r \times 10^{-3} = 20\log 1000 + 1(7.6) = 60 + 7.6 = 67.6$ dB re 1 meter.

5. Use the formula for echo level from Urick (and handout 2): EL = SL - 2 TL + TS = 207 - 2(67.6) = 72 dB re 1 μPa at 1 meter.

6. When ship rolls $\pm$ 10 degrees, the insonified area will vary between $\pm$ (10 + θ_{3dB}) = $\pm$11.5 degrees in the across track direction. $1000\tan(11.5^\circ) = 203$ meters. Hence the ensonified region of the bottom will be up to $\pm$ 203 meters wide.

   Note also that it takes 2 $\times$ 1000m $\div$ 1500m/sec $\approx$ 1.3 seconds for a sound ping to travel from the ship to the bottom and back. Depending on the roll rate of the ship, there is a good chance the beam will be pointing in a different direction at reception than for transmission of the pulse. This would result in a much weaker detection from one of the sidelobes of the beam pattern, or in no detection at all.

6.(c) Expand $D\rho/Dt$ using the definition of substantial derivative,

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + (\vec{u}\cdot\nabla)\rho$$

and insert in Equation 1 to yield

$$\frac{\partial\rho}{\partial t} + (\vec{u}\cdot\nabla)\rho + \rho(\nabla\cdot\vec{u}) = 0.$$

The last two terms are the result of applying the chain rule to the divergence of a scalar times a vector, i.e.

$$\nabla\cdot(\rho\vec{u}) = (\vec{u}\cdot\nabla)\rho + \rho(\nabla\cdot\vec{u})$$

and we get the result:

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\vec{u}).$$

(d) We can neglect the effect of gravity in deriving the wave equation because we are assuming infinitesimal motion. Applying the definition of the substantial derivative Du/Dt and applying Equation 4 yields:

$$\rho\left[\frac{\partial u}{\partial t} + (\vec{u}\cdot\nabla)u\right] = \left\{-\frac{\partial p}{\partial x} + 0 + 0\right\}$$

$$\rho\left[\frac{\partial v}{\partial t} + (\vec{u}\cdot\nabla)v\right] = \left\{0 - \frac{\partial p}{\partial y} + 0\right\}$$

$$\rho\left[\frac{\partial w}{\partial t} + (\vec{u}\cdot\nabla)w\right] = \left\{0 + 0 - \frac{\partial p}{\partial z}\right\}$$

(f) The perturbations for pressure and density are $p = p_0 + p'$, $\rho = \rho_0 + \rho'$. For the three components of velocity we have $u = u_0 + u'$, $v = v_0 + v'$, and $w = w_0 + w'$, however, we are told that $u_0 = v_0 = w_0 = 0$. Hence, the velocity vector $\vec{u}$ is simply $\vec{u'} = u'\vec{i} + v'\vec{j} + v'\vec{k}$. Applying these to the mass equation yields:

$$\frac{\partial(\rho_0 + \rho')}{\partial t} = -\nabla\cdot((\rho_0 + \rho')\vec{u'})$$

$$\frac{\partial\rho_0}{\partial t} + \frac{\partial\rho'}{\partial t} = -\rho_0\nabla\cdot\vec{u'} - \nabla\cdot(\rho'\vec{u'})$$

Since ρ_0 is constant, $\partial\rho_0/\partial t =$ zero. Further, $\nabla\cdot(\rho'\vec{u'})$ is a second-order term, and can be dropped, yielding:

$$\frac{\partial\rho'}{\partial t} = -\rho_0\nabla\cdot\vec{u'}$$

For the momentum equations, first consider the equation for x:

$$\rho\left[\frac{\partial u'}{\partial t} + (\vec{u'}\cdot\nabla)u'\right] = \frac{\partial(p_0+p')}{\partial x}$$

$$\rho\left[\frac{\partial u'}{\partial t} + (u'\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y} + w'\frac{\partial u'}{\partial z})\right] = \frac{\partial p_0}{\partial x} + \frac{\partial p'}{\partial x}$$

The three terms that comprise the 2nd part of left hand side of this equation are all quadratic, and hence can be dropped. Since p_0 is constant, $\partial p_0/\partial x$ is zero, and we are left with:

$$\rho\frac{\partial u'}{\partial t} = \frac{\partial p'}{\partial x}$$

We apply the same process to the y and z momentum equations to yield:

$$\rho\frac{\partial v'}{\partial t} = \frac{\partial p'}{\partial y}$$

$$\rho\frac{\partial w'}{\partial t} = \frac{\partial p'}{\partial z}$$

Which can be combined into a single vector equation:

$$\rho\frac{\partial \vec{u'}}{\partial t} = \nabla\cdot p'$$

(g) Differentiate the new mass equation with respect to time

$$\frac{\partial}{\partial t}\left(\frac{\partial \rho'}{\partial t}\right) = \frac{\partial}{\partial t}\left(-\rho_0\nabla\cdot\vec{u'}\right)$$

$$\frac{\partial^2 p'}{\partial t^2} = -\rho_0\frac{\partial}{\partial t}(\nabla\cdot\vec{u'}) \tag{1}$$

and take the divergence of the momentum equations:

$$\nabla\cdot\left(\rho\frac{\partial u'}{\partial t}\right) = \nabla\cdot\left(\frac{\partial p'}{\partial x}\right)$$

$$\rho(\nabla\cdot\frac{\partial\vec{u'}}{\partial t}) = \nabla^2 p' \tag{2}$$

Because $\nabla\cdot(\partial\vec{u'}/\partial t) = \partial/\partial t(\nabla\cdot\vec{u'})$, the right hand side of Equation 1 equals the left hand side of Equation 2, yielding the desired result:

$$\frac{\partial^2}{\partial t^2}\rho' = \nabla^2 p'.$$

(i) Manipulate Equation 3 to solve for p:

$$\rho = \frac{\rho_\phi}{1-p/k}$$

$$\rho(1 - p/k) = \rho_\phi$$
$$k\rho - \rho p = k\rho_\phi$$
$$p = k\left[1 - \frac{\rho_\phi}{\rho}\right].$$

Subsititute $k = c^2\rho$ into this equation to yield

$$p = c^2\rho\left[1 - \frac{\rho_\phi}{\rho}\right] = c^2\rho - c^2\rho_\phi$$

and take the derivative with respect to ρ

$$\frac{\partial p}{\partial \rho} = c^2$$

(j) From the chain rule, we can expand $\partial^2\rho'/\partial t^2$:

$$\frac{\partial^2\rho'}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial\rho'}{\partial t}\right] = \frac{\partial}{\partial t}\left[\frac{\partial p'}{\partial t}\frac{\partial\rho'}{\partial p'}\right]$$
$$= \frac{\partial p'}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial\rho'}{\partial p'}\right) + \frac{\partial\rho'}{\partial p'}\frac{\partial}{\partial t}\left(\frac{\partial p'}{\partial t}\right)$$

but

$$\frac{\partial}{\partial t}(\frac{\partial\rho'}{\partial p'}) = 0,$$

so we can drop the first term, and we are left with:

$$\frac{\partial^2\rho'}{\partial t^2} = \frac{\partial\rho'}{\partial p'}\frac{\partial^2 p'}{\partial t^2}.$$

Using our result of c^2 for $\partial p/\partial\rho$, we get:

$$\frac{\partial^2\rho'}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2}.$$

Finally, substituting this into Equation 8, we obtain the desired result:

$$\frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} = \nabla^2 p'$$