13.00 Introduction to Ocean Science and Technology
Problem Set 4 Solutions

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1. In a restricted area, the shape of the sea surface is given by

$$z = \zeta(x, y) = Axy$$

where A is a constant. Assume that the density ρ is uniform throughout the region (even with respect to z). Assume that we are in the northern hemisphere.

(a.) Find the horizontal velocity components, $u(x, y, z)$ and $v(x, y, z)$ and show in a sketch what this motion looks like.

(b.) Assuming the Coriolis parameter f is constant over the region of concern, what can you say about $w(x, y, z)$?

Solution: (a) since density is uniform, the z equation, (7.1c) in the notes, can be integrated easily:

$$p = -\rho g z + C,$$

where C is a constant, which we may as well evaluate on the surface:

$$p(x, y, z) = -\rho g[z - \zeta(x, y)],$$

i.e., pressure is zero if z = ζ. From this formula, we can obtain the gradient of the pressure, which we then substitute into the horizontal-flow geostrophic equations:

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial \zeta}{\partial x} = \rho g A y = \rho f v;$$

$$\frac{\partial p}{\partial y} = \rho g \frac{\partial \zeta}{\partial y} = \rho g A x = -\rho f u;$$

A slight rearrangement gives us:

$$u = -\frac{gAx}{f}; \qquad v = \frac{gAy}{f}.$$

In the resulting flow, the motion is a "corner flow" in each of the four quadrants.

(b) From the continuity equation, we know that:

$$\frac{\partial w}{\partial z} = -\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right].$$

If f is a constant, the right hand side is zero, so w is a constant, which can be evaluated at the surface. Since the surface is not moving, there can be no velocity component normal to it, and w is necessarily zero. (If f is not a constant, it causes the derivatives in the continuity equation to be non-zero, and w may not be zero.)

2. Temperature and salinity measurements were acquired by the *R.V. Atlantis* in the Gulf Stream at a number of different locations along a cruise from North Carolina to Bermuda in June, 1955. Your goal in this problem is to process the data from two of the locations:

- latitude 36.38 degrees North, longitude 73.75 degrees West (file: `one.dat`), and
- latitude 36.25 degrees North, longitude 73.48 degrees West (file: `two.dat`).

The data from these locations from the surface to 1000 meters depth is presented on the table on the next page, and is available in the directory `/mit/13.00/exam` of the athena course locker. You can assume that there is no motion at 1000 meters depth. For your convenience, the density and specific volume anomaly are provided to you.

(a.) Process the data to calculate the average geostrophic velocity in the direction perpendicular to the two stations as a function of depth. If you use matlab, be sure to include a full printout of your program. Plot the current velocity and indicate the direction of the current (e.g., north/south/east/west) on your plot.

Solution: see values inserted in tables and attached matlab script and plots. You need to be careful calculating the distance between the two stations, since there is a change in both the east and the north coordinate. It is safe to use the Coriolis parameter for the average of the two latitudes.

The geopotential is higher at station 2 than at station 1. Hence, the flow is "to the right" as you travel from station 2 to station 1, and the flow is to the north/nort-east (the direction of the Gulf stream.)

(b.) Find the slope of the water surface between the two stations. Draw a sketch showing how you define the slope. What is the actual difference in water elevation between the two stations?

Solution: the surface slope is give by:

$$i = \frac{1}{g\rho}\left(\frac{\partial p}{\partial x}\right)_z = \frac{1}{g}\left(\frac{\partial \Phi}{\partial x}\right)_p$$

(see (7.17) and (7.18) in the class notes). The constant-pressure surface on which the right- hand side is to be evaluated is the ocean surface itself. As usual, we use a finite-difference approximation for the derivative, and we subtract out a "standard ocean," so that Φ is replaced by $\Delta\Phi$. From the solution to part (a.), at the surface we have $\Delta\Phi_1 = 9.927$ m^3 kg^{-1} Pa, and $\Delta\Phi_2 = 15.122$.

The slope from 1 to 2 is thus

$$\begin{aligned} i_{12} &= \frac{\Delta\Phi_2 - \Delta\Phi_1}{g\Delta x} \\ &= \frac{15.122 - 9.927}{9.8 \times 28132} = 1.884 \times 10^{-5}. \end{aligned}$$

Hence, the rise in the surface is 0.5301 m = 53.01 cm. (Station 2 is 53.01 cm higher than station 1.)

(c.) From your data, estimate the flow rate in m^3/sec for this part of the gulf stream as it passes between these two locations. If the entire gulf stream had the same velocity profile as you determined from this data, what would you estimate for the flow rate in m^3/sec for the entire gulf stream? (300 km is a reasonable value to use for the width of the Gulf Stream). Do you think the flow rate that you calcuate in this way is an accurate value? If not, do you think it is too high or too low? Why?

To calculate the flow rate in m^3/sec, integrate the velocity as a function of depth, and multiply by the distance between the stations. Using the trapezoidal rule to integrate the velocity profile obtained in part (a.), we get 908.2 m $\times$ m/sec. Multiplying by the distance between stations of 28132, we get 25.5×10^6 m^3/sec for the flow rate between the two stations. If we extrapolate this to to entire 300 km width of the gulf stream, we get 272.4×10^6 m^3/sec. This value is too large, because the data is taken from a place where the gulf stream is strongest. But it is still a "reasonable" value.

Station One

Depth (m)	T (°C)	S (psu)	σ_t (kg/m^3)	δ (10^{-8} m^3/kg)	$\bar{\delta}\Delta p$ (Pa m^3/kg)	$\Delta\Phi$ (Pa m^3/kg)
0	26.00	36.21	1023.95	394.95	2.9835	9.9271
100	19.18	36.36	1026.02	201.74	1.6400	6.9436
200	13.87	35.80	1026.83	126.25	1.1281	5.3035
300	10.76	35.40	1027.13	99.36	0.9179	4.1754
400	8.56	35.14	1027.30	84.21	0.7677	3.2576
500	7.15	35.08	1027.46	69.32	0.6100	2.4899
600	5.55	35.03	1027.63	52.67	0.5136	1.8799
700	5.13	35.01	1027.67	50.03	0.4847	1.3663
800	4.73	35.00	1027.71	46.89	0.4539	0.8817
900	4.34	34.99	1027.74	43.87	0.4278	0.4278
1000	4.02	34.98	1027.77	41.68		0

Station Two

Depth (m)	T (°C)	S (psu)	σ_t (kg/m^3)	δ (10^{-8} m^3/kg)	$\bar{\delta}\Delta p$ (Pa m^3/kg)	$\Delta\Phi$ (Pa m^3/kg)
0	26.60	36.06	1023.64	423.80	3.5316	15.1222
100	22.98	36.62	1025.17	282.50	2.2783	11.5907
200	18.46	36.56	1026.35	173.16	1.7118	9.3123
300	17.85	36.46	1026.42	169.19	1.6852	7.6006
400	17.53	36.42	1026.47	167.85	1.4740	5.9153
500	13.13	35.70	1026.90	126.94	1.1860	4.4413
600	11.02	35.41	1027.09	110.24	1.0431	3.2553
700	9.56	35.25	1027.22	98.37	0.9107	2.2122
800	8.11	35.15	1027.37	83.77	0.7400	1.3015
900	6.47	35.10	1027.57	64.23	0.5614	0.5614
1000	4.82	35.03	1027.72	48.05		0

Calculation of water speed			
Depth (m)	$\Delta\Phi_1$ (Pa m^3/kg)	$\Delta\Phi_2$ (Pa m^3/kg)	V_{1-2} (cm/sec)
0	9.9271	15.1222	213.85
100	6.9436	11.5907	191.29
200	5.3035	9.3123	165.01
300	4.1754	7.6006	140.99
400	3.2576	5.9153	109.40
500	2.4899	4.4413	80.33
600	1.8799	3.2553	56.62
700	1.3663	2.2122	34.82
800	0.8817	1.3015	17.28
900	0.4278	0.5614	5.50
1000	0	0	0