13.00 Introduction to Ocean Science and Technology
Problem Set 2 Solutions

1. The velocity must satisfy the continuity equation, which is, for an incompressible fluid,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Since $w = 0$, the third term vanishes, and substituting the u that we are given,

$$2ax + by + \frac{\partial v}{\partial y} = 0.$$

This is a differential equation that can be solved for v:

$$v = -2axy - \frac{1}{2}by^2 + f(x, z),$$

where $f(x, z)$ is an arbitrary function (the only requirement is that its partial derivative with respect to y is zero.) However, we are also told that $v = 0$ at $y = 0$, so we can deduce that $f(x, z)$ is zero. So:

$$v = -2axy - \frac{1}{2}by^2.$$

2. If the prescribed velocity distributions are valid for the motion of an incompressible fluid, the simplified form of the continuity equation, as given above, must be satisfied.

(a.) The relevant velocity derivatives are as follows:

$$\frac{\partial u}{\partial x} = \frac{2Cxy}{(x^2 + y^2)^2}; \frac{\partial v}{\partial y} = -\frac{2Cxy}{(x^2 + y^2)^2}; \frac{\partial w}{\partial z} = 0.$$

Their sum is obviously zero, and so the continuity equation for the flow of an incompressible fluid is satisfied. The flow is everywhere along circles centered on the origin, and the magnitude of the velocity is inversely proportional to distance from the center:

$$\mid \vec{u} \mid = (u^2 + v^2)^{\frac{1}{2}} = \left[\frac{C^2y^2 + C^2x^2}{(x^2 + y^2)^2}\right]^{\frac{1}{2}} = \frac{\mid C \mid}{(x^2 + y^2)^{\frac{1}{2}}} = \frac{\mid C \mid}{r},$$

where $r = (x^2 + y^2)^{\frac{1}{2}}$. The flow is undefined near the origin, where the magnitude of the velocity goes to infinity. See the sketch for Problem 3(a).

(b) For $x^2 + y^2 > 1$, everything is the same as in (a). In the region $x^2 + y^2 < 1$, the continuity equation is satisfied in a trivial way, *i.e.*, each term is separately equal to zero. So continuity is satisfied everywhere. At the boundary between the two regions, $x^2 + y^2 = 1$, the veloity expressions in the two regions are consistent (give the same values). In the inner region, the magnitude of the velocity is $C(x^2 + y^2)^{\frac{1}{2}}$. See the sketch for Problem 3(b).

(c) A uniform flow in the z direction is added to that specified in (b). The continuity equation is still satisfied, since $\frac{\partial w}{\partial z} = 0$. for uniform w. The motion in the x-y plane is the same as in part (b).

(d) The first two terms in the continuity equation are unchanged from (b) and (c), and thus their sum is again zero. But the third term, $\frac{\partial w}{\partial z}$, is not zero, and so the continuiry equation for an incompressible fluid is not satisfied.

Sketch for problem 3(a).

Sketch for problem 3(b).

Sketch for problem 3(e).

3. For the Eulerian description of a flow field,

$$\mathbf{\vec{u}} = 2z\vec{i} + xt\vec{j} + xy^2\vec{k} \text{ m/sec}$$

the acceleration $\vec{a}$ of a fluid particle is given by the substantial derivative $\frac{D\vec{u}}{Dt}$:

$$\frac{Du}{Dt} = \vec{u} \cdot \vec{\nabla} u + \frac{\partial u}{\partial t} = 2xy^2$$

$$\frac{Dv}{Dt} = \vec{u} \cdot \vec{\nabla} v + \frac{\partial v}{\partial t} = 2zt + x$$

$$\frac{Dw}{Dt} = \vec{u} \cdot \vec{\nabla} w + \frac{\partial w}{\partial t} = 2zy^2 + 2x^2yt$$

$$\vec{a} = \frac{D\vec{u}}{Dt} = 2xy^2\vec{i} + (2zt + x)\vec{j} + (2zy^2 + 2x^2yt)\vec{k}$$

4.

There were several way to do this based on the ambiguity in the problem statement. This solution assumes the two rows in the time array were swapped.

x	y	u	v
0	0	20	10
1	0	25	15
0	1	10	5

t	u	v
0	20	10
0.5	10	5

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

We need to evaluate the terms numerically:

u at $x = 0, y = 0 = 20$

v at $x = 0, y = 0 = 10$

$\frac{\partial u}{\partial x}$ at $x = 0 = 5$

$\frac{\partial v}{\partial x}$ at $x = 0 = 5$

$\frac{\partial u}{\partial y}$ at $x = 0 =$ -10

$\frac{\partial v}{\partial y}$ at $x = 0 =$ -5

$\frac{\partial u}{\partial t}$ at $t = 0.0 =$ -20

$\frac{\partial v}{\partial t}$ at $t = 0.0 =$ -10

$$\frac{Du}{Dt} = 20(5) + 10(-10) - (20) = -20$$

Similarly,

$$\frac{Dv}{Dt} = 20(50 + 10(-5) + -10 = 40$$

5. A remotely operated vehicle (ROV) measuring water salinity is moving with velocity $3xt\vec{i} + 3y^2\vec{j} - 2t\vec{k}$. The salinity S of the water changes with the tidal currents, and is given by $S(x, y, z, t) = 2x\sin(at)$. Find the rate of change of salinity of the water as measured by the ROV.

The rate of change is given by $\vec{U} \cdot \vec{\nabla} S + \frac{\partial S}{\partial t}$, where $\vec{U} = U\hat{\imath} + V\hat{\jmath} + W\hat{k}$ is the velocity of the ROV. This is:

$$U\frac{\partial S}{\partial x} + V\frac{\partial S}{\partial y} + W\frac{\partial S}{\partial z} + \frac{\partial S}{\partial t} = 6xt\sin(at) + 2ax\cos(at).$$