Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.641 Electromagnetic Fields, Forces, and Motion
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Lecture Notes 2, 2/6/03

Figures all drawn from Electromagnetic Fields and Energy by Hermann A. Haus and James R. Melcher.

I Conditions for Electroquasistatic Fields

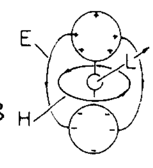
A. Order of Magnitude Estimate [Characteristic Length L, characteristic time τ]

$$\nabla \cdot \bar{E} = \rho/\epsilon \Rightarrow \frac{E}{L} = \rho/\epsilon \Rightarrow E = \frac{\rho L}{\epsilon}$$

$$\nabla \times \bar{H} = \epsilon \frac{\partial \bar{E}}{\partial t} \Rightarrow \frac{H}{L} = \frac{\epsilon E}{\tau} \Rightarrow H = \frac{\epsilon E L}{\tau} = \frac{L^2 \rho}{\tau}$$

$$\nabla \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \Rightarrow \frac{E_{error}}{L} = \frac{\mu H}{\tau} = \frac{\mu \rho L^2}{\tau^2} \Rightarrow E_{error} = \frac{\mu \rho L^3}{\tau}$$

Figure 3.3.1 Prototype systems involving one typical length. (a) EQS system in which source of EMF drives a pair of perfectly conducting spheres having radius and spacing on the order of L.



(a)

$$\frac{E_{error}}{E} = \frac{\mu \rho L^3 \epsilon}{\tau \rho L} = \frac{\mu \epsilon L^2}{\tau^2} = \frac{L^2}{(c\tau)^2}$$

$$\frac{E_{error}}{E} \ll 1 \Rightarrow \frac{L}{c\tau} \ll 1$$

B. Estimate of Error Introduced by EQS approximation

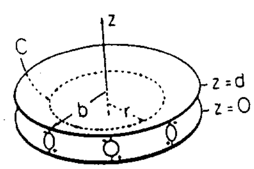


Figure 3.3.2 Plane parallel electrodes having no resistance, driven at their outer edges by a distribution of sources of EMF.

$$\bar{E} = \frac{V}{d}\bar{i}_z = E_o \bar{i}_z$$

$$\sigma_{su} = \begin{cases} -\epsilon E_o & z = d \\ +\epsilon E_o & z = 0 \end{cases}$$

$$K_r 2\pi b + \pi b^2 \frac{d\sigma_{su}}{dt} = 0 \Rightarrow K_r = -\frac{b}{2}\frac{d\sigma_{su}}{dt} = -\frac{b}{2}\epsilon \frac{dE_o}{dt}$$

$$\oint_C \bar{H} \cdot d\bar{s} = \int_S \frac{\partial}{\partial t}(\epsilon \bar{E}) \cdot d\bar{a} \Rightarrow H_\phi 2\pi r = \pi r^2 \epsilon \frac{dE_o}{dt} \Rightarrow H_\phi = \frac{r}{2}\epsilon \frac{dE_o}{dt}$$

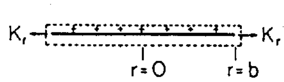


Figure 3.3.3 Parallel plates of Figure 3.3.2, showing volume containing lower plate and radial surface current density at its periphery.

$$\oint_C \bar{E} \cdot d\bar{s} = -\int_S \mu \frac{\partial \bar{H}}{\partial t} \cdot d\bar{a}$$

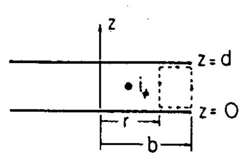


Figure 3.3.4 Cross-section of system shown in Figure 3.3.2 showing surface and contour used in evaluating correction E field.

$$[E_z(b) - E_z(r)]d = +\frac{\mu\epsilon}{2}\int_r^b r'\,dr'\,d\,\frac{d^2\mathcal{E}_0}{dt^2}$$

$$= \frac{\mu\epsilon d}{4}(b^2 - r^2)\frac{d^2\mathcal{E}_0}{dt^2}$$

$$E_z(r) = E_0 + \frac{\epsilon\mu}{4}\frac{d^2\mathcal{E}_0}{dt^2}(r^2 - b^2)$$

If $E_0(t) = A\cos\omega t$

$$\frac{|E_{error}|}{E_0} = \frac{\epsilon\mu}{4E_0}\frac{d^2E_0}{dt^2}(b^2 - r^2) = \frac{1}{4}\omega^2\epsilon\mu(b^2 - r^2)$$

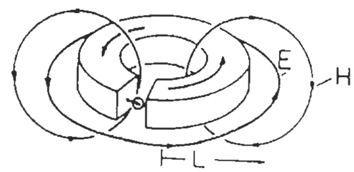
$$\frac{|E_{error}|}{E_0} \ll 1 \Rightarrow \frac{\omega^2\epsilon\mu b^2}{4} \ll 1$$

$$f\lambda = c = \frac{1}{\sqrt{\epsilon\mu}}$$

$$\frac{\omega}{2\pi}\lambda = c \Rightarrow \omega = \frac{2\pi c}{\lambda} \Rightarrow \frac{\omega^2\epsilon\mu b^2}{4} = \frac{\pi^2}{\lambda^2}b^2 \ll 1 \Rightarrow b \ll \frac{\lambda}{\pi}$$

$f = 1\,\text{MHz}$ in free space $\Rightarrow \lambda = \frac{3\times10^8}{10^6} = 300\,\text{m}$

If $b \ll 100\,\text{m}$ EQS approximation is valid.



(b)

(b) MQS system consisting of perfectly conducting loop driven by current source. The radius of the loop and diameter of its cross-section are on the order of L.

II. Conditions for Magnetoquasistatic Fields

$$\nabla\times\bar{H} = \bar{J} \Rightarrow \frac{H}{L} \approx J \Rightarrow H = JL$$

$$\nabla\times\bar{E} = -\mu\frac{\partial\bar{H}}{\partial t} \Rightarrow \frac{E}{L} = \frac{\mu H}{\tau} \Rightarrow E = \frac{\mu HL}{\tau} = \frac{\mu JL^2}{\tau}$$

$$\nabla\times\bar{H}_{error} = \epsilon\frac{\partial\bar{E}}{\partial t} \Rightarrow \frac{\bar{H}_{error}}{L} = \frac{\epsilon E}{\tau} \Rightarrow H_{error} = \frac{\epsilon EL}{\tau^2} = \frac{\epsilon\mu JL^3}{\tau^2}$$

$$\frac{H_{error}}{H} = \frac{\epsilon\mu JL^3}{\tau^2 JL} = \frac{\epsilon\mu L^2}{\tau^2} = \frac{L^2}{(c\tau)^2} \ll 1 \Rightarrow L \ll c\tau$$

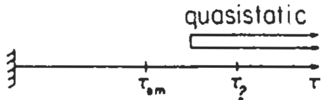


Figure 3.4.1 Range of characteristic times over which quasistatic approximation is valid. The transit time of an electromagnetic wave is τ_{em} while $\tau_?$ is a time characterizing the dynamics of the quasistatic system.

$$\tau_{em} = \frac{L}{c} = L\sqrt{\epsilon\mu}$$

I. Quasistatics

Electroquasistatics (EQS)

$$\nabla \times \overline{E} = -\cancel{\frac{\partial}{\partial t}(\mu_0 \overline{H})}^{\,0} \approx 0$$

$$\nabla \cdot (\epsilon_0 \overline{E}) = \rho$$

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial}{\partial t}(\epsilon_0 \overline{E})$$

$$\nabla \cdot \overline{J} + \frac{\partial \rho}{\partial t} = 0$$

Magnetoquasistatics (MQS)

$$\nabla \times \overline{E} = -\frac{\partial}{\partial t}(\mu_0 \overline{H})$$

$$\nabla \times \overline{H} = \overline{J} + \cancel{\frac{\partial}{\partial t}(\epsilon_0 \overline{E})}^{\,0}$$

$$\nabla \cdot (\mu_0 \overline{H}) = 0$$

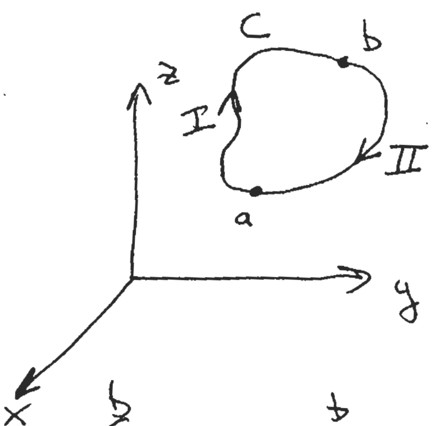
$$\nabla \cdot \overline{J} = 0$$

$$\nabla \cdot (\epsilon_0 \overline{E}) = \rho$$

II. Irrotational EQS Electric Field

1. Conservative Electric Field

$$\oint_C \overline{E} \cdot \overline{ds} = -\cancel{\frac{d}{dt}\int_S \mu_0 \overline{H} \cdot \overline{da}}^{\,0} \approx 0$$



$$\oint_C \overline{E} \cdot \overline{ds} = \underset{\mathrm{I}}{\int_a^b} \overline{E} \cdot \overline{ds} + \underset{\mathrm{II}}{\int_b^a} \overline{E} \cdot \overline{ds} = 0 \Rightarrow \underbrace{\underset{\mathrm{I}}{\int_a^b} \overline{E} \cdot \overline{ds}}_{\text{Electromotive force (EMF)}} = \underset{\mathrm{II}}{\int_a^b} \overline{E} \cdot \overline{ds}$$

EMF between 2 points (a, b) independent of path

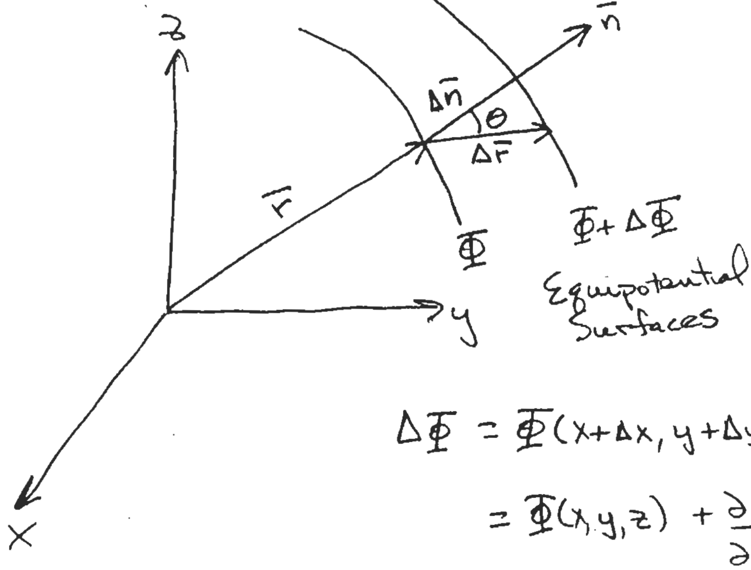
$\overline{E}$ field is <u>conservative</u>

$$\Phi(\overline{r}) - \Phi(\overline{r}_{ref}) = \int_{\overline{r}}^{\overline{r}_{ref}} \overline{E} \cdot \overline{ds}$$

Φ: Scalar electric potential

$$\int_a^b \overline{E} \cdot \overline{ds} = \int_a^{\overline{r}_{ref}} \overline{E} \cdot \overline{ds} + \int_{\overline{r}_{ref}}^b \overline{E} \cdot \overline{ds} = \Phi(a) - \Phi(\overline{r}_{ref}) + \Phi(\overline{r}_{ref}) - \Phi(b) = \Phi(a) - \Phi(b)$$

2. The Electric Scalar Potential



$$\bar{r} = x\bar{i}_x + y\bar{i}_y + z\bar{i}_z$$
$$\Delta\bar{r} = \Delta x\,\bar{i}_x + \Delta y\,\bar{i}_y + \Delta z\,\bar{i}_z$$
$$\Delta n = \Delta r\cos\theta$$

$$\Delta\Phi = \Phi(x+\Delta x, y+\Delta y, z+\Delta z) - \Phi(x,y,z)$$
$$= \Phi(x,y,z) + \frac{\partial\Phi}{\partial x}\Delta x + \frac{\partial\Phi}{\partial y}\Delta y + \frac{\partial\Phi}{\partial z}\Delta z - \Phi(x,y,z)$$
$$= \frac{\partial\Phi}{\partial x}\Delta x + \frac{\partial\Phi}{\partial y}\Delta y + \frac{\partial\Phi}{\partial z}\Delta z$$
$$= \underbrace{\left[\frac{\partial\Phi}{\partial x}\bar{i}_x + \frac{\partial\Phi}{\partial y}\bar{i}_y + \frac{\partial\Phi}{\partial z}\bar{i}_z\right]}_{\text{grad}\,\Phi = \nabla\Phi}\cdot\Delta\bar{r}$$

$$\nabla = \bar{i}_x\frac{\partial}{\partial x} + \bar{i}_y\frac{\partial}{\partial y} + \bar{i}_z\frac{\partial}{\partial z}$$

$$\text{grad}\,\Phi = \nabla\Phi = \bar{i}_x\frac{\partial\Phi}{\partial x} + \bar{i}_y\frac{\partial\Phi}{\partial y} + \bar{i}_z\frac{\partial\Phi}{\partial z}$$

$$\int_{\bar{r}}^{\bar{r}+\Delta\bar{r}} \bar{E}\cdot d\bar{s} = \Phi(\bar{r}) - \Phi(\bar{r}+\Delta\bar{r}) = -\Delta\Phi = -\nabla\Phi\cdot\Delta\bar{r} = \bar{E}\cdot\Delta\bar{r}$$
$$\boxed{\bar{E} = -\nabla\Phi}$$

$$\Delta\Phi = \frac{\Delta\Phi}{\Delta n}\Delta r\cos\theta = \frac{\Delta\Phi}{\Delta n}\bar{n}\cdot\Delta\bar{r} = \nabla\Phi\cdot\Delta\bar{r}$$

$$\nabla\Phi = \frac{\Delta\Phi}{\Delta n}\bar{n} = \frac{\partial\Phi}{\partial n}\bar{n}$$

The gradient is in the direction perpendicular to the equipotential surfaces

III. Vector Identity

$$\nabla\times\bar{E} = 0$$
$$\bar{E} = -\nabla\Phi$$
$$\nabla\times(\nabla\Phi) = 0$$

IV. Sample Problem

$$\Phi(x,y) = \frac{V_0\, x y}{a^2} \qquad (\text{Equipotential lines hyperbolas: } xy = \text{constant})$$

$$\bar{E} = -\nabla\Phi = -\left[\frac{\partial\Phi}{\partial x}\bar{i}_x + \frac{\partial\Phi}{\partial y}\bar{i}_y\right]$$

$$= -\frac{V_0}{a^2}\left(y\,\bar{i}_x + x\,\bar{i}_y\right)$$

Electric Field Lines [lines tangent to electric field]

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{x}{y} \Rightarrow y\,dy = x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

$$y^2 - x^2 = y_0^2 - x_0^2 \quad (\text{lines pass through point } (x_0, y_0))$$

(hyperbolas orthogonal to xy)

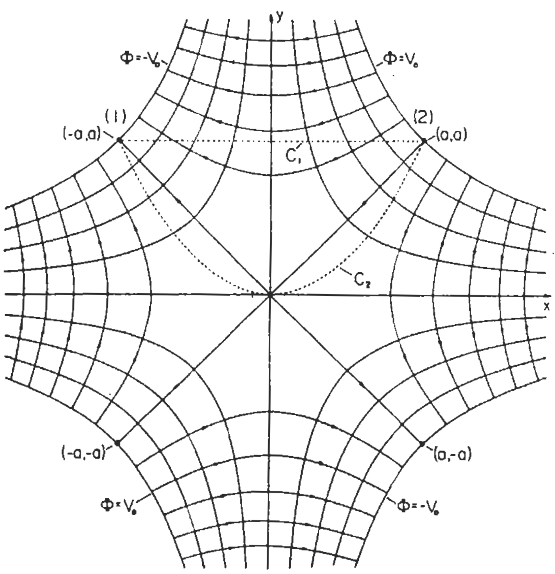


Figure 4.1.3 Cross-sectional view of surfaces of constant potential for two-dimensional potential given by (18).

V. Poisson's Equation

$$\nabla\cdot\bar{E} = \nabla\cdot(-\nabla\Phi) = \rho/\epsilon_0 \Rightarrow \nabla^2\Phi = -\rho/\epsilon_0$$

$$\nabla^2\Phi = \nabla\cdot(\nabla\Phi) = \left[\bar{i}_x\frac{\partial}{\partial x} + \bar{i}_y\frac{\partial}{\partial y} + \bar{i}_z\frac{\partial}{\partial z}\right]\cdot\left[\frac{\partial\Phi}{\partial x}\bar{i}_x + \frac{\partial\Phi}{\partial y}\bar{i}_y + \frac{\partial\Phi}{\partial z}\bar{i}_z\right]$$

$$= \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

VI. Coulomb Superposition Integral

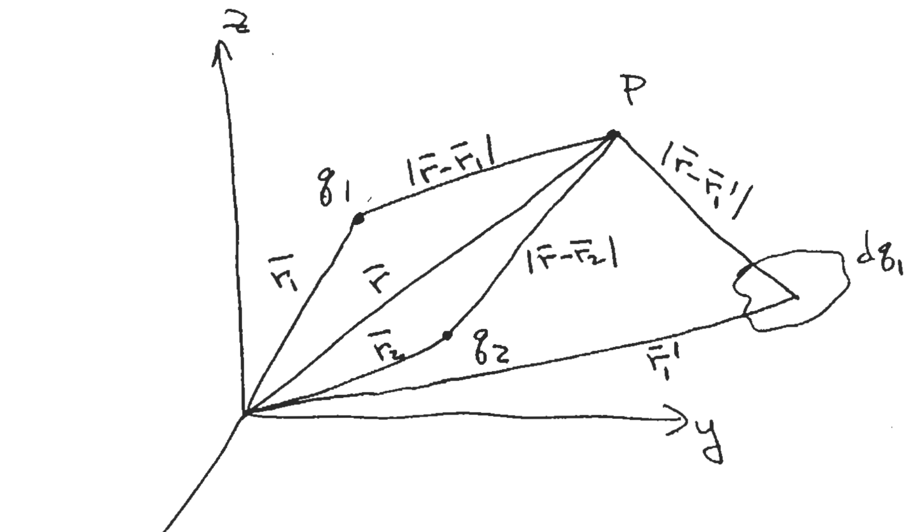
1. Point Charge

$$E_r = -\frac{\partial \Phi}{\partial r} = \frac{q}{4\pi\epsilon_0 r^2} \Rightarrow \Phi = \frac{q}{4\pi\epsilon_0 r} + C$$

Take reference $\Phi(r \to \infty) = 0 \Rightarrow C = 0$

$$\Phi = \frac{q}{4\pi\epsilon_0 r}$$

2. Superposition of Charges



$$d\Phi_T(P) = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{|\bar{r}-\bar{r}_1|} + \frac{q_2}{|\bar{r}-\bar{r}_2|} + \cdots \frac{dq_1}{|\bar{r}-\bar{r}_1'|} + \frac{dq_2}{|\bar{r}-\bar{r}_2'|} + \cdots\right]$$

$$\Phi_T(P) = \frac{1}{4\pi\epsilon_0}\left[\sum_{n=1}^{N} \frac{q_n}{|\bar{r}-\bar{r}_n|} + \int \frac{dq}{|\bar{r}-\bar{r}'|}\right]$$

all line, surface, and volume charges

$$= \frac{1}{4\pi\epsilon_0}\left[\sum_{n=1}^{N} \frac{q_n}{|\bar{r}-\bar{r}_n|} + \int_L \frac{\lambda(\bar{r}')\,dl'}{|\bar{r}-\bar{r}'|} + \int_S \frac{\sigma_s(\bar{r}')\,da'}{|\bar{r}-\bar{r}'|} + \int_V \frac{\rho(\bar{r}')\,dV'}{|\bar{r}-\bar{r}'|}\right]$$

Short-hand notation

$$\Phi(\bar{r}) = \int_V \frac{\rho(\bar{r}')\,dV'}{4\pi\epsilon_0|\bar{r}-\bar{r}'|}$$