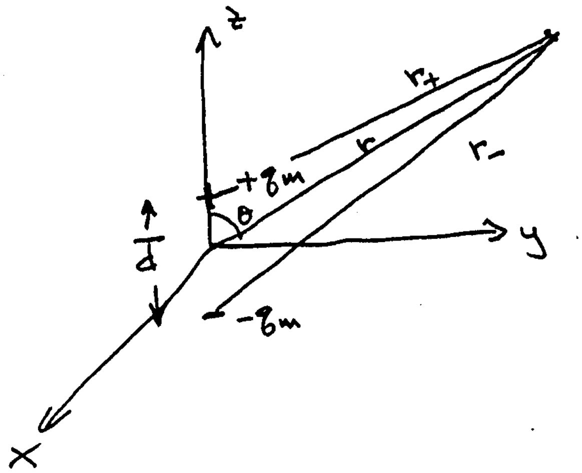
6.641 Lecture 7 Supplement
3/7/03

I. Magnetic Dipole

A. Magnetic Charge Model



$$\nabla \times \overline{H} = 0 \quad (\overline{J} = 0) \Rightarrow \overline{H} = -\nabla \Psi \qquad (\Psi = \text{magnetic scalar potential})$$

$$\nabla \cdot \mu_0 \overline{H} = \rho_m = -\nabla^2 \Psi \mu_0$$

$$\nabla^2 \Psi = -\frac{\rho_m}{\mu_0}$$

By analogy to the electrostatic scalar potential with electric charge

$$\Psi(\overline{r}) = \int_{V'} \frac{\rho_m(\overline{r}')\, dV'}{4\pi\mu_0 |\overline{r} - \overline{r}'|}$$

For 2 point magnetic charges

$$\Psi(r) = \frac{q_m}{4\pi\mu_0}\left[\frac{1}{r_+} - \frac{1}{r_-}\right]$$

$$r_+ \approx r - \frac{d}{2}\cos\theta \approx r\left(1 - \frac{d}{2r}\cos\theta\right)$$

$$r_- \approx r + \frac{d}{2}\cos\theta \approx r\left(1 + \frac{d}{2r}\cos\theta\right)$$

magnetic dipole moment

$$\lim_{\substack{d \to 0 \\ q_m \to \infty}} \Psi(r) = \frac{q_m d}{4\pi\mu_0 r^2}\cos\theta \; ; \; m = \frac{q_m d}{\mu_0} \Rightarrow \Psi(r) = \frac{m\cos\theta}{4\pi r^2}$$

$$\overline{H} = -\nabla\Psi = -\left[\frac{\partial\Psi}{\partial r}\overline{i}_r + \frac{1}{r}\frac{\partial\Psi}{\partial\theta}\overline{i}_\theta\right] = \frac{m}{4\pi r^3}\left(2\cos\theta\,\overline{i}_r + \sin\theta\,\overline{i}_\theta\right)$$

B. Current Loop Model

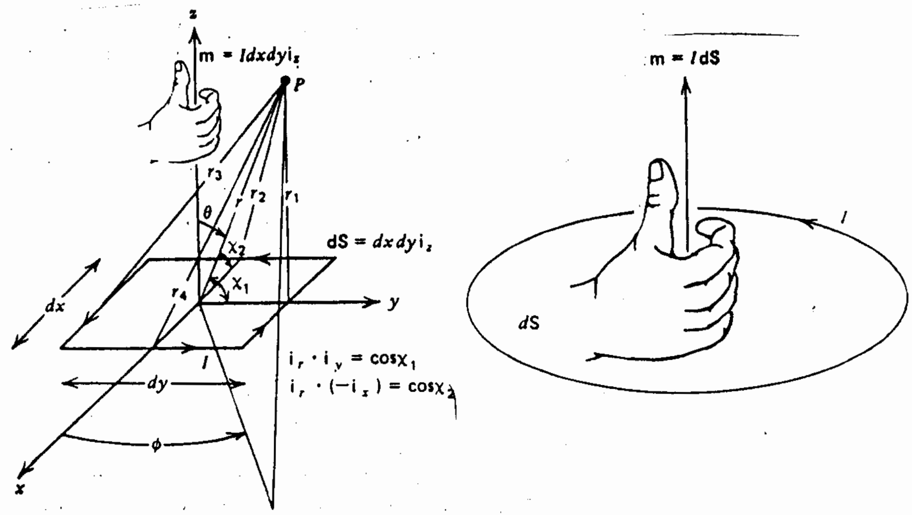


Figure 5-14 A magnetic dipole consists of a small circulating current loop. The magnetic moment is in the direction normal to the loop by the right-hand rule.

1. Rectangular loop

$$\nabla^2 \bar{A} = -\mu_0 \bar{J} \Rightarrow \bar{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\bar{J}(\bar{r}')\,dV'}{|\bar{r}-\bar{r}'|}$$

$$\bar{A} \approx \frac{\mu_0 I}{4\pi}\left[dx\left(\frac{1}{r_3}-\frac{1}{r_1}\right)\bar{i}_x + dy\left(\frac{1}{r_4}-\frac{1}{r_2}\right)\bar{i}_y\right]$$

$$r_1 \approx r\left(1-\frac{dy}{2r}\cos\chi_1\right), \quad r_2 \approx r\left(1-\frac{dx}{2r}\cos\chi_2\right)$$

$$r_3 \approx r\left(1+\frac{dy}{2r}\cos\chi_1\right), \quad r_4 \approx r\left(1+\frac{dx}{2r}\cos\chi_2\right)$$

$$\bar{i}_r = \sin\theta\cos\phi\,\bar{i}_x + \sin\theta\sin\phi\,\bar{i}_y + \cos\theta\,\bar{i}_z$$

$$\cos\chi_1 = \bar{i}_r\cdot\bar{i}_y = \sin\theta\sin\phi$$

$$\cos\chi_2 = -\bar{i}_r\cdot\bar{i}_x = -\sin\theta\cos\phi$$

$$\lim_{\substack{r\gg dx\\ r\gg dy}} \bar{A} = \frac{\mu_0 I}{4\pi}\left[\frac{dx}{r}\left(\frac{1}{1+\frac{dy}{2r}\cos\chi_1} - \frac{1}{1-\frac{dy}{2r}\cos\chi_1}\right)\bar{i}_x + \frac{dy}{r}\left(\frac{1}{1+\frac{dx}{2r}\cos\chi_2} - \frac{1}{1-\frac{dx}{2r}\cos\chi_2}\right)\bar{i}_y\right]$$

$$= \frac{\mu_0 I}{4\pi}\left[\frac{dx}{r}\left(-\frac{dy}{r}\cos\chi_1\right)\bar{i}_x - \frac{dy}{r}\left(\frac{dx}{2r}\cos\chi_2\right)\bar{i}_y\right]$$

$$= -\frac{\mu_0 I\,dx\,dy}{4\pi r^2}\left[\cos\chi_1\,\bar{i}_x + \cos\chi_2\,\bar{i}_y\right]$$

$$= -\frac{\mu_0 I\,dx\,dy}{4\pi r^2}\left[\sin\theta\sin\phi\,\bar{i}_x - \sin\theta\cos\phi\,\bar{i}_y\right]$$

$$\overline{A} = -\frac{\mu_0 I dx dy}{4\pi r^2}\sin\theta\left[\sin\phi\,\overline{i_x} - \cos\phi\,\overline{i_y}\right]$$

$$\overline{i_\phi} = -\sin\phi\,\overline{i_x} + \cos\phi\,\overline{i_y}$$

$$\overline{A} = \frac{\mu_0 I S}{4\pi r^2}\sin\theta\,\overline{i_\phi}, \quad S = dx\,dy = \text{area of current loop}$$

$$\overline{m} = I S\,\overline{i_z} = I\overline{S} \quad \text{magnetic dipole moment}$$

$$\mu_0\overline{H} = \nabla\times\overline{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta)\,\overline{i_r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\,\overline{i_\theta}$$

$$\overline{H} = \frac{m}{4\pi r^3}\left[2\cos\theta\,\overline{i_r} + \sin\theta\,\overline{i_\theta}\right]$$

2. Circular Loop

$$\overline{J}(\overline{r}')dV' \to \overline{I\,dl}$$

$$= I a\, d\phi\,\overline{i_\phi}$$

$$= I a\, d\phi(-\sin\phi\,\overline{i_x} + \cos\phi\,\overline{i_y})$$

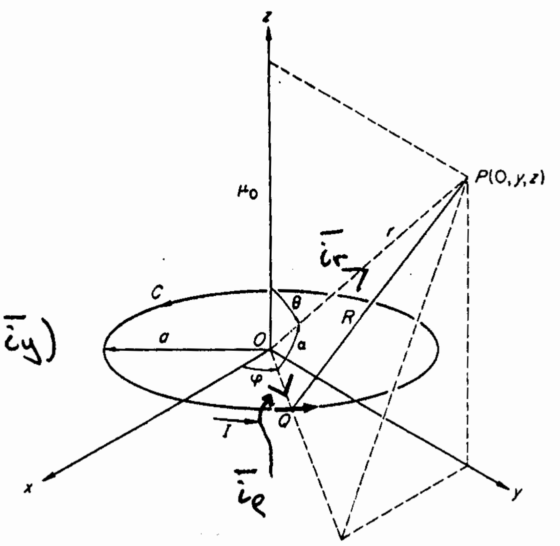


FIGURE. 4-4 A circular loop in free space.

$$\overline{A} = \frac{\mu_0 I a}{4\pi}\int_{\phi=0}^{2\pi}\frac{d\phi\,(-\sin\phi\,\overline{i_x} + \cos\phi\,\overline{i_y})}{R}$$

Without loss of generality take the field point in the $x=0$ plane. Using the law of cosines

$$R^2 = a^2 + r^2 - 2ar\cos\alpha$$

$$\begin{aligned}\overline{i_r} &= \sin\theta\,\overline{i_y} + \cos\theta\,\overline{i_z}\\ \overline{i_\rho} &= \cos\phi\,\overline{i_x} + \sin\phi\,\overline{i_y}\end{aligned} \Rightarrow \cos\alpha = \overline{i_r}\cdot\overline{i_\rho} = \sin\theta\sin\phi$$

$$\lim_{r\gg a} R \approx r\left(1 - \frac{a}{r}\cos\alpha\right)$$

$$\bar{A} = \frac{\mu_0 I a}{4\pi} \int_{\phi=0}^{2\pi} \frac{d\phi(-\sin\phi\,\bar{i}_x + \cos\phi\,\bar{i}_y)}{r(1-\frac{a}{r}\cos\alpha)}$$

$$\approx \frac{\mu_0 I a}{4\pi r} \int_{\phi=0}^{2\pi} \left(1 + \frac{a\cos\alpha}{r}\right)(-\sin\phi\,\bar{i}_x + \cos\phi\,\bar{i}_y)\,d\phi$$

$$\approx \frac{\mu_0 I a}{4\pi r} \int_{\phi=0}^{2\pi} \left(1 + \frac{a}{r}\sin\theta\sin\phi\right)\left(-\sin\phi\,\bar{i}_x + \cos\phi\,\bar{i}_y\right)d\phi$$

$$\approx \frac{\mu_0 I a^2}{4\pi r^2}\sin\theta \int_{\phi=0}^{2\pi} \left[-\sin^2\phi\,\bar{i}_x + \sin\phi\cos\phi\,\bar{i}_y\right]d\phi$$

$$\approx \frac{\mu_0 I \pi a^2}{4\pi r^2}\sin\theta\,\underbrace{\bar{i}_x}_{-\bar{i}_\phi \text{ for field point in } x=0 \text{ plane}}$$

$$A_\phi = \frac{\mu_0 I S}{4\pi r^2}\sin\theta \quad , \quad S = \pi a^2 \quad , \quad m = \pi a^2 I$$

$$A_\phi = \frac{\mu_0 m}{4\pi r^2}\sin\theta$$

$$\mu_0 \bar{H} = \nabla \times \bar{A} = \bar{i}_r \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \bar{i}_\theta \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi)$$

$$= \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\bar{i}_r + \sin\theta\,\bar{i}_\theta\right)$$

II. Perfectly Conducting Sphere In Uniform Magnetic Field

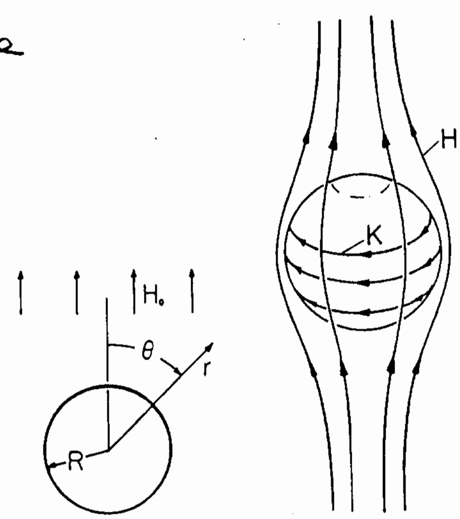


Figure 8.4.5 Immersed in a uniform magnetic field, a perfectly conducting sphere has the same effect as an oppositely directed magnetic dipole.

(5)

$$\bar{H} = \underbrace{H_0 (\cos\theta\, \bar{i}_r - \sin\theta\, \bar{i}_\theta)}_{\text{Uniform imposed z directed field}} + \underbrace{\frac{m}{4\pi r^3}(2\cos\theta\, \bar{i}_r + \sin\theta\, \bar{i}_\theta)}_{\text{Induced effective dipole}} \qquad r > R$$

$$\text{B.C.:}\ \mu_0 H_r(r=R) = 0 \Rightarrow H_0 + \frac{2m}{4\pi R^3} = 0 \Rightarrow m = -2\pi H_0 R^3$$

$$\bar{H} = H_0 \left[\left(1 - \left(\frac{r}{R}\right)^3\right)\cos\theta\, \bar{i}_r - \left(1 + \frac{1}{2}\left(\frac{r}{R}\right)^3\right)\sin\theta\, \bar{i}_\theta\right]$$

$$K_\phi = H_\theta(r=R) = -\frac{3}{2} H_0 \sin\theta$$

Fig. 5.14 From Electromagnetic Field Theory: A Problem Solving Approach, by Markus Zahn, (c) 1987, Robert E. Krieger Publishing Company. Used with permission. Fig. 8.4.5 from Electromagnetic Fields and Energy, by Hermann A. Haus and James R. Melcher.