Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.641 Electromagnetic Fields, Forces, and Motion
Spring 2003

Lecture Notes 6, 2/27/03

Conduction

See last page for credit language.

I. Plasma Conduction Model (Classical)

$$m_+ \frac{d\bar{v}_+}{dt} = q_+ \bar{E} - m_+ \nu_+ \bar{v}_+ - \frac{\nabla P_+}{n_+}$$

$$m_- \frac{d\bar{v}_-}{dt} = -q_- \bar{E} - m_- \nu_- \bar{v}_- - \frac{\nabla P_-}{n_-}$$

$$P_+ = n_+ kT \ , \ P_- = n_- kT$$

$k = 1.38 \times 10^{-23}$ joules/$^\circ$K Boltzmann Constant

A. London Model of Superconductivity $[T \to 0, \ \nu_\pm \to 0]$

$$m_+ \frac{d\bar{v}_+}{dt} = q_+ \bar{E} \ , \ m_- \frac{d\bar{v}_-}{dt} = -q_- \bar{E}$$

$$\bar{J}_+ = q_+ n_+ \bar{v}_+ \ , \ \bar{J}_- = -q_- n_- \bar{v}_-$$

$$\frac{d\bar{J}_+}{dt} = \frac{d}{dt}(q_+ n_+ \bar{v}_+) = q_+ n_+ \frac{d\bar{v}_+}{dt} = q_+ n_+ \left(\frac{q_+ \bar{E}}{m_+}\right) = \underbrace{\frac{q_+^2 n_+}{m_+}}_{\omega_{p+}^2 \epsilon} \bar{E}$$

$$\frac{d\bar{J}_-}{dt} = -\frac{d}{dt}(q_- n_- \bar{v}_-) = -q_- n_- \frac{d\bar{v}_-}{dt} = -q_- n_- \left(\frac{-q_- \bar{E}}{m_-}\right) = \underbrace{\frac{q_-^2 n_-}{m_-}}_{\omega_{p-}^2 \epsilon} \bar{E}$$

$$\omega_{p+}^2 = \frac{q_+^2 n_+}{m_+ \epsilon} \ , \ \omega_{p-}^2 = \frac{q_-^2 n_-}{m_- \epsilon} \quad (\omega_p = \text{plasma frequency})$$

For electrons: $q_- = 1.6 \times 10^{-19}$ Coulombs, $m_- = 9.1 \times 10^{-31}$ kg

$n_- = 10^{20}/m^3$, $\epsilon = \epsilon_0 \approx 8.854 \times 10^{-12}$ farads/m

$$\omega_{p-} = \sqrt{\frac{q_-^2 n_-}{m_- \epsilon}} \approx 5.6 \times 10^{11} \text{ rad/s}$$

$$f_{p-} = \frac{\omega_{p-}}{2\pi} \approx 9 \times 10^{10} \text{ Hz}$$

B. Drift-Diffusion Conduction [Neglect inertia]

(2)

$$\cancel{m_+ \frac{d\bar{v}_+}{dt}}^{\to 0} = q_+ \bar{E} - m_+ \nu_+ \bar{v}_+ - \frac{\nabla(n_+ kT)}{n_+} \Rightarrow \bar{v}_+ = \frac{q_+}{m_+ \nu_+} \bar{E} - \frac{kT}{m_+ \nu_+ n_+} \nabla n_+$$

$$\cancel{m_- \frac{d\bar{v}_-}{dt}}^{\to 0} = -q_- \bar{E} - m_- \nu_- \bar{v}_- - \frac{\nabla(n_- kT)}{n_-} \Rightarrow \bar{v}_- = \frac{-q_-}{m_- \nu_-} \bar{E} - \frac{kT}{m_- \nu_- n_-} \nabla n_-$$

$$\bar{J}_+ = q_+ n_+ \bar{v}_+ = \frac{q_+^2 n_+}{m_+ \nu_+} \bar{E} - \frac{q_+ kT}{m_+ \nu_+} \nabla n_+$$

$$\bar{J}_- = -q_- n_- \bar{v}_- = \frac{q_-^2 n_-}{m_- \nu_-} \bar{E} + \frac{q_- kT}{m_- \nu_-} \nabla n_-$$

$$\rho_+ = q_+ n_+, \quad \rho_- = -q_- n_-$$

$$\bar{J}_+ = \rho_+ \mu_+ \bar{E} - D_+ \nabla \rho_+$$

$$\bar{J}_- = -\rho_- \mu_- \bar{E} - D_- \nabla \rho_-$$

$$\underbrace{\mu_+ = \frac{q_+}{m_+ \nu_+}, \quad \mu_- = \frac{q_-}{m_- \nu_-}}_{\text{charge mobilities}} \qquad \underbrace{D_+ = \frac{kT}{m_+ \nu_+}, \quad D_- = \frac{kT}{m_- \nu_-}}_{\text{Molecular Diffusion Coefficients}}$$

$$\underbrace{\frac{D_+}{\mu_+} = \frac{D_-}{\mu_-} = \frac{kT}{q}}_{\text{Einstein's Relation}} = \text{thermal voltage } (25\,mV @ T \approx 300°K)$$

C. Drift Diffusion Conduction Equilibrium ($\bar{J}_+ = \bar{J}_- = 0$)

$$\bar{J}_+ = 0 = \rho_+ \mu_+ \bar{E} - D_+ \nabla \rho_+ = -\rho_+ \mu_+ \nabla \Phi - D_+ \nabla \rho_+$$

$$\bar{J}_- = 0 = -\rho_- \mu_- \bar{E} - D_- \nabla \rho_- = \rho_- \mu_- \nabla \Phi - D_- \nabla \rho_-$$

$$\nabla \Phi = -\frac{D_+}{\rho_+ \mu_+} \nabla \rho_+ = -\frac{kT}{q} \nabla(\ln \rho_+)$$

$$\nabla \Phi = \frac{D_-}{\rho_- \mu_-} \nabla \rho_- = \frac{kT}{q} \nabla(\ln \rho_-)$$

$$\left.\begin{aligned} \rho_+ &= \rho_0\, e^{-q\Phi/kT} \\ \rho_- &= -\rho_0\, e^{+q\Phi/kT} \end{aligned}\right\}\ \text{Boltzmann Distributions}$$

$$\rho_+(\Phi=0) = -\rho_-(\Phi=0) = \rho_0$$ [Potential is zero when system is charge neutral]

$$\nabla^2\Phi = -\frac{\rho}{\epsilon} = -\frac{(\rho_+ + \rho_-)}{\epsilon} = -\frac{\rho_0}{\epsilon}\left[e^{-q\Phi/kT} - e^{+q\Phi/kT}\right] = \frac{2\rho_0}{\epsilon}\sinh\frac{q\Phi}{kT}$$

(Poisson-Boltzmann Equation)

Small Potential Approximation: $\frac{q\Phi}{kT} \ll 1$

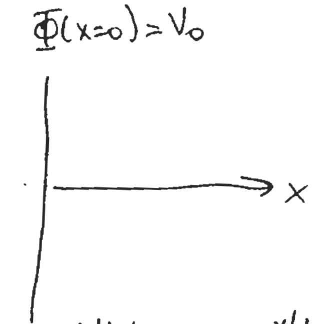
$$\sinh\frac{q\Phi}{kT} \approx \frac{q\Phi}{kT}$$

$$\nabla^2\Phi - \frac{2\rho_0 q}{\epsilon kT}\Phi = 0$$

$$\nabla^2\Phi - \frac{\Phi}{L_d^2} = 0 \quad ; \quad L_d = \sqrt{\frac{\epsilon kT}{2\rho_0 q}}$$ Debye Length

D. Case Studies

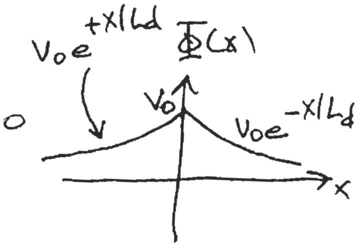
1. Planar Sheet



$$\frac{d^2\Phi}{dx^2} - \frac{\Phi}{L_d^2} = 0 \Rightarrow \Phi = A_1 e^{x/L_d} + A_2 e^{-x/L_d}$$

B.C. $\Phi(x \to \pm\infty) = 0$
$\Phi(x=0) = V_0$

$$\Rightarrow \Phi(x) = \begin{cases} V_0\, e^{-x/L_d} & x > 0 \\ V_0\, e^{+x/L_d} & x < 0 \end{cases}$$



$$E_x = -\frac{d\Phi}{dx} = \begin{cases} \frac{V_0}{L_d}\, e^{-x/L_d} & x > 0 \\ -\frac{V_0}{L_d}\, e^{x/L_d} & x < 0 \end{cases}$$

$$\rho = \epsilon \frac{dE_x}{dx} = \begin{cases} -\dfrac{\epsilon V_0}{L_d^2} e^{-x/L_d} & x > 0 \\[2ex] -\dfrac{\epsilon V_0}{L_d^2} e^{+x/L_d} & x < 0 \end{cases}$$

$$\sigma_s(x=0) = \epsilon \left[E_x(x=0_+) - E_x(x=0_-) \right] = \frac{2\epsilon V_0}{L_d}$$

2. Point Charge (Debye Shielding)

$$\underbrace{\nabla^2 \Phi}_{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)} - \frac{\Phi}{L_d^2} = 0 \Rightarrow \frac{d^2}{dr^2}(r\Phi) - \frac{r\Phi}{L_d^2} = 0$$

$$r\Phi = A_1 e^{-r/L_d} + \overset{0}{A_2} e^{+r/L_d}$$

$$\Phi(r) = \frac{Q}{4\pi\epsilon r} e^{-r/L_d}$$

II. Ohmic Conduction

$$\bar{J}_+ = \rho_+ \mu_+ \bar{E} - D_+ \nabla \rho_+$$

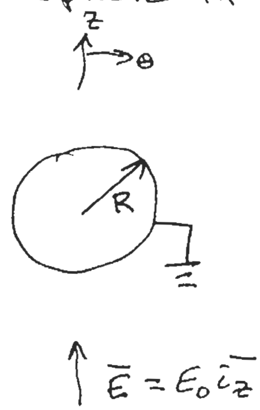
$$\bar{J}_- = -\rho_- \mu_- \bar{E} - D_- \nabla \rho_-$$

If charge density gradients small, then $\nabla \rho_\pm$ negligible $\Rightarrow \rho_+ = -\rho_- = \rho_0$

$$\bar{J} = \bar{J}_+ + \bar{J}_- = (\rho_+ \mu_+ - \rho_- \mu_-) E = \underbrace{\rho_0 (\mu_+ + \mu_-)}_{\sigma = \text{ohmic conductivity}} \bar{E} = \sigma \bar{E}$$

$$\bar{J} = \sigma \bar{E} \quad (\text{Ohm's Law})$$

III. Perfectly Conducting Sphere In Uniform Electric Field



$r > R$

$$\Phi = -E_0 r\cos\theta + \frac{A\cos\theta}{r^2}$$

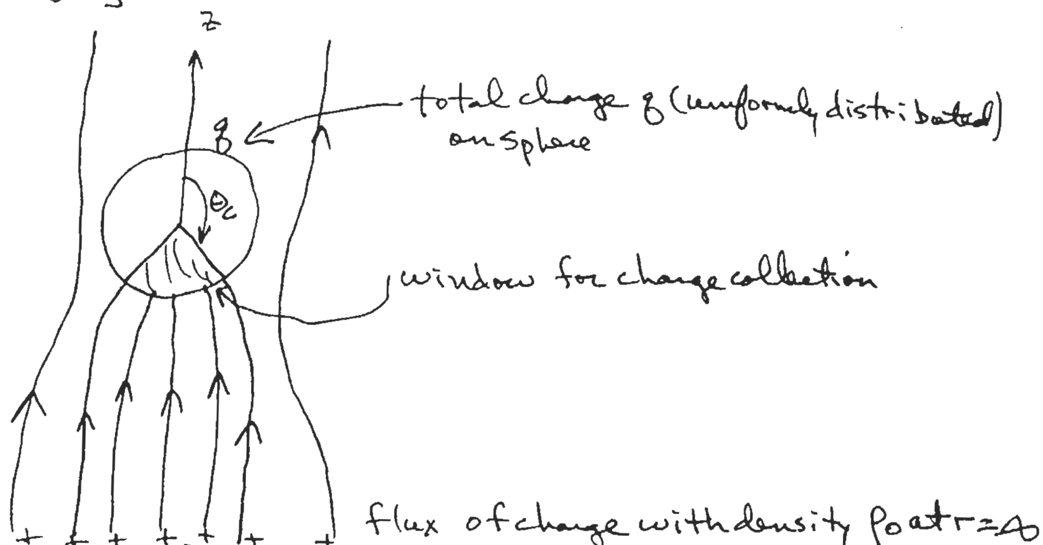
$$\Phi(r=R) = 0 = \left(-E_0 R + \frac{A}{R^2}\right)\cos\theta \Rightarrow A = E_0 R^3$$

$$\Phi(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta$$

$$\bar{E} = -\nabla\Phi = -\left[\frac{\partial\Phi}{\partial r}\bar{i_r} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\bar{i_\theta}\right]$$

$$= E_0\left[\left(1+2\left(\frac{R}{r}\right)^3\right)\cos\theta\,\bar{i_r} - \left(1-\left(\frac{R}{r}\right)^3\right)\sin\theta\,\bar{i_\theta}\right]$$

IV Ion-Impact Charging of Macroscopic Particles



$$\bar{E} = \frac{q}{4\pi\varepsilon_0 r^2}\bar{i_r} + E_0\left[\left(1+2\left(\frac{R}{r}\right)^3\right)\cos\theta\,\bar{i_r} - \left(1-\left(\frac{R}{r}\right)^3\right)\sin\theta\,\bar{i_\theta}\right]$$

[neglect self-field of charge ρ_0 compared to E_0]

Charge will collect on sphere only where $E_r < 0$

Drift Dominated Current: $\bar{J} = \rho\mu\bar{E}$ (unipolar)

mobility

charge density

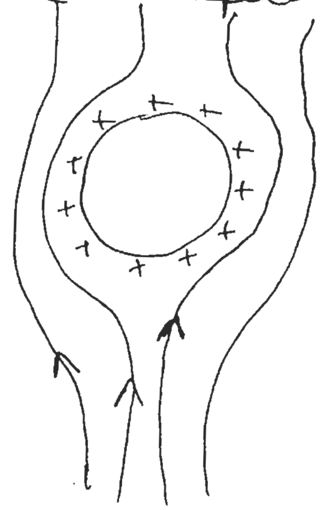
$$E_r(r=R) = \frac{q}{4\pi\epsilon_0 R^2} + 3E_0\cos\theta = 0$$

$$\cos\theta_c = -\frac{q}{12\pi\epsilon_0 E_0 R^2}$$

When $q=0 \Rightarrow \cos\theta_c = 0 \Rightarrow \theta_c = \pi/2$ [Window for collection $\frac{\pi}{2} < \theta < \pi$]

As q increases, θ_c increases so window for charge collection decreases until $\theta_c = \pi$, $\cos\theta_c = -1 \Rightarrow q_c = 12\pi\epsilon_0 E_0 R^2$

q_c is maximum amount of charge that can collect on sphere as $E_r \geq 0$ everywhere on sphere and repels additional charge from collecting



To find the rate of charge build-up, the current charging sphere is:

$$i = \int_{\theta=\theta_c}^{\pi} -\rho_0 \mu E_r(r=R,\theta)\, 2\pi R^2 \sin\theta\, d\theta$$

$$= -\rho_0\mu 2\pi R^2 \int_{\theta=\theta_c}^{\pi} \left[3E_0\cos\theta + \frac{q}{4\pi\epsilon_0 R^2}\right]\sin\theta\, d\theta$$

$$= -\rho_0\mu 6\pi E_0 R^2 \int_{\theta_c}^{\pi} \Big[\cos\theta + \underbrace{\frac{q}{12\pi\epsilon_0 E_0 R^2}}_{-\cos\theta_c}\Big]\sin\theta\, d\theta$$

$$\cos\theta_c = -\frac{q}{q_c}$$

$$i = -\rho_0 \mu 6\pi E_0 R^2 \left[-\frac{\cos 2\theta}{4} + \cos\theta_c \cos\theta\right]\Bigg|_{\theta=\theta_c}^{\pi}$$

$$= -\rho_0 \mu 6\pi E_0 R^2 \underbrace{\left[-\frac{1}{4} + \frac{\cos 2\theta_c}{4} - \cos\theta_c - \cos^2\theta_c\right]}_{\left[-\frac{1}{2} - \frac{1}{2}\cos^2\theta_c - \cos\theta_c\right] = -\frac{1}{2}\left[1+\cos\theta_c\right]^2}$$

$$i = \rho_0 \mu 3\pi E_0 R^2 (1+\cos\theta_c)^2 = \rho_0 \mu 3\pi E_0 R^2 \left(1 - \frac{q}{q_c}\right)^2 \quad ; \quad q_c = 12\pi\epsilon_0 E_0 R^2$$

$$\cos\theta_c = -\frac{q}{q_c}$$

$$i = \frac{dq}{dt} = \rho_0 \mu 3\pi E_0 R^2 \left(1 - \frac{q}{q_c}\right)^2$$

$$\frac{dq}{\left(1 - \frac{q}{q_c}\right)^2} = \rho_0 \mu 3\pi E_0 R^2 \, dt$$

$$\frac{q_c}{1 - \frac{q}{q_c}} = \rho_0 \mu 3\pi E_0 R^2 t + q_c$$

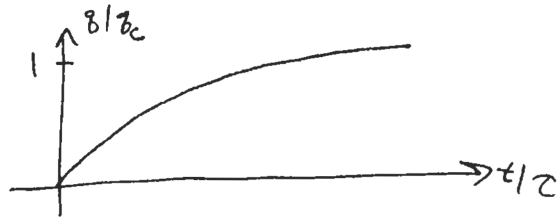
where constant of integration is chosen so that $q(t=0)=0$

$$q_c \left[\frac{1}{1 - \frac{q}{q_c}} - 1\right] = \rho_0 \mu 3\pi E_0 R^2 t = \frac{\rho_0 \mu}{4\epsilon_0} q_c t$$

$$\frac{q}{q_c} = \frac{\rho_0 \mu}{4\epsilon_0} t \left(1 - \frac{q}{q_c}\right) \quad ; \quad \tau = \frac{4\epsilon_0}{\rho_0 \mu}$$

$$\frac{q}{q_c} = \frac{t}{\tau}\left(1 - \frac{q}{q_c}\right) \Rightarrow \frac{q}{q_c}\left(1 + \frac{t}{\tau}\right) = t/\tau$$

$$\frac{q}{q_c} = \frac{t/\tau}{1 + t/\tau}$$



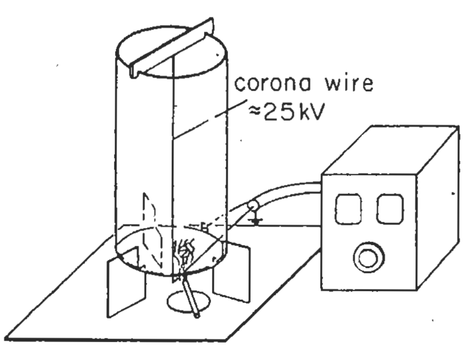


Figure 7.7.5 Electrostatic precipitator consisting of fine wire at high voltage relative to surrounding conducting transparent coaxial cylinder. Ions created in corona discharge in the immediate vicinity of the wire follow field lines toward outer wall, some terminating on smoke particles. Once charged by the mechanism described in Example 7.7.2, the smoke particles are precipitated on the outer wall.

Ohmic Conduction

I. Ohmic Conduction Constitutive Law

$$\bar{f}_+ = q_+ \bar{E} = \underset{\text{mass}}{m_+} \underset{\text{collision frequency}}{\nu_+} \underset{\text{velocity}}{\bar{V}_+}$$

$$\bar{f}_- = -q_- \bar{E} = m_- \nu_- \bar{V}_-$$

} collision dominated model

$$\bar{V}_+ = \underbrace{\frac{q_+}{m_+ \nu_+}}_{\substack{\text{mobility} \\ \mu_+}} \bar{E} = \mu_+ \bar{E}$$

$$\bar{V}_- = \underbrace{\frac{-q_-}{m_- \nu_-}}_{-\mu_-} \bar{E} = -\mu_- \bar{E}$$

$\longrightarrow \bar{E}$

$\oplus \longrightarrow \bar{V}_+$

$\bar{V}_- \longleftarrow \ominus$

μ = mobility - units of $m^2/(\text{Volt-sec})$

N_+ = number per unit volume of positive charges

N_- = " " " " " negative charges

$$\bar{J}_u = N_+ q_+ \bar{V}_+ - N_- q_- \bar{V}_- = \underbrace{(N_+ q_+ \mu_+ + N_- q_- \mu_-)}_{\sigma \text{ = ohmic conductivity}} \bar{E}$$

σ = ohmic conductivity units Siemens/meter

$$\rho_u = N_+ q_+ - N_- q_-$$

Material	σ @ 20°C (Siemens/m)
Aluminum	3.54×10^7
Copper	$5.65 - 5.8 \times 10^7$
Glass	10^{-12}
Teflon	$< 10^{-16}$
Water, Distilled	2×10^{-4}

II. Steady Conduction – No variation with time ($\frac{\partial}{\partial t} = 0$)

$$\nabla \cdot \quad | \quad \nabla \times \overline{H} = \overline{J}_u + \frac{\partial \overline{D}}{\partial t}$$

$$\nabla \cdot \overline{D} = \rho_u$$

$$\nabla \cdot (\nabla \times \overline{H}) = 0 = \nabla \cdot \overline{J}_u + \frac{\partial}{\partial t}(\nabla \cdot \overline{D}) = \nabla \cdot \overline{J}_u + \frac{\partial \rho_u}{\partial t} = 0$$ Conservation of Charge

For steady conduction ($\frac{\partial}{\partial t} = 0$) $\Rightarrow$ $\nabla \cdot \overline{J}_u = 0$

Ohm's Law: $\overline{J}_u = \sigma \overline{E}$, If $\sigma =$ constant

$$\nabla \cdot \overline{J}_u = 0 = \sigma \nabla \cdot \overline{E} \Rightarrow \nabla \cdot \overline{E} = 0$$

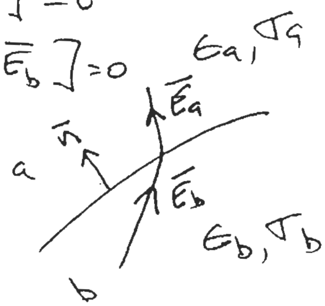
$$\nabla \times \overline{E} = 0 \Rightarrow \overline{E} = -\nabla \Phi$$

$$\nabla \cdot \overline{E} = 0 \Rightarrow \nabla^2 \Phi = 0 \quad \text{(Laplace's Equation)}$$

Boundary Conditions: $\nabla \cdot \overline{J}_u = 0 \Rightarrow \overline{n} \cdot [\overline{J}_{ua} - \overline{J}_{ub}] = 0$

$$\Rightarrow \overline{n} \cdot [\sigma_a \overline{E}_a - \sigma_b \overline{E}_b] = 0$$

$$\overline{n} \times [\overline{E}_a - \overline{E}_b] = 0 \Rightarrow \Phi^a - \Phi^b = 0$$



$$\sigma_{su} = \overline{n} \cdot [\epsilon_a \overline{E}_a - \epsilon_b E_b]$$

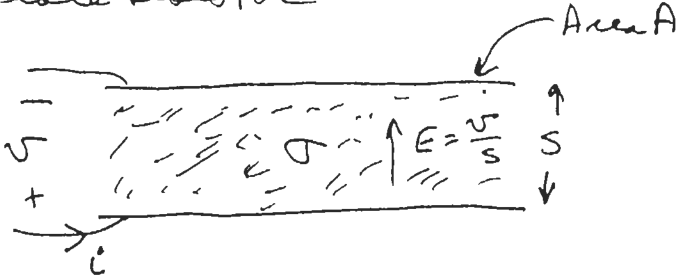
$$\overline{n} \cdot \overline{E}_b = \overline{n} \cdot \overline{E}_a \left(\frac{\sigma_a}{\sigma_b}\right)$$

$$\sigma_{su} = \overline{n} \cdot \epsilon_a \overline{E}_a \left[1 - \frac{\epsilon_b}{\epsilon_a} \frac{\sigma_a}{\sigma_b}\right]$$

In a uniform ohmic conductor there is no steady state volume charge but there can be steady state surface charge at interfaces between dissimilar materials.

II. One Dimensional Resistors ($\frac{\partial}{\partial t} = 0$)

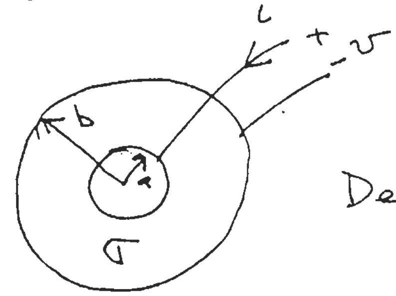
A. Parallel Plate Resistor



$$E = \frac{v}{s}, \quad J = \sigma E = \frac{\sigma v}{s} \Rightarrow i = JA = \frac{\sigma A}{s} v$$

$$R = \frac{v}{i} = \frac{s}{\sigma A} \; ; \quad G = \frac{1}{R} = \frac{i}{v} = \frac{\sigma A}{s}$$

B. Coaxial Resistor



$$\nabla \cdot \bar{E} = \frac{1}{r}\frac{\partial}{\partial r}(r E_r) = 0$$

$$\Rightarrow E_r = \frac{C}{r}$$

$$\int_a^b E_r dr = \int_a^b \frac{C}{r} dr = C \ln r \Big|_a^b = C \ln \frac{b}{a} = v$$

$$C = \frac{v}{\ln b/a}$$

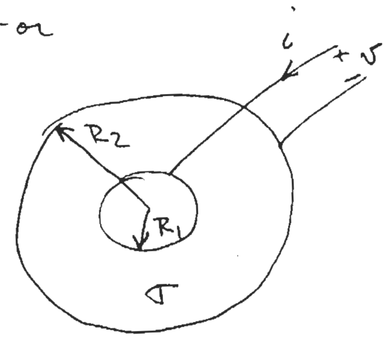
$$E_r = \frac{v}{r \ln \frac{b}{a}} \Rightarrow J_r = \sigma E_r = \frac{\sigma v}{r \ln b/a}$$

$$i = \int_{z=0}^{l} \int_{\phi=0}^{2\pi} J_r \, r \, d\phi \, dz = \frac{\sigma v 2\pi l}{\ln b/a}$$

$$R = \frac{v}{i} = \frac{\ln b/a}{2\pi \sigma l}$$

$$G = \frac{1}{R} = \frac{i}{v} = \frac{2\pi\sigma l}{\ln b/a}$$

C. Spherical Resistor



$$\nabla \cdot \bar{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) = 0 \Rightarrow E_r = \frac{C}{r^2}$$

$$\int_{R_1}^{R_2} E_r dr = -\frac{C}{r}\Big|_{R_1}^{R_2} = -C\left(\frac{1}{R_2} - \frac{1}{R_1}\right) = v$$

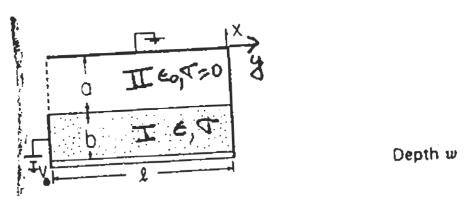
$$C = \frac{v}{\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

$$E_r = \frac{v}{r^2\left(\frac{1}{R_1} - \frac{1}{R_2}\right)} \Rightarrow J_r = \sigma E_r = \frac{\sigma v}{r^2\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

$$i = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} J_r r^2 \sin\theta \, d\theta \, d\phi = \frac{4\pi\sigma v}{\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

$$R = \frac{v}{i} = \frac{\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}{4\pi\sigma} \Rightarrow G = \frac{1}{R} = \frac{i}{v} = \frac{4\pi\sigma}{\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

IV. Resistor In An Open Box



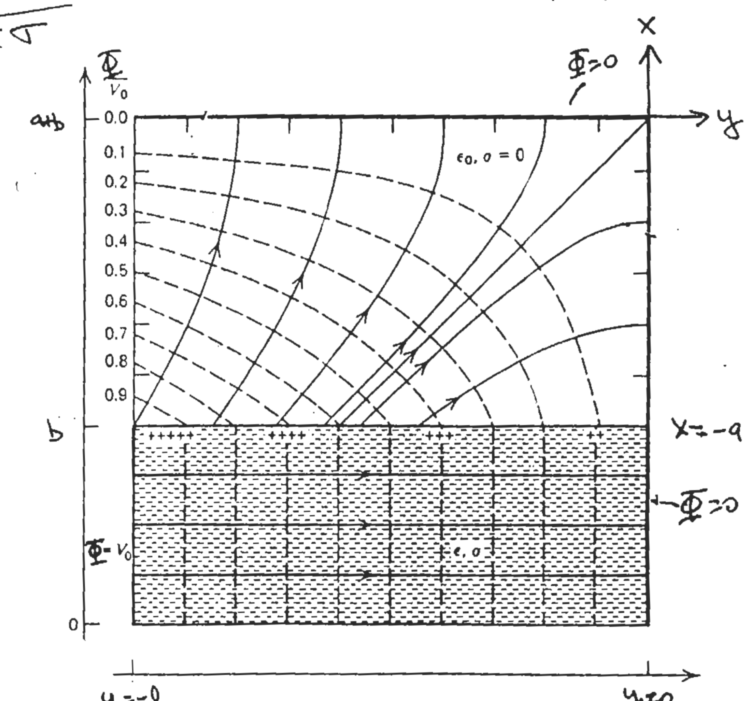


Figure 4-2 A resistive medium partially fills an open conducting box.

Boundary Conditions

Region I

$$\Phi(y=-l)=V_0$$
$$\Phi(y=0)=0$$

Region II

$$\Phi(y=0)=0$$
$$\Phi(x=0)=0$$
$$\Phi(x=-a_-)=\Phi(x=-a_+)$$

Try for both regions

$$\Phi = a+bx+cy+dxy$$

$$\Phi(y=-l)=V_0=a+bx-cl-dly$$
$$\Phi(y=0)=0=a+bx$$
$$a=b=d=0$$
$$c=-V_0/l$$
$$\Phi(y)=-\frac{V_0 y}{l}$$

$$\Phi(y=0)=0=a+bx \Rightarrow a=b=0$$
$$\Phi(x=0)=0=cy \quad \Rightarrow c=0$$
$$\Phi(x=-a_-)=\Phi(x=-a_+)\Rightarrow -\frac{V_0 y}{l}=-d$$
$$d=\frac{V_0}{al}$$
$$\Phi(x,y)=\frac{V_0xy}{al}$$

$$\bar{E}=-\nabla\Phi=\frac{V_0}{l}\bar{i_y}$$

$$\bar{E}=-\nabla\Phi=-\left[\frac{\partial\Phi}{\partial x}\bar{i}_x+\frac{\partial\Phi}{\partial y}\bar{i}_y\right]$$
$$=\frac{-V_0}{al}\left[y\,\bar{i}_x+x\,\bar{i}_y\right]$$

$$\sigma_s(x=a)=\epsilon_0 E_x(x=a_+)-\epsilon E_x(x=a_-)=-\frac{\epsilon_0 V_0}{al}y$$

Interfacial Shear Force

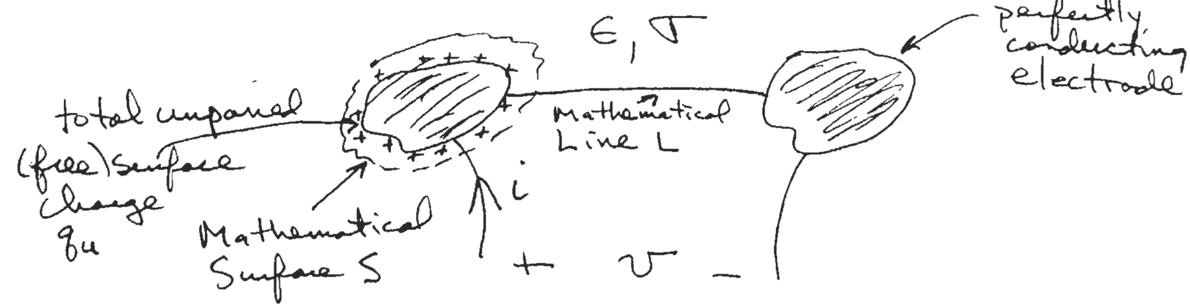
$$F_y=\int_{-l}^{0}\sigma_s(x=a)E_y(x=a)w\,dy=\int_{-l}^{0}-\frac{\epsilon_0V_0^2}{al^2}yw\,dy$$

$$=-\frac{\epsilon_0V_0^2wy^2}{2al^2}\Big|_{-l}^{0}=\frac{\epsilon_0V_0^2w}{2a}$$

Region II field lines that pass through point (x_0,y_0)

$$\frac{dy}{dx}=\frac{E_y}{E_x}=\frac{x}{y}\Rightarrow x\,dx=y\,dy\Rightarrow x^2-y^2=x_0^2-y_0^2$$

I. Relationship Between Resistance and Capacitance In Uniform Media Described by ϵ and σ.



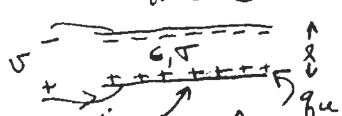
$$C = \frac{q_u}{v} = \frac{\oint_S \bar{D}\cdot\bar{da}}{\int_L \bar{E}\cdot\bar{ds}} = \frac{\epsilon\oint_S \bar{E}\cdot\bar{da}}{\int_L \bar{E}\cdot\bar{ds}}$$

$$R = \frac{v}{i} = \frac{\int_L \bar{E}\cdot\bar{ds}}{\oint_S \bar{J}\cdot\bar{da}} = \frac{\int_L \bar{E}\cdot\bar{ds}}{\sigma\oint_S \bar{E}\cdot\bar{da}}$$

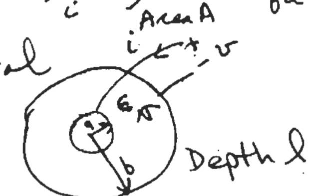
$$RC = \frac{\int_L \bar{E}\cdot\bar{ds}}{\sigma\oint_S \bar{E}\cdot\bar{da}} \; \frac{\epsilon\oint_S \bar{E}\cdot\bar{da}}{\int_L \bar{E}\cdot\bar{ds}} = \frac{\epsilon}{\sigma}$$

Check:

Parallel Plate Electrodes: $R = \dfrac{l}{\sigma A}$, $C = \dfrac{\epsilon A}{l} \Rightarrow RC = \epsilon/\sigma$



Coaxial



$R = \dfrac{\ln b/a}{2\pi\sigma l}$, $C = \dfrac{2\pi\epsilon l}{\ln b/a} \Rightarrow RC = \epsilon/\sigma$

Concentric Spherical

$R = \dfrac{\frac{1}{R_1} - \frac{1}{R_2}}{4\pi\sigma}$, $C = \dfrac{4\pi\epsilon}{\frac{1}{R_1} - \frac{1}{R_2}} \Rightarrow RC = \epsilon/\sigma$

II. Charge Relaxation in Uniform Conductors

$$\nabla \cdot \bar{J}_u + \frac{\partial \rho_u}{\partial t} = 0$$

$$\nabla \cdot \bar{E} = \rho_u / \epsilon$$

$$\bar{J}_u = \sigma \bar{E}$$

$$\sigma \underbrace{\nabla \cdot \bar{E}}_{\rho_u/\epsilon} + \frac{\partial \rho_u}{\partial t} = 0 \Rightarrow \frac{\partial \rho_u}{\partial t} + \frac{\sigma}{\epsilon}\rho_u = 0$$

$$\tau_e = \epsilon/\sigma = \text{dielectric relaxation time}$$

$$\frac{\partial \rho_u}{\partial t} + \frac{\rho_u}{\tau_e} = 0 \Rightarrow \rho_u = \rho_0(\bar{r}, t=0)\, e^{-t/\tau_e}$$

III. Demonstration 7.7.1 – Relaxation of Charge on Particle in Ohmic Conductor

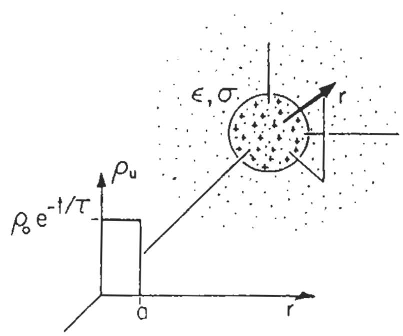


Figure 7.7.1 Within a material having uniform conductivity and permittivity, initially there is a uniform charge density ρ_u in a spherical region, having radius a. In the surrounding region the charge density is given to be initially zero and found to be always zero. Within the spherical region, the charge density is found to decay exponentially while retaining its uniform distribution.

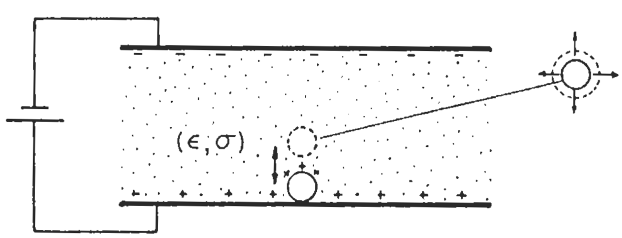
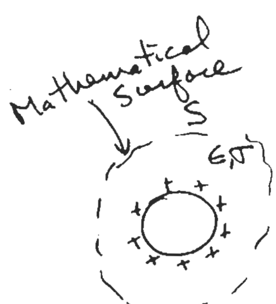


Figure 7.7.2 The region between plane parallel electrodes is filled by a semi-insulating liquid. With the application of a constant potential difference, a metal particle resting on the lower plate makes upward excursions into the fluid. [See footnote 1.]



$$\oint_S \bar{J} \cdot d\bar{a} = \sigma \oint_S \bar{E} \cdot d\bar{a} = \frac{\sigma q_u}{\epsilon} = -\frac{dq}{dt}$$

$$\frac{dq}{dt} + \frac{q}{\tau_e} = 0 \Rightarrow q = q(t=0)\, e^{-t/\tau_e} \quad (\tau_e = \epsilon/\sigma)$$

Partially Uniformly Charged Sphere

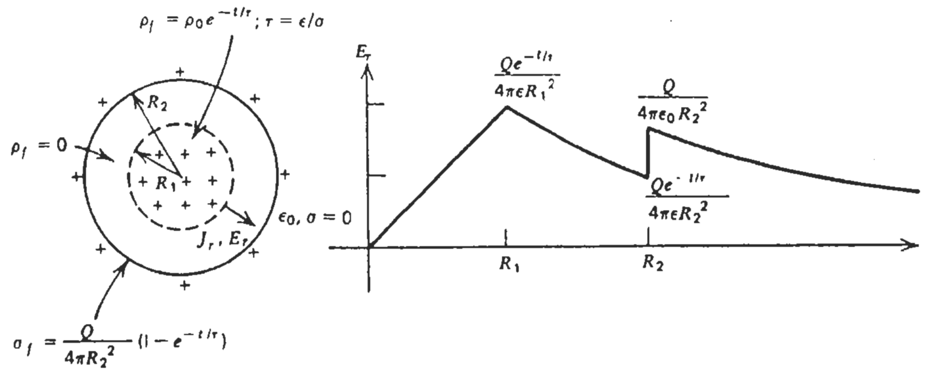


Figure 3-21 An initial volume charge distribution within an Ohmic conductor decays exponentially towards zero with relaxation time $\tau = \epsilon/\sigma$ and appears as a surface charge at an interface of discontinuity. Initially uncharged regions are always uncharged with the charge transported through by the current.

$$\rho_u(t=0) = \begin{cases} \rho_0 & r < R_1 \\ 0 & r > R_1 \end{cases} \qquad Q_T = \frac{4}{3}\pi R_1^3 \rho_0$$

$$\rho_u(t) = \begin{cases} \rho_0 e^{-t/\tau_e} & r < R_1 \\ 0 & r > R_1 \end{cases} \qquad (\tau_e = \epsilon/\sigma)$$

$$E_r(r,t) = \begin{cases} \dfrac{\rho_0 r e^{-t/\tau_e}}{3\epsilon} = \dfrac{Q r e^{-t/\tau_e}}{4\pi\epsilon R_1^3} & 0 < r < R_1 \\[2ex] \dfrac{Q e^{-t/\tau_e}}{4\pi\epsilon r^2} & R_1 < r < R_2 \\[2ex] \dfrac{Q}{4\pi\epsilon_0 r^2} & r > R_2 \end{cases}$$

$$\sigma_{su}(r=R_2) = \epsilon_0 E_r(r=R_{2+}) - \epsilon E_r(r=R_{2-})$$
$$= \frac{Q}{4\pi R_2^2}\left(1 - e^{-t/\tau_e}\right)$$

I. Self-Excited Water Dynamos

A. DC High Voltage Generation (Self-Excited)

$$-nC_i v_1 = C\frac{dv_2}{dt} \qquad\Rightarrow\qquad v_1 = \hat{V}_1 e^{st} \qquad\Rightarrow\qquad -nC_i\hat{V}_1 = Cs\hat{V}_2$$

$$-nC_i v_2 = C\frac{dv_1}{dt} \qquad\qquad v_2 = \hat{V}_2 e^{st} \qquad\qquad -nC_i\hat{V}_2 = Cs\hat{V}_1$$

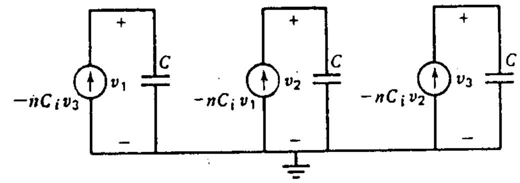
$$\underbrace{\begin{bmatrix} \frac{nC_i}{Cs} & 1 \\ 1 & \frac{nC_i}{Cs} \end{bmatrix}}_{\mathrm{Det}=0} \begin{bmatrix} \hat{V}_1 \\ \hat{V}_2 \end{bmatrix} = 0$$

$$\left(\frac{nC_i}{Cs}\right)^2 = 1 \quad\Rightarrow\quad s = \pm\frac{nC_i}{C}$$

$\oplus$ root blows up $e^{\frac{nC_i}{C}t}$

Any perturbation grows exponentially with time

B. AC High Voltage Self-Excited Generation

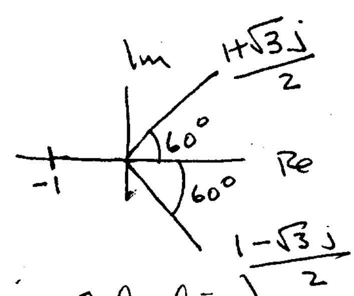


$$-nC_i v_1 = C\frac{dv_2}{dt} \quad ; \quad v_1 = \hat{V}_1 e^{st}$$

$$-nC_i v_2 = C\frac{dv_3}{dt} \qquad v_2 = \hat{V}_2 e^{st}$$

$$-nC_i v_3 = C\frac{dv_1}{dt} \qquad v_3 = \hat{V}_3 e^{st}$$

$$\underbrace{\begin{bmatrix} nC_i & Cs & 0 \\ 0 & nC_i & Cs \\ Cs & 0 & nC_i \end{bmatrix}}_{\det = 0} \begin{bmatrix} \hat{V}_1 \\ \hat{V}_2 \\ \hat{V}_3 \end{bmatrix} = 0$$



$$(nC_i)^3 + (Cs)^3 = 0 \Rightarrow s = \left(\frac{nC_i}{C}\right)(-1)^{1/3}$$

$s_1 = -nC_i/C$ (exponentially decaying solution)

$$s_{2,3} = \frac{nC_i}{2C}\left[1 \pm \sqrt{3}\,j\right]$$

$$(-1)^{1/3} = -1,\ \frac{1 \pm \sqrt{3}j}{2}$$

(blows up exponentially because $s_{real} > 0$; but also oscillates at frequency $s_{imag} \neq 0$)