Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.641 Electromagnetic Fields, Forces, and Motion
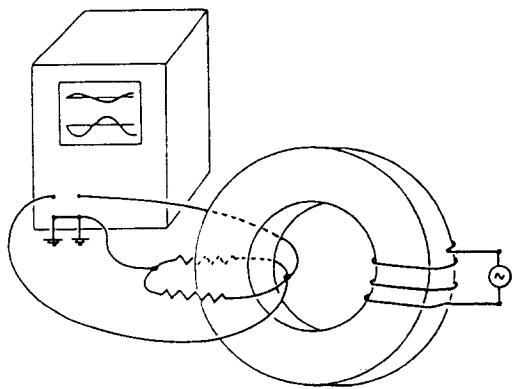Spring 2003

Lecture Notes 8, 3/6/03

I. Nonuniqueness of Voltage in an MQS System

Figure 10.0.1 A pair of unequal resistors are connected in series around a magnetic circuit. Voltages measured between the terminals of the resistors by connecting the nodes to the dual-trace oscilloscope, as shown, differ in magnitude and are 180 degrees out of phase.
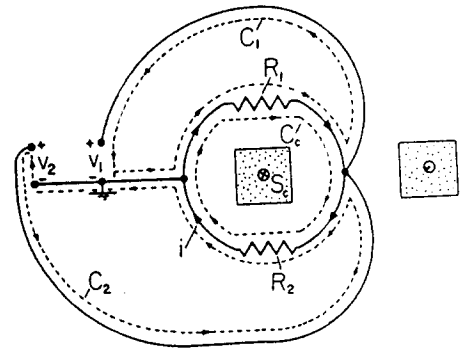

Figure 10.0.2 Schematic of circuit for experiment of Figure 10.0.1, showing contours used with Faraday's law to predict the differing voltages v_1 and v_2.

$$\Phi_\lambda = \int_{S_c} \overline{B} \cdot \overline{da}$$

$$\oint_{C_1} \overline{E} \cdot \overline{ds} = v_1 + i R_1 = 0$$

$$\oint_{C_2} \overline{E} \cdot \overline{ds} = -v_2 + i R_2 = 0$$

$$\oint_{C_c} \overline{E} \cdot \overline{ds} = -\frac{d\Phi_\lambda}{dt} = i(R_1 + R_2)$$

$$i = -\frac{1}{(R_1 + R_2)} \frac{d\Phi_\lambda}{dt}$$

$$v_1 = -i R_1 = +\frac{R_1}{R_1 + R_2} \frac{d\Phi_\lambda}{dt}$$

$$v_2 = i R_2 = -\frac{R_2}{R_1 + R_2} \frac{d\Phi_\lambda}{dt}$$

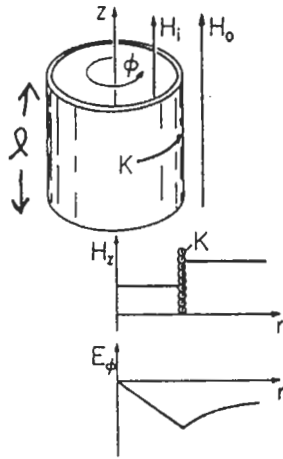$$\frac{v_1}{v_2} = -\frac{R_1}{R_2}$$

Figure 10.3.2 Circular cylindrical conducting shell with external axial field intensity $H_o(t)$ imposed. The response to a step in applied field is a current density that initially shields the field from the inner region. As this current decays, the field penetrates into the interior and is finally uniform throughout.

$$K_\phi = H_i - H_o = J_\phi \Delta = \Delta \sigma E_\phi \implies E_\phi = \frac{K_\phi}{\sigma \Delta}$$

$$\oint_C \vec{E} \cdot \vec{ds} = E_\phi 2\pi a = -\frac{d}{dt}\left[\mu_o \pi a^2 H_i\right] = \frac{2\pi a \, K_\phi}{\sigma \Delta} = \frac{2\pi a}{\sigma \Delta}(H_i - H_o)$$

$$\frac{d H_i}{dt} + \frac{2\pi a \sigma}{\sigma \Delta \mu_o \pi a^2}(H_i - H_o) = 0$$

$$\frac{d H_i}{dt} + \frac{H_i}{\tau_m} = \frac{H_o}{\tau_m} \quad ; \quad \tau_m = \frac{\mu_o \sigma \Delta a}{2} \left[\text{Magnetic Diffusion Time}\right]$$

$$H_i = H_o\left[1 - e^{-t/\tau_m}\right]$$

$$K_\phi = H_i - H_o = -H_o \, e^{-t/\tau_m}$$

Note: $L = \dfrac{\Phi}{K_\phi \ell} = \dfrac{\mu_o H_i \pi a^2}{H_i \ell} = \dfrac{\mu_o \pi a^2}{\ell}$

$R = \dfrac{2\pi a}{\sigma \ell \Delta} \implies \tau_m = \dfrac{L}{R} = \dfrac{\mu_o \pi a^2 \sigma \ell \Delta}{\ell 2\pi a}$

$\qquad = \dfrac{\mu_o \sigma \Delta a}{2}$

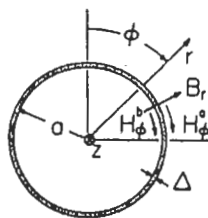III. Diffusion of <u>Transverse</u> Magnetic Fields Through Thin Conductors

Figure 10.4.1 Cross-section of circular cylindrical conducting shell having its axis perpendicular to the magnetic field.

$$\vec{n} \cdot \left[\vec{B}^a - \vec{B}^b\right] = 0 \Rightarrow B_r^a = B_r^b$$

$$K_z = \Delta J_z = \Delta \sigma E_z = H_\phi^a - H_\phi^b$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \frac{1}{a}\frac{\partial E_z}{\partial \phi} = -\frac{\partial B_r}{\partial t}$$

$$E_z = \frac{1}{\Delta}\left[H_\phi^a - H_\phi^b\right]$$

$$\frac{1}{\Delta \sigma a}\frac{\partial}{\partial \phi}\left[H_\phi^a - H_\phi^b\right] = -\frac{\partial B_r}{\partial t}$$

II. Diffusion of Transverse Magnetic Field into Circular Cylindrical Conducting Shell with a Permeable Core
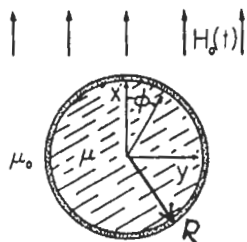

Figure 10.4.2 Circular cylindrical conducting shell filled by insulating material of permeability μ and surrounded by free space. A magnetic field $H_o(t)$ that is uniform at infinity is imposed transverse to the cylinder axis.

$$\vec{B}^o = \mu_o \vec{H}^o \; ; \; \vec{B}^i = \mu \vec{H}^i$$

$$\vec{H} = -\nabla \Psi \Rightarrow \nabla^2 \Psi = 0$$

$$\Psi = \begin{cases} C r \cos\phi & r < R \\[2em] \left[-H_o r + \frac{A}{r}\right]\cos\phi & r > R \end{cases}$$

$$\vec{H} = -\nabla\Psi = -\left[\frac{\partial \Psi}{\partial r}\vec{i}_r + \frac{1}{r}\frac{\partial \Psi}{\partial \phi}\vec{i}_\phi\right]$$

$$= \begin{cases} -C\left[\cos\phi\,\vec{i}_r - \sin\phi\,\vec{i}_\phi\right] & r < R \\[2em] -\left[\left(-H_o - \frac{A}{r^2}\right)\cos\phi\,\vec{i}_r - \left(-H_o + \frac{A}{r^2}\right)\sin\phi\,\vec{i}_\phi\right] & r > R \end{cases}$$

B.C.

$$B_r(r=R_-) = B_r(r=R_+) \Rightarrow \mu H_r(r=R_-) = \mu_0 H_r(r=R_+)$$

$$-\mu C = \mu_0\left(H_0 + \frac{A}{R^2}\right)$$

$$\frac{1}{\Delta T R}\frac{\partial}{\partial \phi}\left(H_\phi^a - H_\phi^b\right) = -\frac{\partial B_r}{\partial t} \Rightarrow \frac{1}{\Delta T R}\frac{\partial}{\partial \phi}\left[H_\phi(r=R_+) - H_\phi(r=R_-)\right] = -\frac{\partial B_r}{\partial t}\Big|_{=R}$$

$$\frac{1}{\Delta T R}\frac{\partial}{\partial \phi}\left[\left(-H_0 + \frac{A}{R^2} - c\right)\sin\phi\right] = -\mu\frac{dC}{dt}\cos\phi$$

$$\frac{1}{\Delta T R}\left[-H_0 + \frac{A}{R^2} - c\right] = \mu\frac{dC}{dt} = -\mu_0\frac{d}{dt}\left(H_0 + \frac{A}{R^2}\right)$$
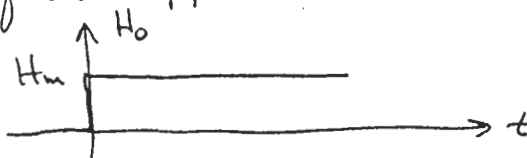
$$\frac{\mu_0}{R^2}\frac{dA}{dt} + \frac{A}{\Delta T R^3}\left(1 + \frac{\mu_0}{\mu}\right) = -\mu_0\frac{dH_0}{dt} + \frac{H_0}{\Delta T R}\left(1 - \frac{\mu_0}{\mu}\right)$$

$$\frac{dA}{dt} + \frac{\left(1 + \frac{\mu_0}{\mu}\right)}{\mu_0\Delta T R}A = -R^2\frac{dH_0}{dt} + \frac{H_0 R^2}{\Delta T R \mu_0}\left(1 - \frac{\mu_0}{\mu}\right)$$

$$\tau_m = \frac{\mu_0 \Delta T R}{1 + \frac{\mu_0}{\mu}}$$

$$\frac{dA}{dt} + \frac{A}{\tau_m} = -R^2\frac{dH_0}{dt} + \frac{H_0 R}{\Delta T \mu_0}\left(1 - \frac{\mu_0}{\mu}\right)$$

Solution for a Stepped Magnetic Field

$$A(t=0_+) = -H_m R^2$$

$$A(t \to \infty) = \frac{H_m R}{\Delta T \mu_0}\left(1 - \frac{\mu_0}{\mu}\right)\tau_m = \frac{H_m R}{\Delta T \mu_0}\left(1 - \frac{\mu_0}{\mu}\right)\frac{\mu_0 \Delta T R}{1 + \frac{\mu_0}{\mu}} = \frac{H_m R^2\left(1 - \frac{\mu_0}{\mu}\right)}{1 + \frac{\mu_0}{\mu}}$$

$$A(t) = A(t \to \infty) + \left[A(t=0_+) - A(t \to \infty)\right]e^{-t/\tau_m}$$

$$= H_m R^2\left[\frac{\mu - \mu_0}{\mu + \mu_0}\left(1 - e^{-t/\tau_m}\right) - e^{-t/\tau_m}\right]$$

$$C(t) = -\frac{\mu_0}{\mu}\left[H_m + \frac{A(t)}{R^2}\right]$$

Surface Current Distribution:

$$K_z(r=R) = H_\phi(r=R_+) - H_\phi(r=R_-)$$

$$= \left[-H_0 + \frac{A}{R^2} - c \right] \sin\phi$$

$$= \left[-H_m + \frac{A}{R^2} + \frac{\mu_0}{\mu}\left(H_m + \frac{A}{R^2}\right) \right] \sin\phi$$

$$= \left[H_m\left(-1 + \frac{\mu_0}{\mu}\right) + \frac{A}{R^2}\left(1 + \frac{\mu_0}{\mu}\right) \right] \sin\phi$$

$$= -2H_m e^{-t/\tau_m} \sin\phi$$

$t/\tau = 0$

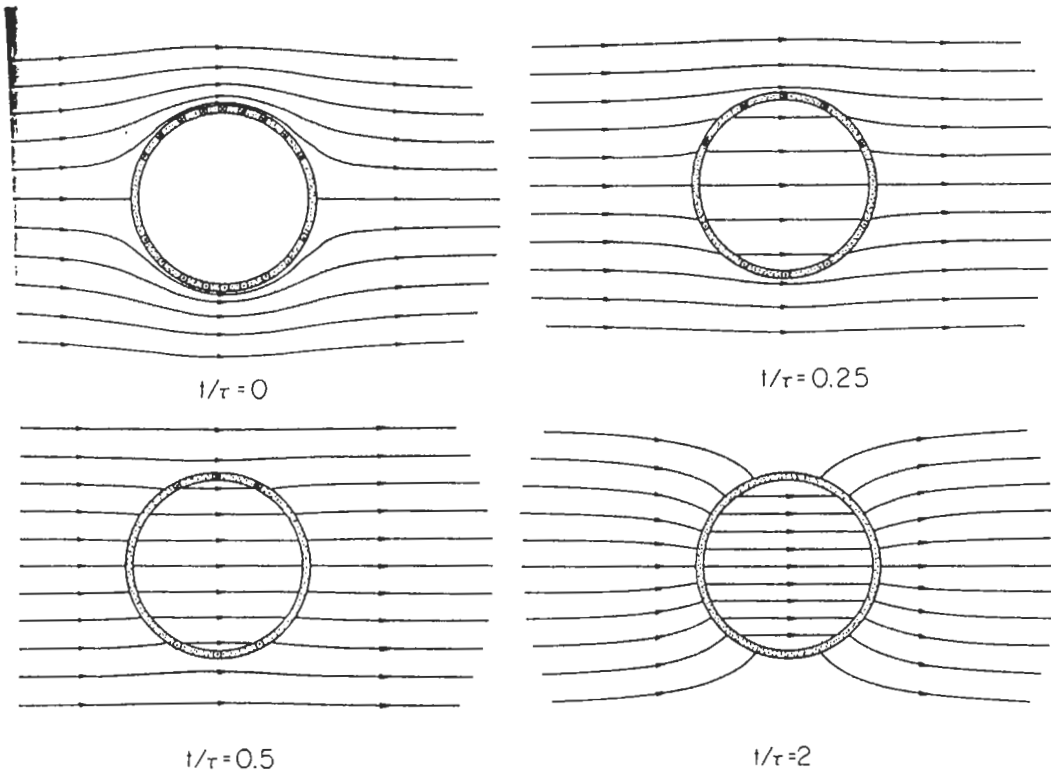$t/\tau = 0.25$

$t/\tau = 0.5$

$t/\tau = 2$

Figure 10.4.3 When $t = 0$, a magnetic field that is uniform at infinity is suddenly imposed on the circular cylindrical conducting shell. The cylinder is filled by an insulating material of permeability $\mu = 200\mu_0$. When $t/\tau = 0$, an instant after the field is applied, the surface currents completely shield the field from the central region. As time goes on, these currents decay, until finally the field is no longer influenced by the conducting shell. The final field is essentially perpendicular to the highly permeable core. In the absence of this core, the final field would be uniform.