Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.641 Electromagnetic Fields, Forces, and Motion
Spring 2003

Lecture Notes 13, 4/1/03

I. EQS Energy Method of Forces
A. Circuit Point of View

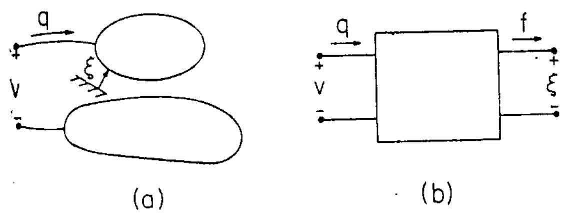


Figure 11.6.1 (a) Electroquasistatic system having one electrical terminal pair and one mechanical degree of freedom. (b) Schematic representation of EQS subsystem with coupling to external mechanical system represented by a mechanical terminal pair.

$$q = C(\xi)v$$

$$i = \frac{dq}{dt} = \frac{d}{dt}\left[C(\xi)v\right] = C(\xi)\frac{dv}{dt} + v\frac{dC(\xi)}{dt}$$

$$= C(\xi)\frac{dv}{dt} + v\frac{dC}{d\xi}\frac{d\xi}{dt}$$

$$P_{in} = vi = v\frac{d}{dt}\left[C(\xi)v\right] = C(\xi)v\frac{dv}{dt} + v^2\frac{dC}{d\xi}\frac{d\xi}{dt}$$

$$= C(\xi)\frac{d}{dt}\left(\frac{1}{2}v^2\right) + v^2\frac{dC}{d\xi}\frac{d\xi}{dt}$$

$$= \frac{d}{dt}\left[\frac{1}{2}C(\xi)v^2\right] + \frac{1}{2}v^2\frac{dC}{d\xi}\frac{d\xi}{dt}$$

$$= \underbrace{\frac{dW}{dt}}_{W = \text{energy storage}} + \underbrace{f_\xi\frac{d\xi}{dt}}_{\text{mechanical power (force} \times \text{velocity)}}$$

$$W = \frac{1}{2}C(\xi)v^2, \quad f_\xi = \frac{1}{2}v^2\frac{dC}{d\xi}$$

$$= \frac{1}{2}\frac{q^2}{C^2(\xi)}\frac{dC}{d\xi} = -\frac{1}{2}q^2\frac{d}{d\xi}\left(\frac{1}{C(\xi)}\right)$$

B. Energy Point of View

$$v i = v \frac{dq}{dt} = \frac{dW_e}{dt} + f_\xi \frac{d\xi}{dt}$$

$$v\,dq = dw_e + f_\xi\,d\xi \Rightarrow dw_e = v\,dq - f_\xi\,d\xi$$

$$f_\xi = -\frac{\partial w_e}{\partial \xi}\bigg|_{q=\text{constant}} ; \quad v = \frac{\partial w_e}{\partial q}\bigg|_{\xi=\text{constant}}$$

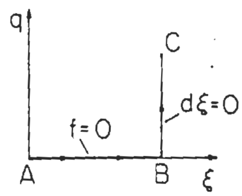


Figure 11.6.2 Path of line integration in state space (q,ξ) used to find energy at location C.

$$w_e = -\int_{q=0} \cancelto{0}{f_\xi}\,d\xi + \int_{\xi=\text{constant}} v\,dq$$

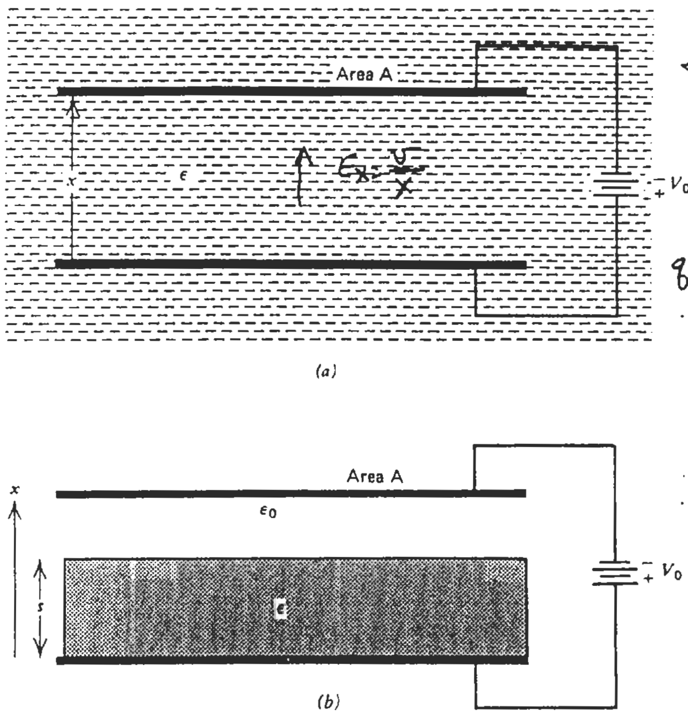
$$v = \frac{q}{C(\xi)}$$

$$w_e = \int_{\xi=\text{constant}} \frac{q}{C(\xi)}\,dq = \frac{1}{2}\frac{q^2}{C(\xi)}$$

$$f = -\frac{\partial w_e}{\partial \xi}\bigg|_{q=\text{constant}} = -\frac{1}{2} q^2 \frac{d}{d\xi}\left(\frac{1}{C(\xi)}\right) = \frac{1}{2}\frac{q^2}{C^2(\xi)}\frac{dC(\xi)}{d\xi}$$

$$= \frac{1}{2} v^2 \frac{dC(\xi)}{d\xi}$$

II Forces In Capacitors



$\sigma_s = -\epsilon E_x = -\frac{\epsilon v}{x}$

$q = \sigma_s A = \epsilon E_x A = \frac{\epsilon v A}{x} = C(x)\, v$

$C(x) = \frac{\epsilon A}{x}$

Figure 3-36 A parallel plate capacitor (*a*) immersed within a dielectric fluid or with (*b*) a free space region in series with a solid dielectric.

(a) Coulombic force method on upper electrode:

$$f_x = \frac{1}{2}\sigma_s E_x A = -\frac{1}{2}\epsilon E_x^2 A = -\frac{1}{2}\frac{\epsilon v^2}{x^2}A$$

$\frac{1}{2}$ because E in electrode = 0, E outside electrode = E_x
take average

Energy method: $C(x) = \frac{\epsilon A}{x}$

$$f_x = \frac{1}{2}v^2\frac{dC}{dx} = \frac{1}{2}v^2\epsilon A\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{2}\frac{v^2\epsilon A}{x^2}$$

$$v = \frac{q}{C(x)} = \frac{q x}{\epsilon A} \Rightarrow f_x = -\frac{1}{2}\frac{\cancel{\epsilon A}}{\cancel{x^2}}\frac{q^2\cancel{x^2}}{\epsilon^{\cancel{2}}A^{\cancel{2}}} = -\frac{1}{2}\frac{q^2}{\epsilon A}$$

(b)

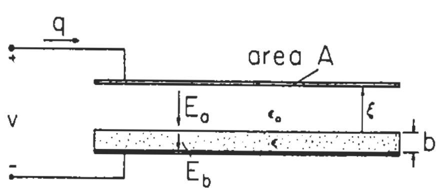


Figure 11.6.3 Specific example of EQS systems having one electrical and one mechanical terminal pair.

$$\frac{1}{C(\xi)} = \frac{1}{C_a} + \frac{1}{C_b} \quad ; \quad C_a = \frac{\epsilon_0 A}{\xi}, \quad C_b = \frac{\epsilon A}{b}$$

$$= \frac{\xi}{\epsilon_0 A} + \frac{b}{\epsilon A}$$

$$= \frac{\epsilon\xi + \epsilon_0 b}{\epsilon\epsilon_0 A}$$

$$f_\xi = -\frac{1}{2} q^2 \frac{d}{d\xi}\left(\frac{1}{C(\xi)}\right) = -\frac{1}{2}\frac{q^2}{\epsilon\epsilon_0 A}\frac{d}{d\xi}(\epsilon\xi+\epsilon_0 b) = -\frac{1}{2}\frac{q^2}{\epsilon_0 A}$$

$$f_\xi = \frac{1}{2} v^2 \frac{d(C(\xi))}{d\xi} = \frac{1}{2} v^2 \frac{d}{d\xi}\left[\frac{\epsilon\epsilon_0 A}{\epsilon\xi+\epsilon_0 b}\right] = -\frac{1}{2}\frac{v^2\epsilon^2\epsilon_0 A}{(\epsilon\xi+\epsilon_0 b)^2}$$

III Energy Conversion Cycles

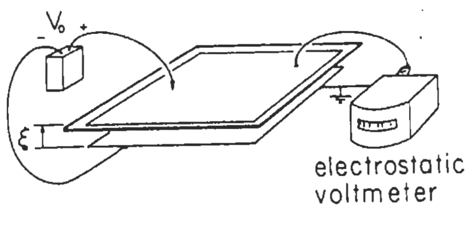


Figure 11.6.4 Apparatus used to demonstrate amplification of voltage as the upper electrode is raised. (The electrodes are initially charged and then the voltage source is removed so q = constant.) The electrodes, consisting of foil mounted on insulating sheets, are about 1 m × 1 m, with the upper one insulated from the frame, which is used to control its position. The voltage is measured by the electrostatic voltmeter, which "loads" the system with a capacitance that is small compared to that of the electrodes and (at least on a dry day) a negligible resistance.

$$\oint v\,dq = \cancel{\oint dW_e}^{\,0} + \oint f_\xi\,d\xi$$

$$\oint v\,dq = \oint f_\xi\,d\xi$$

$\oint v\,dq, \oint f_\xi\,d\xi > 0$

electric energy in, mechanical energy out.

$\oint v\,dq, \oint f_\xi\,d\xi < 0$

Electric power out, mechanical energy in.

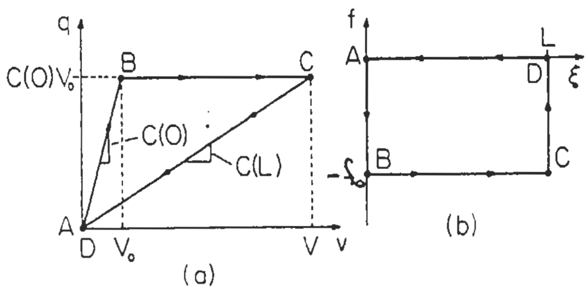


Figure 11.6.5 Closed paths followed in cyclic conversion of energy from mechanical to electrical form: (a) in (q, v) plane; and (b) in (f, ξ) plane.

- $A \to B$. With $v = 0$, the upper electrode rests on the plastic sheet. A voltage V_o is applied.
- $B \to C$. With the voltage source removed so that the upper electrode is electrically isolated, it is raised to the position $\xi = L$.
- $C \to D$. The upper electrode is shorted, so that its voltage returns to zero.
- $D \to A$. The upper electrode is returned to its original position at $\xi = 0$.

Is electrical energy converted to mechanical form, or vice versa?

$$\oint v\,dq = \int_A^B v\,dq + \int_C^D v\,dq = \frac{1}{2}C(0)V_0^2 - \frac{1}{2}C(L)V^2$$

$$C(0)V_0 = C(L)V$$

$$\oint v\,dq = \frac{1}{2}C(0)V_0^2\left[1 - \frac{\cancel{C(L)}\,C(0)}{C^{\cancel{2}}(L)}\right] = \frac{1}{2}C(0)V_0^2\left[1 - \frac{C(0)}{C(L)}\right]$$

$$\frac{C(0)}{C(L)} = \frac{\epsilon A\left(L + b\frac{\epsilon_0}{\epsilon}\right)}{b(\epsilon_0 A)}$$

5

$$\oint v\,dq = \frac{1}{2} C(0) V_0^2 \left[1 - \frac{\left(L + b\frac{\epsilon_0}{\epsilon}\right)\epsilon}{\epsilon_0 b} \right] = -\frac{1}{2} C(0) V_0^2 \frac{\epsilon}{\epsilon_0} \frac{L}{b} < 0$$

(electric energy out)

$$\oint f\,d\xi = -f_0 L$$

$$f_0 = +\frac{1}{2}\frac{q^2}{\epsilon_0 A} = +\frac{1}{2}\frac{C^2(0) V_0^2}{\epsilon_0 A} = +\frac{1}{2} C(0) V_0^2 \left[\frac{\epsilon \cancel{A}}{b\,\epsilon_0 \cancel{A}}\right]$$

$$\oint f\,d\xi = -\frac{1}{2} C(0) V_0^2 \frac{\epsilon L}{\epsilon_0 b} = \oint v\,dq$$

$\oint f\,d\xi < 0 \Rightarrow$ mechanical energy out is negative means mechanical energy is put in

Mechanical energy is converted to electrical energy

III. Force On a Dielectric Material

$$C(\xi) = \frac{\epsilon_0 (b-\xi) c}{a} + \frac{\epsilon \xi c}{a}$$

$$f_\xi = \frac{1}{2} v^2 \frac{dC(\xi)}{d\xi} = \frac{1}{2} v^2 \frac{c}{a}(\epsilon - \epsilon_0)$$

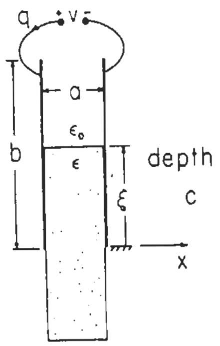


Figure 11.6.6 Slab of dielectric partially extending between capacitor plates. The spacing, a, is much less than either b or the depth c of the system into the paper. Further, the upper surface at ξ is many spacings a away from the upper and lower edges of the capacitor plates, as is the lower surface as well.

In equilibrium:

$$f_\xi = \frac{1}{2} v^2 \frac{c}{a}(\epsilon - \epsilon_0) = \underbrace{\overbrace{\rho}^{\text{mass density}} g\,\xi a c}_{\text{fluid weight}}$$

$$\xi = \frac{1}{2}\frac{v^2 (\epsilon - \epsilon_0)}{\rho g a^2}$$

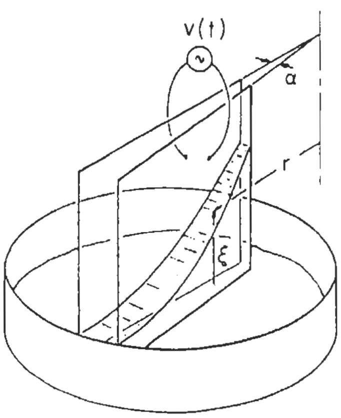


Figure 11.6.7 In a demonstration of the polarization force, a pair of conducting transparent electrodes are dipped into a liquid (corn oil dyed with food coloring). They are closer together at the upper right than at the lower left, so when a voltage is applied, the electric field intensity decreases with increasing distance, r, from the apex. As a result, the liquid is seen to rise to a height that varies as $1/r^2$. The electrodes are about 10 cm × 10 cm, with an electric field exceeding the nominal breakdown strength of air at atmospheric pressure, 3×10^6 V/m. The experiment is therefore carried out under pressurized nitrogen.

$$a \to \alpha r$$

$$\xi = \frac{1}{2}\frac{v^2 (\epsilon - \epsilon_0)}{\rho g \alpha^2 r^2}$$

IV. Physical Model of Forces on Dielectrics

$$\bar{f}_{dipole} = q\left[\bar{E}(\bar{r}+\bar{d}) - \bar{E}(\bar{r})\right]$$
$$= q\left[\bar{E}(\bar{r}) + \bar{d}\cdot\nabla\bar{E}(\bar{r}) - \bar{E}(\bar{r})\right]$$
$$= q(\bar{d}\cdot\nabla)\,\bar{E}$$
$$= (\bar{p}\cdot\nabla)\,\bar{E} \quad \text{Kelvin force}$$

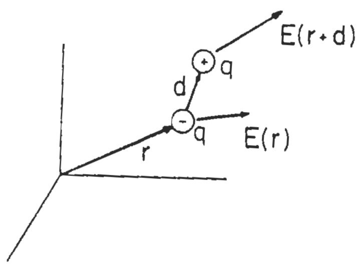


Figure 11.8.1 An electric dipole experiences a net electric force if the positive charge q is subject to an electric field $\mathbf{E}(\mathbf{r}+\mathbf{d})$ that differs from $\mathbf{E}(\mathbf{r})$ acting on the negative charge q.

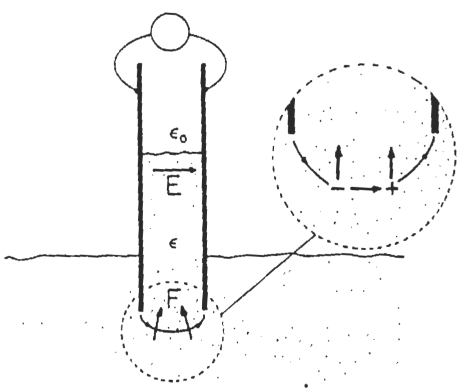
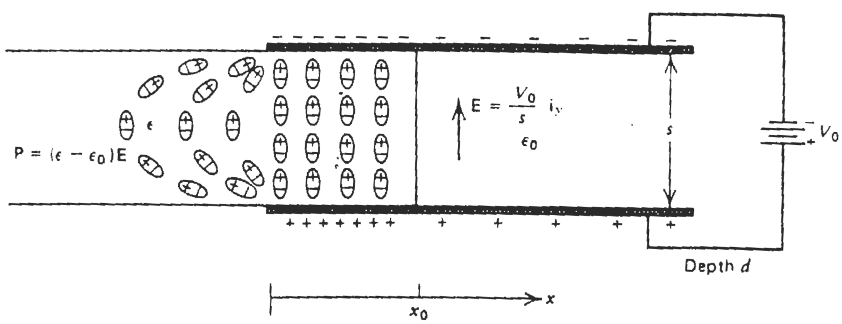


Figure 11.9.4 In terms of the Kelvin force density, the dielectric liquid is pushed into the field region between capacitor plates because of the forces on individual dipoles in the fringing field.



A linear dielectric is always attracted into a free space capacitor because of the net force on dipoles in the nonuniform field. The dipoles are now aligned with the electric field, no matter the voltage polarity.

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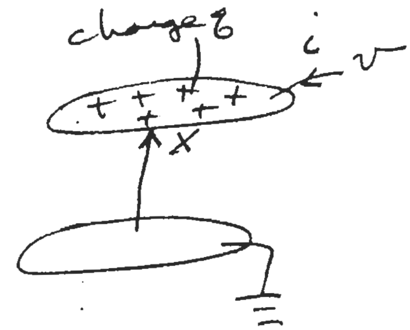
Lecture Notes 13 Supplement, 4/1/03

I. EQS Forces

A. Energy Method

$$q = C(x)\,v$$



$$i = \frac{dq}{dt} = \frac{d}{dt}\left[C(x)\,v\right] = C(x)\frac{dv}{dt} + v\frac{dC(x)}{dt}$$

$$p = vi = C(x)\,v\frac{dv}{dt} + v^2\frac{dC(x)}{dt}$$

$$= \frac{C(x)}{2}\frac{d}{dt}v^2 + v^2\frac{dC(x)}{dt}$$

$$= \frac{d}{dt}\left[\frac{1}{2}C(x)\,v^2\right] + \frac{1}{2}v^2\frac{dC(x)}{dt}$$

$$= \frac{d}{dt}\left[\frac{1}{2}C(x)\,v^2\right] + \frac{1}{2}v^2\frac{dC}{dx}\frac{dx}{dt}$$

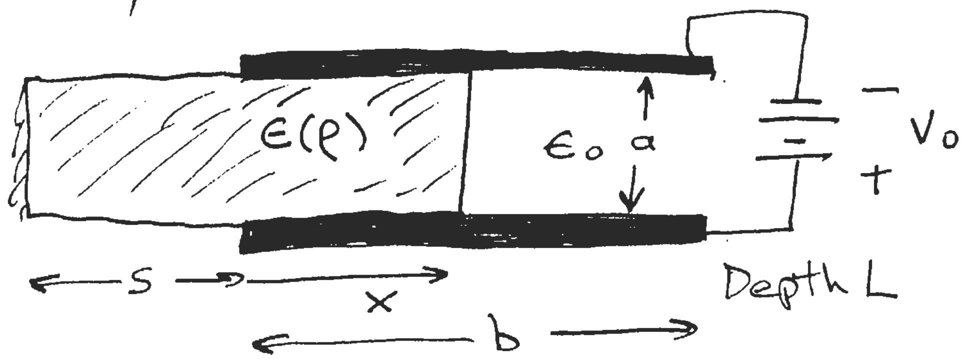
$$= \frac{dW}{dt} + f_x\frac{dx}{dt}$$

$$W = \frac{1}{2}C(x)\,v^2, \quad f_x = \frac{1}{2}v^2\frac{dC}{dx}$$

$$f_x = \frac{1}{2}\frac{q^2}{C^2(x)}\frac{dC}{dx} = -\frac{1}{2}q^2\frac{d\left[1/C(x)\right]}{dx}$$

B. Polarization and Electrostrictive Forces

ρ = mass density



$$C = \frac{\epsilon(\rho) x L}{a} + \frac{\epsilon_0 (b-x) L}{a}$$

$$f_x = \frac{1}{2} V_0^2 \frac{dC}{dx} = \frac{1}{2} \frac{V_0^2 L}{a} \frac{d}{dx}\left[\epsilon(\rho) x - \epsilon_0 x + \epsilon_0 b\right]$$

$$= \frac{1}{2} \frac{V_0^2 L}{a}\left[\epsilon(\rho) - \epsilon_0 + x \frac{d\epsilon}{d\rho}\frac{d\rho}{dx}\right]$$

$$M = \rho a L (s+x)$$

$$\frac{dM}{dx} = 0 = aL\left[(s+x)\frac{d\rho}{dx} + \rho\right] \Rightarrow \frac{d\rho}{dx} = -\frac{\rho}{s+x}$$

$$f_x = \frac{1}{2} \frac{V_0^2 L}{a}\left[\epsilon(\rho) - \epsilon_0 - \frac{\rho x}{s+x}\frac{d\epsilon}{d\rho}\right]$$

C. Continuum Force Density

$$\bar{F} = \rho_f \bar{E} - \frac{1}{2}\bar{E}\cdot\bar{E}\,\nabla\epsilon + \nabla\left(\frac{1}{2}\rho\frac{\partial\epsilon}{\partial\rho}\bar{E}\cdot\bar{E}\right)$$

MQS Forces

A. Energy Method

$$\lambda = L(x)\,i$$

$$v = \frac{d\lambda}{dt} = \frac{d[L(x)\,i]}{dt}$$

$$= L(x)\frac{di}{dt} + i\frac{dL(x)}{dt}$$

$$P = vi = L(x)\,i\frac{di}{dt} + i^2\frac{dL(x)}{dt}$$

$$= \frac{L(x)}{2}\frac{di^2}{dt} + i^2\frac{dL(x)}{dt}$$

$$= \frac{d}{dt}\left[\frac{1}{2}L(x)\,i^2\right] + \frac{1}{2}i^2\frac{dL(x)}{dt}$$

$$= \frac{d}{dt}\left[\frac{1}{2}L(x)\,i^2\right] + \frac{1}{2}i^2\frac{dL}{dx}\frac{dx}{dt}$$

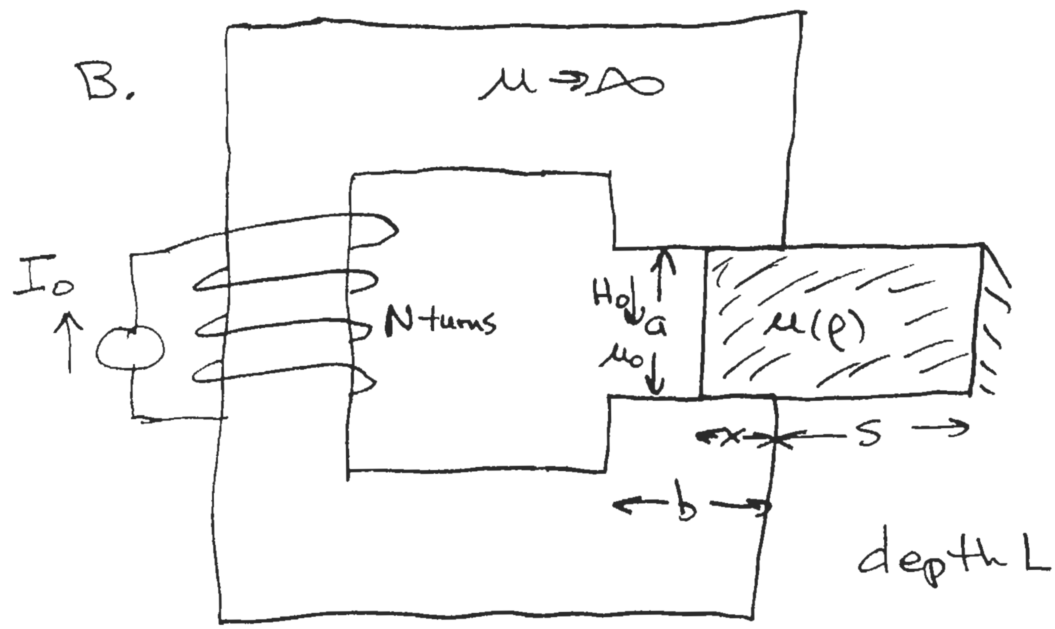
$$= \frac{dW}{dt} + f_x\frac{dx}{dt}$$

$$W = \frac{1}{2}L(x)\,i^2\,; \quad f_x = \frac{1}{2}i^2\frac{dL}{dx}$$

$$= \frac{1}{2}\frac{\lambda^2}{L^2}\frac{dL}{dx}$$

$$= -\frac{1}{2}\lambda^2\frac{d(1/L(x))}{dx}$$

B.



$$\lambda = \left[\mu_0 H_0 (b-x) L + \mu(\rho) H_0 x L\right] N$$

$$H_0 = \frac{Ni}{a}$$

$$\lambda = \frac{N^2 i}{a} L \left[\mu_0 (b-x) + \mu(\rho) x\right]$$

$$\frac{\lambda}{i} = L(x) = \frac{N^2 L}{a}\left[\mu_0 (b-x) + \mu(\rho) x\right]$$

$$f_x = \frac{1}{2} i^2 \frac{dL}{dx} = \frac{1}{2} I_0^2 \frac{N^2 L}{a} \frac{d}{dx}\left[\mu(\rho) x + \mu_0 (b-x)\right]$$

$$= \frac{1}{2} I_0^2 \frac{N^2 L}{a}\left[\mu(\rho) - \mu_0 + x \frac{d\mu}{d\rho}\frac{d\rho}{dx}\right]$$

$$M = \rho_a L (s+x) \Rightarrow \frac{d\rho}{dx} = \frac{-\rho}{s+x}$$

$$f_x = \frac{1}{2} I_0^2 \frac{N^2 L}{a}\left[\mu(\rho) - \mu_0 - \frac{\rho x}{s+x}\frac{d\mu}{d\rho}\right]$$