Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.641 Electromagnetic Fields, Forces, and Motion
Spring 2003

Lecture Notes 14, 4/3/03

I. MQS Energy Method of Forces

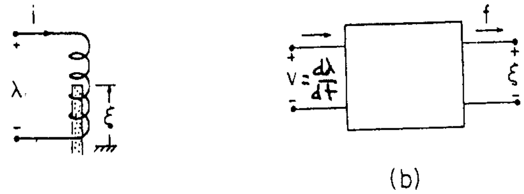


Figure 11.6.1 (a) Magnetoquasistatic system having one electrical terminal pair and one mechanical degree of freedom. (b) Schematic representation of MQS subsystem with coupling to external mechanical system represented by a mechanical terminal pair.

A. Circuit Approach

$$v = \frac{d\lambda}{dt} = \frac{d}{dt}\left[L(\xi)i\right] = L(\xi)\frac{di}{dt} + i\frac{dL(\xi)}{dt}$$

$$p = vi = L(\xi)i\frac{di}{dt} + i^2\frac{dL(\xi)}{dt}$$

$$= L(\xi)\frac{d}{dt}\left(\tfrac{1}{2}i^2\right) + i^2\frac{dL(\xi)}{dt}$$

$$= \frac{d}{dt}\left[\tfrac{1}{2}L(\xi)i^2\right] + \tfrac{1}{2}i^2\frac{dL(\xi)}{dt}$$

$$= \frac{d}{dt}\left[\tfrac{1}{2}L(\xi)i^2\right] + \tfrac{1}{2}i^2\frac{dL(\xi)}{d\xi}\frac{d\xi}{dt}$$

$$vi = \frac{dW}{dt} + f_\xi\frac{d\xi}{dt} \Rightarrow W = \tfrac{1}{2}L(\xi)i^2, \quad f_\xi = \tfrac{1}{2}i^2\frac{dL(\xi)}{d\xi}$$

$$\lambda = L(\xi)i \Rightarrow f_\xi = \tfrac{1}{2}i^2\frac{dL(\xi)}{d\xi}$$

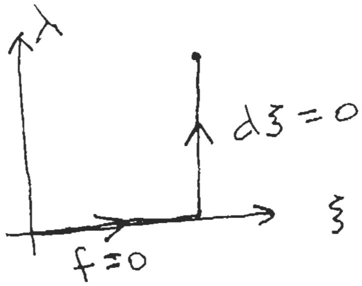
$$= \frac{1}{2}\frac{\lambda^2}{L^2(\xi)}\frac{dL(\xi)}{d\xi}$$

$$= -\frac{1}{2}\lambda^2\frac{d}{d\xi}\left[1/L(\xi)\right]$$

B. Energy Method

$$vi = i\frac{d\lambda}{dt} = \frac{dW_m}{dt} + f_\xi\frac{d\xi}{dt} \Rightarrow dW_m = i\,d\lambda - f_\xi\,d\xi$$

$$f_\xi = -\frac{\partial W_m}{\partial \xi}\bigg|_{\lambda = \text{constant}}, \quad i = \frac{\partial W_m}{\partial \lambda}\bigg|_{\xi = \text{constant}}$$



$$w_m = \int_{\lambda=0} -\cancel{f_\xi}^{\,0}\, d\xi + \int_{\xi=\text{constant}} i\, d\lambda$$

$$i = \frac{\lambda}{L(\xi)}$$

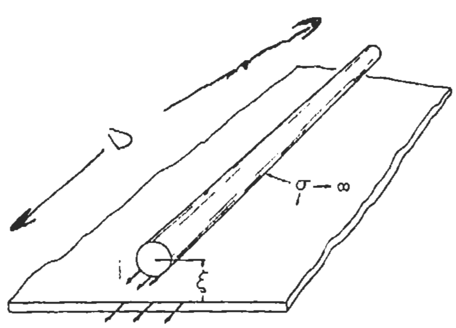
$$w_m = \int_{\xi=\text{constant}} \frac{\lambda}{L(\xi)}\, d\lambda = \frac{\lambda^2}{2L(\xi)}$$

$$f_\xi = -\left.\frac{\partial w_m}{\partial \xi}\right|_{\lambda=\text{constant}} = -\frac{1}{2}\lambda^2 \frac{d\left(1/L(\xi)\right)}{d\xi}$$

$$= -\frac{1}{2}\lambda^2 \left(-\frac{1}{L^2(\xi)}\right)\frac{dL(\xi)}{d\xi}$$

$$= \frac{1}{2} i^2 \frac{dL(\xi)}{d\xi}$$

II. Force On a Wire over a Perfectly Conducting Plane



Depth D

Figure 11.7.3 Cross-section of perfectly conducting current-carrying wire over a perfectly conducting ground plane.

$$L(\xi) = \frac{\mu_0 D}{2\pi} \ln\left[\frac{\xi}{R} + \sqrt{\left(\frac{\xi}{R}\right)^2 - 1}\right]$$

[See p. 343, take $\frac{1}{2}$ of Eq. (12) which is the inductance between 2 cylinders]

A. Energy Method

$$f_\xi = \frac{1}{2} i^2 \frac{dL(\xi)}{d\xi} = \frac{\mu_0 i^2 D}{4\pi R} \frac{1}{\sqrt{(\xi/R)^2 - 1}}$$

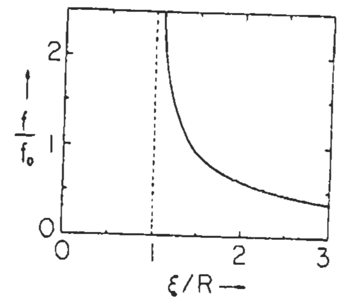
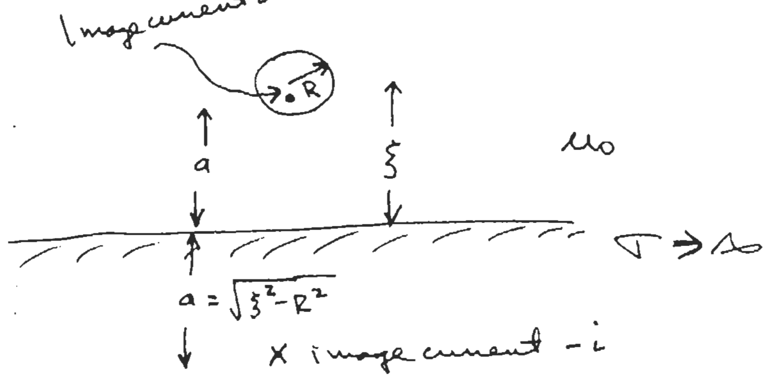


Figure 11.7.4 The force tending to levitate the wire of Figure 11.7.3 as a function of the distance to the ground plar normalized to the radius R of wire.

B. Method of Images Approach with Lorentz Force



$$f_\xi = iD\left(\frac{\mu_0 i}{2\pi(2a)}\right) = \frac{\mu_0 i^2 D}{4\pi a} = \frac{\mu_0 i^2 D}{4\pi\sqrt{\xi^2 - R^2}}$$

C. Demonstration: Steady State Magnetic Levitation

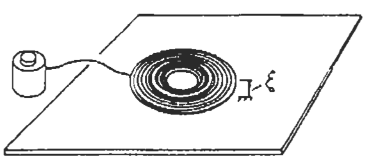


Figure 11.7.5 When the pancake coil is driven by an ac current, it floats above the aluminum plate. In this experiment, the coil consists of 250 turns of No. 10 aluminum wire with an outer radius of 16 cm and an inner one of 2.5 cm. The aluminum sheet has a thickness of 1.3 cm. With a 60 Hz current i of about 20 amp rms, the height above the plate is 2 cm.

III. One Turn Loop

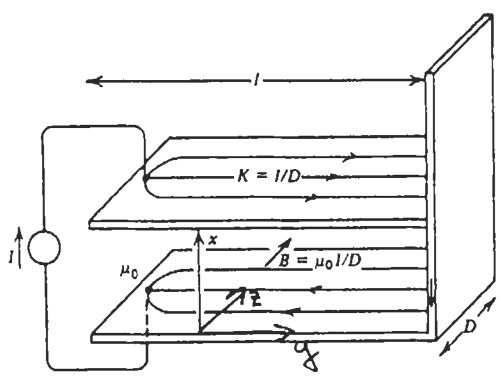


Figure 6-35 The magnetic force on a current-carrying loop tends to expand the loop.

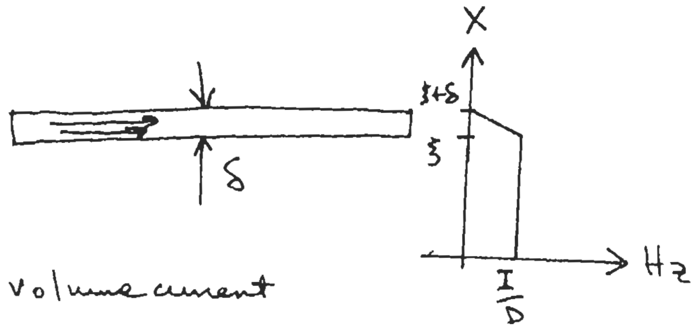
$$H_z = \frac{I}{D}, \quad \Phi = \mu_0 H_z x l = \frac{\mu_0 x l}{D} I, \quad L(x) = \Phi / I = \frac{\mu_0 x l}{D}$$

A. Energy Method

$$f_x = \frac{1}{2} I^2 \frac{dL(x)}{dx} = \frac{1}{2} I^2 \frac{\mu_0 l}{D}$$

B. Lorentz Force Law

$$\bar{f} = \int_V \bar{J} \times \bar{B} \, dV$$



Model surface current $K_y = \frac{I}{D}$ as volume current $J_y = \frac{I}{D\delta}$.

$$\nabla \times \bar{H} = \bar{J} \Rightarrow \frac{\partial H_z}{\partial x} = -J_y = -\frac{I}{D\delta} \Rightarrow H_z = -\frac{I}{D\delta}\left(x - (\xi+\delta)\right)$$

$$f_x = \int_V J_y \mu_0 H_z \, dx\, dy\, dz$$

$$= \int_{x=\xi}^{\xi+\delta} \frac{I}{D\delta}\left(-\frac{\mu_0 I}{\cancel{D}\delta}\right)\left(x-(\xi+\delta)\right) l \cancel{D} \, dx$$

$$= \frac{-\mu_0 I^2 l}{D\delta^2}\left[\frac{x^2}{2} - (\xi+\delta)x\right]\Bigg|_{x=\xi}^{\xi+\delta}$$

$$= \frac{-\mu_0 I^2 l}{D\delta^2}\left[\frac{(\xi+\delta)^2}{2} - \frac{\xi^2}{2} - (\xi+\delta)^2 + \xi(\xi+\delta)\right]$$

$$= \frac{-\mu_0 I^2 l}{D\delta^2}\left[-\frac{1}{2}(\xi+\delta)^2 + \frac{\xi^2}{2} + \xi\delta\right]$$

$$= -\frac{\mu_0 I^2 l}{D\delta^2}\left[-\frac{1}{2}\delta^2\right]$$

$$= -\frac{1}{2}\frac{\mu_0 I^2 l}{D} \quad \Rightarrow \quad \bar{f} = \int_S \frac{1}{2}\bar{K} \times \bar{B} \, d$$

$\frac{1}{2}$ comes from integrating uniform volume current over small thickness δ

General formula: $\bar{f} = \int_S \bar{K} \times \bar{B}_{av} \, dS$

For our case: $B_{av} = \frac{\cancel{B_{metal}}^{\,0} + B_{air}}{2} = \frac{1}{2} B_{air}$

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Lecture Notes 14 Supplements, 4/3/03

$$\lambda_{as} = L_s i_{as} + M i_r \cos\theta$$

$$\lambda_{bs} = L_s i_{bs} + M i_r \sin\theta$$

$$\lambda_r = L_r i_r + M(i_{as}\cos\theta + i_{bs}\sin\theta)$$

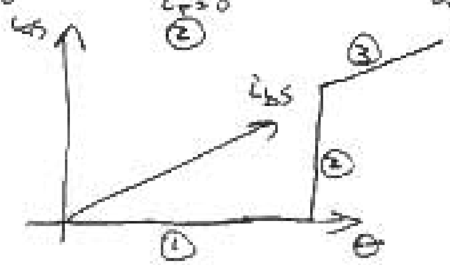
$$dW = i_{as}\,d\lambda_{as} + i_{bs}\,d\lambda_{bs} + i_r\,d\lambda_r - T^e\,d\theta$$

$$d(W - i_{as}\lambda_{as} - i_{bs}\lambda_{bs} - i_r\lambda_r) = -dW'$$

$$W' = i_{as}\lambda_{as} + i_{bs}\lambda_{bs} + i_r\lambda_r - W$$

$$dW' = \lambda_{as}\,di_{as} + \lambda_{bs}\,di_{bs} + \lambda_r\,di_r + T^e\,d\theta$$

$$W' = \underset{\substack{i_{as}=0\\ i_{bs}=0\\ i_r=0\\ ①}}{\int} T\,d\theta + \underset{\substack{\theta=\text{constant}\\ i_{bs}=0\\ i_r=0\\ ②}}{\int} \lambda_{as}\,di_{as} + \underset{\substack{\theta=\text{constant}\\ i_{as}=\text{constant}\\ i_r=0\\ ③}}{\int} \lambda_{bs}\,di_{bs} + \underset{\substack{\theta=\text{constant}\\ i_{as}=\text{constant}\\ i_{bs}=\text{constant}\\ ④}}{\int} \lambda_r\,di_r$$



$$W' = \underset{\substack{i_{as}=0\\ i_{bs}=0\\ i_r=0\\ ①}}{\int} \cancel{T}^{\,0}\,d\theta + \underset{②}{\int} L_s i_{as}\,di_{as} + \underset{③}{\int} L_s i_{bs}\,di_{bs} + \underset{\substack{④\\ \theta=\text{constant}\\ i_{as}=\text{constant}\\ i_{bs}=\text{constant}}}{\int} \left[L_r i_r + M(i_{as}\cos\theta + i_{bs}\sin\theta)\right] di_r$$

$$W' = \frac{1}{2} L_s i_{as}^2 + \frac{1}{2} L_s i_{bs}^2 + \frac{1}{2} L_r i_r^2 + M i_r (i_{as} \cos\theta + i_{bs} \sin\theta)$$

$$T^e = + \left.\frac{\partial W'}{\partial \theta}\right|_{i_{as}, i_{bs}, i_r} = M i_r (-i_{as} \sin\theta + i_{bs} \cos\theta)$$

Balanced 2 phase currents

$$i_{as} = I_s \cos\omega t \;,\; i_{bs} = I_s \sin\omega t \;,\; i_r = I_r \;,\; \theta = \omega_m t + \gamma$$

$$T^e = M I_r I_s (-\cos\omega t \sin\theta + \sin\omega t \cos\theta) = M I_r I_s \sin(\omega t - \theta)$$
$$= M I_r I_s \sin((\omega - \omega_m) t - \gamma)$$

$$\langle T^e \rangle \neq 0 \Rightarrow \omega = \omega_m$$
$$T^e = -M I_r I_s \sin\gamma$$
$$J \frac{d^2\theta}{dt^2} = T^e - \beta \frac{d\theta}{dt}$$

$$\theta = \omega_m t + \gamma_0 + \gamma'(t) \quad , \; \gamma'(t) << \gamma_0$$

$$-M I_r I_s \sin\gamma_0 - \beta\omega = 0$$
$$\sin\gamma_0 = -\frac{\beta\omega}{M I_r I_s}$$

Pullout when $|\sin\gamma_0| = 1 \Rightarrow \beta\omega = M I_r I_s$

Hunting transients: $\sin(\gamma_0 + \gamma') \approx \sin\gamma_0 \cos\gamma' + \cos\gamma_0 \sin\gamma' \approx \sin\gamma_0 + \gamma' \cos\gamma_0$

$$J \frac{d^2\gamma'}{dt^2} = -M I_r I_s \cos\gamma_0 \, \gamma' - \beta\gamma' = -(M I_r I_s \cos\gamma_0 + \beta)\gamma'$$

$$\frac{d^2\gamma'}{dt^2} + \omega_0^2 \gamma' = 0 \quad ; \quad \omega_0^2 = [M I_r I_s \cos\gamma_0 + \beta]/J$$

$$\gamma' = A_1 \sin\omega_0 t + A_2 \cos\omega_0 t$$