

Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.641 Electromagnetic Fields, Forces, and Motion
Spring 2003

Lecture Notes 16, 4/15/03

D-C Magnetic Machines

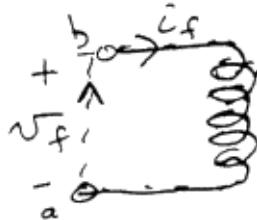
I. Quasi-One Dimensional Description

A. Electrical Equations

(2)

$$\oint_C \bar{E} \cdot d\bar{l} = -\frac{d}{dt} \int_S \bar{B} \cdot \bar{n} da$$

1. Field Winding



$$\oint_C \bar{E} \cdot d\bar{l} = -v_f + \int_{\text{windings}} \frac{i_f}{A\sigma} d\bar{l} = -v_f + i_f R_f$$

Resistance
of field
winding

$$\lambda_f = \int_S \bar{B} \cdot \bar{n} da = L_f i_f$$

$$-v_f + i_f R_f = -L_f \frac{di_f}{dt}$$

$$v_f = L_f \frac{di_f}{dt} + i_f R_f$$

2. Armature Winding

$$\text{Reminder: } \bar{f} = g(\bar{E} + \bar{\tau} \times \bar{B}) = g \bar{E}'$$

$$\bar{E}' = \bar{E} + \bar{\tau} \times \bar{B}$$

Take Stationary Contour through
armature winding

$$\bar{E} = \bar{E}' - \bar{\tau} \times \bar{B}$$

$$\oint_C \bar{E} \cdot d\bar{l} = -v_a + \int (\bar{E}' - \bar{\tau} \times \bar{B}) \cdot d\bar{l}$$

$$= -v_a + \int_a^b \left(\frac{i_a}{A\sigma} + \omega R B_r \right) \cdot d\bar{l}$$

$$; \bar{\tau} = \omega R \bar{i}_\theta$$

$$\bar{B} = i_r B_r (\gamma)$$

$$= -v_a + i_a R_a + \omega R (B_{r\text{far}} \& N)$$

$$= -\frac{d}{dt} \int_S \bar{B} \cdot \bar{d}a = -L_a \frac{di_a}{dt}$$

$$V_a = i_a R_a + L_{\text{dia}} \frac{di_a}{dt} + G\omega i_f \quad (G i_f = \text{LENR}(B_{rf})_{aw})$$

B. Mechanical Equations

$$\bar{F} = \bar{i}_\theta J_2 B_r = \bar{i}_\theta \frac{i_a}{A_w} B_r, \quad \bar{f} = \bar{F} A_w l = \bar{i}_\theta i_{al} B_r$$

$$T = f R = i_{al} B_r R N = G i_f i_a$$

$$J \frac{d^2 \theta}{dt^2} = T = G i_f i_a$$

C. Linear Amplifier

1. Open Circuit

$$V_f = V_f, \quad i_a = 0 \Rightarrow i_f = V_f / R_f$$

$$V_a = G\omega V_f / R_f$$

2. Resistively Loaded Armature (DC Generator)

$$V_a = -i_a R_L = i_a R_a + G\omega V_f / R_f$$

$$i_a = -\frac{G\omega V_f}{R_f (R_a + R_L)}$$

$$V_a = \frac{G\omega V_f R_L}{R_f (R_a + R_L)}$$

D. DC Motors

$$1. \text{ Shunt Excitation: } V_a = V_f = V_t$$

$$V_t = i_f R_f = i_a R_a + G\omega i_f$$

$$i_f (R_f - G\omega) = i_a R_a$$

$$i_f = \frac{V_t}{R_f}, \quad i_a = \frac{V_t}{R_f} \frac{(R_f - G\omega)}{R_a}$$

$$T = G i_f i_a = G \left(\frac{V_t}{R_f} \right)^2 \frac{(R_f - G\omega)}{R_a}$$

2. Series: $i_a = i_f = i_t$

$$i_t (R_f + R_a + G\omega) = V_t$$

$$i_t = \frac{V_t}{(R_f + R_a + G\omega)}$$

$$T = G i_t^2 = G \frac{V_t^2}{(R_f + R_a + G\omega)^2}$$

II. Self-Excited Machines

I. D-C Magnetic Machines

$$\bar{H} = -\nabla \Phi \Rightarrow H_\theta = -\frac{1}{r} \frac{\partial \Phi}{\partial \theta}$$

A. Sources

$$H_r = -\frac{\partial \Phi}{\partial r}$$

On $r \rightarrow \infty$ surfaces of field yoke, $H_\theta = 0 \Rightarrow \Phi = \text{constant}$

$$\oint_C \bar{H} \cdot d\bar{l} = \oint_C -\nabla \Phi \cdot d\bar{l}$$

$$\left. \begin{array}{l} \oint_a \bar{H} \cdot d\bar{l} = \Phi_a - \Phi_b \\ \oint_b \bar{H} \cdot d\bar{l} = \Phi_b - \Phi_c \end{array} \right\} -\nabla \Phi \cdot d\bar{l} = \Phi_a - \Phi_c$$

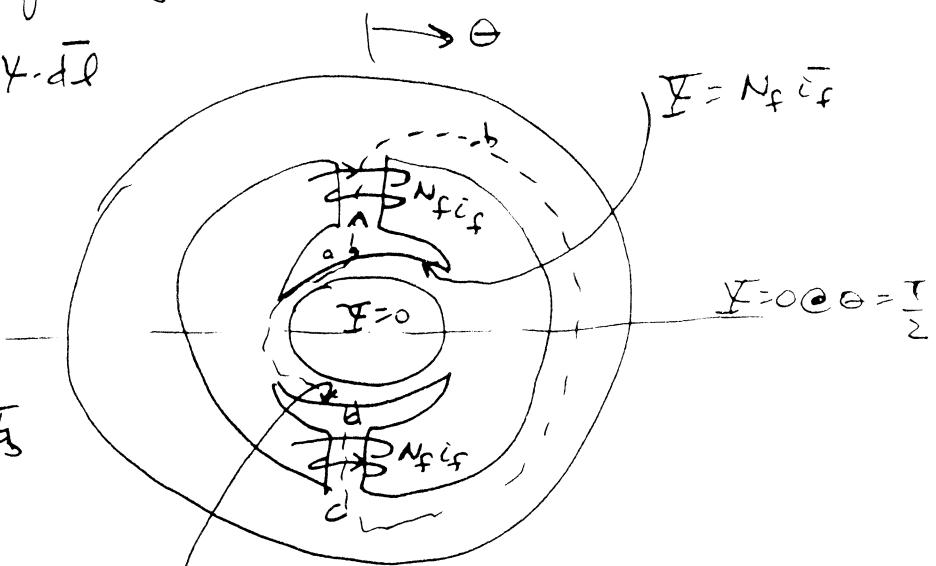
$$\left. \begin{array}{l} \oint_a \bar{H} \cdot d\bar{l} = N_f i_f = \Phi_a - \Phi_d \\ \oint_d \bar{H} \cdot d\bar{l} = N_f i_f \end{array} \right\} \bar{H} \cdot d\bar{l} = N_f i_f = \Phi_a - \Phi_d$$

$$\left. \begin{array}{l} \oint_c \bar{H} \cdot d\bar{l} = 0 \\ \oint_b \bar{H} \cdot d\bar{l} = 0 \end{array} \right\} -\nabla \Phi \cdot d\bar{l} = 0$$

$$\left. \begin{array}{l} \oint_c \bar{H} \cdot d\bar{l} = 0 \\ \oint_d \bar{H} \cdot d\bar{l} = N_f i_f \end{array} \right\} -\nabla \Phi \cdot d\bar{l} = N_f i_f$$

$$\left. \begin{array}{l} \oint_a \bar{H} \cdot d\bar{l} = \Phi^a - \Phi^d \\ \oint_d \bar{H} \cdot d\bar{l} = N_f i_f \end{array} \right\} -\nabla \Phi \cdot d\bar{l} = \Phi^a - \Phi^d$$

$$\left. \begin{array}{l} \oint_c \bar{H} \cdot d\bar{l} = 0 \\ \oint_b \bar{H} \cdot d\bar{l} = 2N_f i_f \end{array} \right\} -\nabla \Phi \cdot d\bar{l} = 2N_f i_f$$



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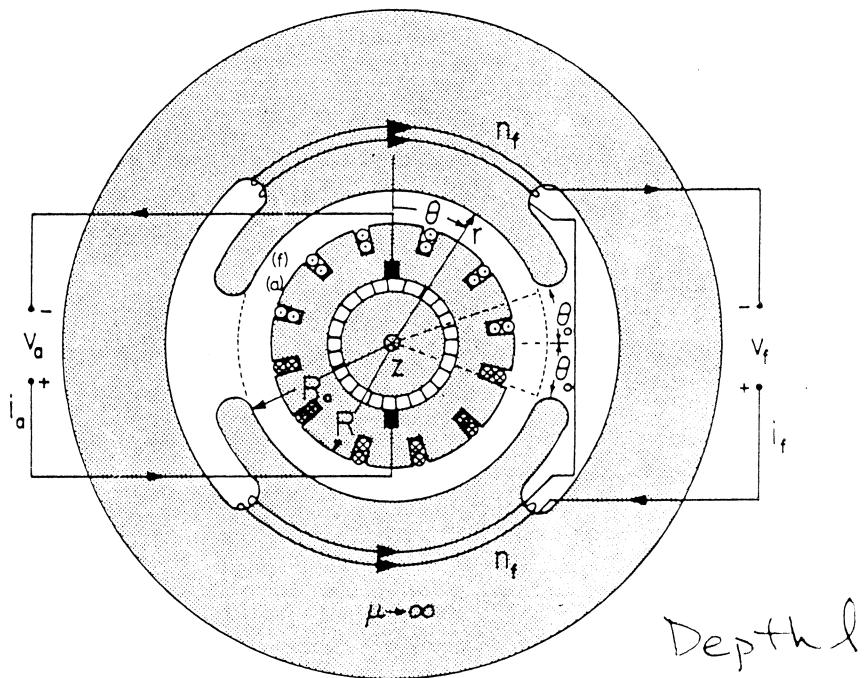


Fig. 4.10.1. Cross section of d-c machine.

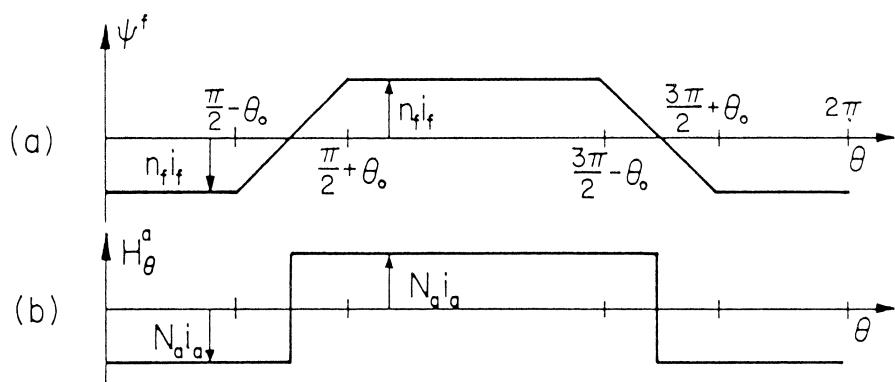


Fig. 4.10.2. Circumferential distribution of magnetic potential at $r = R_o$ and tangential magnetic field intensity at $r = R$.

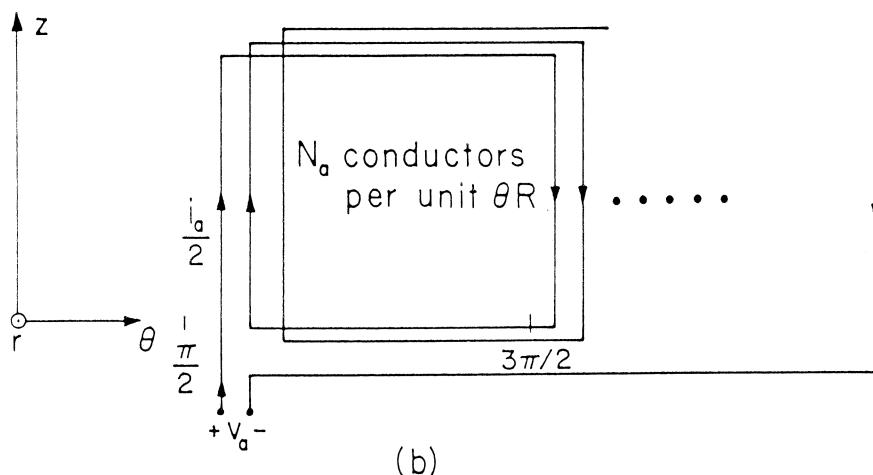
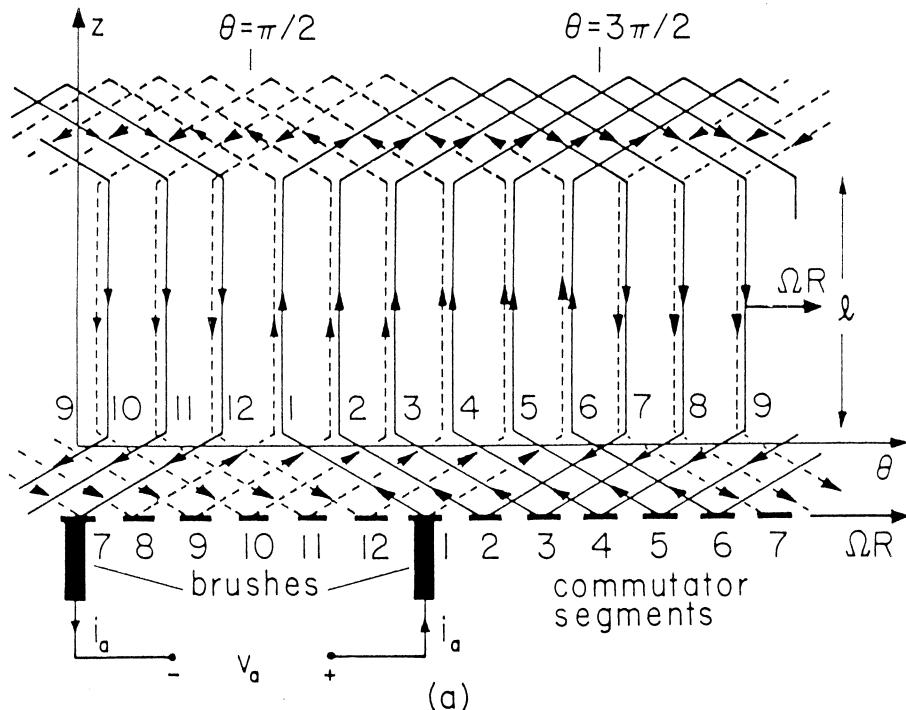


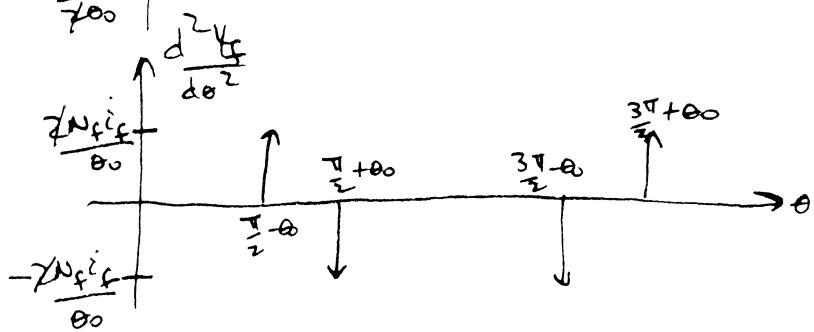
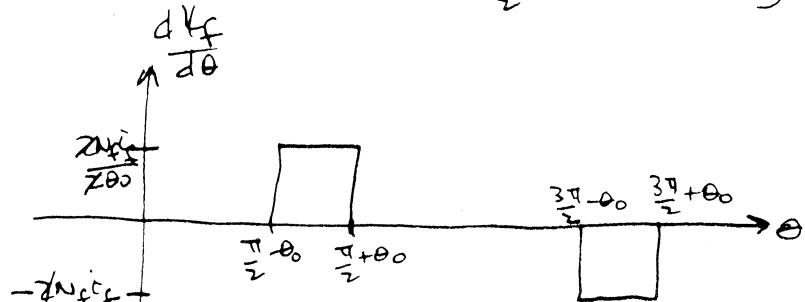
Fig. 4.10.3. (a) Typical winding scheme for armature of d-c machine shown in Fig. 4.10.1. The r axis is directed out of the paper. Brushes make contact with commutator segments which move to the right with armature conductors.² (b) Winding distribution of solid wires.

(4)

$$H_0^a = H_0(r=R) = K_2^a$$

$$\tilde{Y}^f = \tilde{Y}(r=R_0) = \sum_{m=-\infty}^{+\infty} \tilde{Y}_m^f e^{-j m \theta}$$

$$\begin{aligned}\tilde{Y}_m^f &= \frac{1}{2\pi} \int_0^{2\pi} \tilde{Y}^f e^{j m \theta} d\theta \\ &= \frac{1}{2\pi} \left\{ -N_f c_f \int_0^{\frac{\pi}{2}-\theta_0} e^{j m \theta} d\theta + \left(\frac{N_f c_f (\theta - \frac{\pi}{2})}{\theta_0} \right) e^{j m \theta} \Big|_{\frac{\pi}{2}-\theta_0}^{\frac{\pi}{2}+\theta_0} \right. \\ &\quad + N_f i_f \int_{\frac{\pi}{2}+\theta_0}^{\frac{3\pi}{2}-\theta_0} e^{j m \theta} d\theta + \left(\frac{-N_f i_f (\theta - \frac{3\pi}{2})}{\theta_0} \right) e^{j m \theta} \Big|_{\frac{3\pi}{2}-\theta_0}^{\frac{3\pi}{2}+\theta_0} \\ &\quad \left. - N_f i_f \int_{\frac{3\pi}{2}+\theta_0}^{2\pi} e^{j m \theta} d\theta \right\}\end{aligned}$$



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$$\frac{d^2 \tilde{Y}_f}{d\theta^2} = \frac{N_f i_f}{\theta_0} \left[\delta\left(\frac{\pi}{2} - \theta_0\right) - \delta\left(\frac{\pi}{2} + \theta_0\right) - \delta\left(\frac{3\pi}{2} - \theta_0\right) + \delta\left(\frac{3\pi}{2} + \theta_0\right) \right]$$

$$-m^2 \tilde{Y}_m^f = \frac{N_f i_f}{2\pi\theta_0} \int_0^{2\pi} \left[\delta\left(\frac{\pi}{2} - \theta_0\right) - \delta\left(\frac{\pi}{2} + \theta_0\right) - \delta\left(\frac{3\pi}{2} - \theta_0\right) + \delta\left(\frac{3\pi}{2} + \theta_0\right) \right] e^{j m \theta} d\theta$$

$$= \frac{N_f i_f}{2\pi\theta_0} \left[e^{jm\left(\frac{\pi}{2} - \theta_0\right)} - e^{jm\left(\frac{\pi}{2} + \theta_0\right)} - e^{jm\left(\frac{3\pi}{2} - \theta_0\right)} + e^{jm\left(\frac{3\pi}{2} + \theta_0\right)} \right]$$

$$= \frac{N_f i_f}{2\pi\theta_0} e^{jm\pi/2} \left[e^{-jm\theta_0} - e^{jm\theta_0} - e^{jm\pi} e^{-jm\theta_0} + e^{jm\pi} e^{jm\theta_0} \right]$$

$$= \frac{N_f i_f}{2\pi\theta_0} e^{jm\pi/2} \underbrace{\left(e^{-jm\theta_0} - e^{jm\theta_0} \right)}_{-2jsinm\theta_0} \underbrace{\left(1 - e^{jm\pi} \right)}_{\begin{cases} 0 & m \text{ even} \\ 2 & m \text{ odd} \end{cases}}$$

$$-m^2 \tilde{Y}_m^f = \begin{cases} 0 & m \text{ even} \\ -\frac{2N_f i_f}{\pi\theta_0} \sin m\theta_0 j e^{jm\pi/2} & m \text{ odd} \end{cases}$$

$$\tilde{Y}_m^f = \frac{2N_f i_f \sin m\theta_0 j e^{jm\pi/2}}{m^2 \pi \theta_0} \quad m \text{ odd}$$

(6)

$$H_\theta^a = \sum_{m=-\infty}^{+\infty} \tilde{H}_{\theta m}^a e^{-jm\theta}$$

$$\tilde{H}_{\theta m}^a = \frac{2N_a i_a}{m\pi} j e^{jm\pi/2} \quad m \text{ odd} \quad \left[\begin{array}{l} \text{take series for } K_m^f \\ \text{and let } N_f E_f = N_a i_a, \\ \theta \rightarrow 0 \end{array} \right]$$

$$\begin{bmatrix} \tilde{B}_{rm}^f \\ \tilde{B}_{rm}^a \end{bmatrix} = \mu_0 \begin{bmatrix} f_m(R, R_0) & g_m(R_0, R) \\ g_m(R, R_0) & f_m(R_0, R) \end{bmatrix} \begin{bmatrix} \tilde{\mathbb{F}}_m^f \\ \tilde{\mathbb{F}}_m^a \end{bmatrix}$$

$$f_m(x, y) = \frac{m}{y} \frac{[(\frac{x}{y})^m + (\frac{y}{x})^m]}{[(\frac{x}{y})^m - (\frac{y}{x})^m]} \quad \text{even function of } m$$

$$g_m(x, y) = \frac{2m}{x} \frac{1}{[(\frac{x}{y})^m - (\frac{y}{x})^m]} \quad \text{even function of } m$$

$$H_\theta = -\frac{1}{r} \frac{\partial \tilde{\mathbb{F}}}{\partial \theta} \Rightarrow H_{\theta m}^a = \frac{jm}{R} \tilde{\mathbb{F}}_m^a$$

B. Torque

$$\tilde{\mathbb{F}}_m^a = \frac{R H_{\theta m}^a}{jm}$$

$$T = R(R_0) \int_{\theta=0}^{2\pi} T_{r\theta} |_{r=R} d\theta$$

$$T = R(2\pi R_0 l) \langle T_{r\theta} \rangle |_{r=R}$$

$$= R_0(2\pi R_0 l) \langle T_{r\theta} \rangle |_{r=R_0}$$

(7)

$$\langle T_{r\theta} \rangle|_{r=R_0} = \langle [u_0 H_r^f H_\theta^f] \rangle$$

$$= \operatorname{Re} \left[\sum_{m=-\infty}^{+\infty} u_0 H_{rm}^f * \frac{j m}{R_0} \tilde{\mathcal{I}}_m^f \right]$$

$$u_0 H_{rm}^f = u_0 \left[f_m(R, R_0) \tilde{\mathcal{I}}_m^f + g_m(R_0, R) \tilde{\mathcal{I}}_m^g \right]$$

$$\langle T_{r\theta} \rangle|_{r=R_0} = \operatorname{Re} \left[\sum_{m=-\infty}^{+\infty} \frac{u_0 j m}{R_0} \tilde{\mathcal{I}}_m^f [f_m(R, R_0) \tilde{\mathcal{I}}_m^f * + g_m(R_0, R) \tilde{\mathcal{I}}_m^g *] \right]$$

$$= \operatorname{Re} \left[\sum_{m=-\infty}^{+\infty} \frac{u_0 j m}{R_0} g_m(R_0, R) \tilde{\mathcal{I}}_m^f \tilde{\mathcal{I}}_m^g * \right]$$

$$\tilde{\mathcal{I}}_m^g = \frac{R}{j m} \tilde{\mathcal{I}}_m^g = \frac{R}{j m} \frac{2 N_a i q}{m \pi} j e^{j m \pi / 2} \quad \text{modd}$$

$$\tilde{\mathcal{I}}_m^f = \frac{2 N_f i_f \sin m \theta_0}{m^2 \pi \theta_0} (j) e^{j m \pi / 2} \quad \text{m odd}$$

$$\tilde{\mathcal{I}}_m^f \tilde{\mathcal{I}}_m^g * = \frac{2 N_f i_f \sin m \theta_0}{m^2 \pi \theta_0} j e^{j m \pi / 2} \cancel{\frac{R}{-j m} \frac{2 N_a i q}{m \pi} (j) e^{-j m \pi / 2}}$$

$$= \frac{4 N_a N_f i_q i_f \sin m \theta_0 R}{m^4 \pi^2 \theta_0} j$$

$$\tau = 2 \pi R_0 l \frac{u_0}{R_0} \sum_{m=-\infty}^{+\infty} g_m(R_0, R) j m j \frac{4 N_a N_f i_q i_f \sin m \theta_0 R}{m^4 \pi^2 \theta_0} \quad \text{m odd}$$

$$= \frac{R_0 R l u_0 8 N_a N_f i_q i_f}{\pi \theta_0} \sum_{m=-\infty}^{+\infty} - \frac{g_m(R_0, R) \sin m \theta_0}{m^3}$$

$$\tau = -G_m i_f i_q ; \quad G_m = \frac{8}{\pi} R R_0 l u_0 N_a N_f \sum_{m=-\infty}^{+\infty} \frac{g_m(R_0, R) \sin(m \theta_0)}{m^3 \theta_0} \quad \text{m odd}$$

C. Electrical Equations

$$\oint_C \vec{E} \cdot d\vec{l} = - \frac{d}{dt} \int_S \vec{B} \cdot \vec{n} da$$

$$\vec{E}' = \vec{E} + \vec{v} \times \vec{B} = \frac{\vec{j}}{A} \Rightarrow \vec{E} = \frac{\vec{j}}{A} - \vec{v} \times \vec{B}$$

$$-v_a + \underbrace{\int_{\text{wire}} \frac{\vec{j}}{A} \cdot d\vec{l}}_{\sim} + \underbrace{\int_{\text{wire}} -wR \vec{i}_\theta \times \vec{B}_r \vec{i}_r \cdot d\vec{l}}_{wR \vec{B}_r \vec{i}_r \cdot d\vec{l}} = - \frac{d}{dt} \int_S \vec{B}_r \cdot d\vec{a}$$

$$\frac{i_a}{2A_a} \frac{1}{\Phi_a} i_a = i_a R_a$$

$$I_1 = \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} wR \vec{B}_r \vec{i}_r \cdot d\vec{l} = l \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} wR^2 N_a B_r^a d\theta - l \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} wR^2 N_a B_r^a d\theta$$

$$B_r^a = \sum_{m=-\infty}^{+\infty} \tilde{B}_{rm} e^{-jma}$$

$$= l w R^2 N_a \sum_{m=-\infty}^{+\infty} \left[\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \tilde{B}_{rm}^a e^{-jma} d\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tilde{B}_{rm}^a e^{-jma} d\theta \right]$$

$$= l w R^2 N_a \sum_{m=-\infty}^{+\infty} \frac{\tilde{B}_{rm}^a}{-jm} \left[e^{-jm3\pi/2} - e^{-jm\pi/2} - e^{-jm\pi/2} + e^{jm\pi/2} \right]$$

$$= l w R^2 N_a j \sum_{m=-\infty}^{+\infty} \frac{\tilde{B}_{rm}^a}{m} e^{-jm\pi/2} \left[e^{-jm\pi} - 2 + e^{jm\pi} \right]$$

$$= -4 l w R^2 N_a j \sum_{m=-\infty}^{+\infty} \frac{\tilde{B}_{rm}^a}{m} e^{-jm\pi/2}$$

(9)

$$\tilde{B}_m^a = \mu_0 [g_m(R, R_0) \tilde{I}_m^f + f_m(R_0, R) \tilde{I}_m^a]$$

$$\tilde{I}_m^f = \frac{2N_f i_f \sin m\theta_0}{m^2 \pi \theta_0} j e^{jm\pi/2}$$

$$\tilde{I}_m^a = \frac{R}{jm} \frac{2N_a i_a}{m\pi} j e^{jm\pi/2}$$

$$I_1 = -4\ell \omega R^2 N_a j \sum_{\substack{m=0 \\ \text{odd}}}^{\infty} \frac{e^{-jm\pi/2}}{m} \left[\frac{g_m(R, R_0) 2N_f i_f \sin m\theta_0 j e^{jm\pi/2}}{m^2 \pi \theta_0} \right. \\ \left. + \frac{f_m(R_0, R) R 2N_a i_a j e^{jm\pi/2}}{m^2 \pi} \right]$$

$$\frac{f_m(R_0, R)}{m^3} \text{ odd function of } m$$

$$\frac{g_m(R, R_0) \sin m\theta_0}{m^3} \text{ even function of } m$$

$$I_1 = +4\ell \omega R^2 N_a 2N_f i_f \sum_{\substack{m=0 \\ \text{odd}}}^{\infty} \frac{g_m(R, R_0) \sin m\theta_0}{m^3}$$

$$= \frac{8}{\pi} \ell \omega R^2 N_a N_f i_f \sum_{\substack{m=0 \\ \text{odd}}}^{\infty} \frac{g_m(R, R_0) \sin m\theta_0}{m^3}$$

$$R g_m(R, R_0) = -R_0 g_m(R_0, R)$$

$$I_1 = -G \omega i_f ; \quad G = \frac{8}{\pi} R R_0 \ell \mu_0 N_a N_f \sum_{\substack{m=0 \\ \text{odd}}}^{\infty} \frac{g_m(R_0, R) \sin m\theta_0}{m^3 \theta_0}$$

2. Armature Self-Inductance

(10)

$$\int_{\theta' = -\frac{\pi}{2}}^{\frac{3\pi}{2}} B_r d\theta = \left(\int_{\theta'} B_r^a R d\theta \right) N_a R d\theta'$$

$$B_r^a = \sum_{m=-\infty}^{+\infty} \tilde{B}_{rm} e^{-j m \theta}$$

$$\int_{\theta' = -\frac{\pi}{2}}^{\frac{3\pi}{2}} B_r d\theta = l N_a R^2 \int_{\theta' = -\frac{\pi}{2}}^{\frac{3\pi}{2}} \sum_{m=-\infty}^{+\infty} \frac{\tilde{B}_{rm} e^{-j m \theta'}}{-j m} \Big|_{\theta' = \theta} d\theta'$$

$$= j l N_a R^2 \int_{\theta' = -\frac{\pi}{2}}^{\frac{3\pi}{2}} \sum_m \frac{\tilde{B}_{rm}}{m} 2 e^{-j m \theta'} d\theta'$$

$$= 2 j l N_a R^2 \sum_m \frac{\tilde{B}_{rm}}{m} \frac{e^{-j m \theta'}}{-j m} \Big|_{\theta' = \frac{\pi}{2}}$$

$$= -2 l N_a R^2 \sum_m \frac{\tilde{B}_{rm}}{m^2} \left(e^{-j m \frac{3\pi}{2}} - e^{-j m \frac{\pi}{2}} \right)$$

$$= +4 l N_a R^2 \sum_m \frac{\tilde{B}_{rm}}{m^2} e^{-j m \frac{\pi}{2}}$$

$$= 4 l N_a R^2 \sum_{\text{odd } m} \frac{e^{-j m \frac{\pi}{2}}}{m^2} \left[\frac{g_m(R, R_0) 2 N_a \sin m \theta_0}{m^2 \pi \theta_0} j e^{j m \frac{\pi}{2}} \right.$$

$$\left. + f_m(R_0, R) R 2 N_a i_a / e^{j m \frac{\pi}{2}} \right]$$

$\frac{g_m(R, R_0) \sin m \theta_0}{m^4}$ odd function of m

$\frac{f_m(R_0, R)}{m^4}$ even function of m

$$\int_{\theta' = -\infty}^{+\infty} B_r d\theta = N_a^2 l N_a R^2 \sum_{\text{odd } m} \frac{f_m(R_0, R) R 2 N_a i_a}{m^4 \pi} = \frac{8 \mu_0 l N_a^2 R^3}{\pi} \sum_{\text{odd } m} \frac{f_m(R_0, R)}{m^4} i_a$$

L_a

(4)

$$V_a = i_a R_a - \omega w_i f + L_a \frac{di_a}{dt}$$

3. Self-inductance for Field Winding

$$\left\{ \begin{array}{l} B_r d\theta = N_{fl} \int_{-\frac{\pi}{2}+\theta_0}^{\frac{\pi}{2}-\theta_0} B_r^f R_0 d\theta - N_{fl} \int_{\frac{\pi}{2}+\theta_0}^{\frac{3\pi}{2}-\theta_0} B_r^f R_0 d\theta \\ S \end{array} \right.$$

$$B_r^f = \sum_{m=-\infty}^{+\infty} \mu_0 H_m e^{-j m \theta}$$

$$\mu_0 H_m^f = \mu_0 [f_m(R, R_0) \tilde{I}_m^f + g_m(R_0, R) \tilde{I}_m^g]$$

$$\tilde{I}_m^f = \frac{2 N_{fl} f_m \sin m \theta_0}{m^2 \pi \theta_0} j e^{jm\pi/2}$$

$$\tilde{I}_m^g = \frac{R}{\sin \theta_0} \frac{2 N_a i_a}{m \pi} j e^{jm\pi/2}$$

$$\left\{ \begin{array}{l} B_r d\theta = \mu_0 N_{fl} R_0 \left[\frac{\tilde{H}_m^f e^{-jm\theta}}{-jm} \Big|_{m=-\infty}^{+\infty} \right. \\ \left. + \frac{\tilde{H}_m^f e^{-jm\theta}}{+jm} \Big|_{\frac{\pi}{2}+\theta_0}^{\frac{3\pi}{2}-\theta_0} \right] \end{array} \right.$$

$$= \mu_0 N_{fl} R_0 \sum_{m=-\infty}^{+\infty} \frac{\tilde{H}_m^f}{-jm} \left[e^{-jm\frac{\pi}{2}} e^{jm\theta_0} - e^{jm\frac{\pi}{2}} e^{-jm\theta_0} - e^{-jm\frac{3\pi}{2}} e^{jm\theta_0} + e^{-jm\frac{\pi}{2}} e^{jm\theta_0} \right]$$

$$= \mu_0 N_{fl} R_0 \sum_{m=-\infty}^{+\infty} \frac{\tilde{H}_m^f}{-jm} e^{jm\pi/2} \underbrace{\left[e^{jm\theta_0} + e^{-jm\theta_0} + e^{jm\theta_0} + e^{-jm\theta_0} \right]}_{4 \cos m\theta_0}$$

$$= 4 \mu_0 N_{fl} R_0 \sum_{m=-\infty}^{+\infty} \frac{\tilde{H}_m^f}{-jm} e^{-jm\pi/2} \cos m\theta_0$$

(12)

$$\left\{ \begin{array}{l} B_{rda} = 4\mu_0 N_f l R_0 \sum_{m=-\infty}^{+\infty} \frac{e^{-j m \pi/2}}{-jm} \cos m \theta \left[\frac{f_m(R, R_0) 2N_f i_f \sin m \theta_0}{m^2 \pi \theta_0} e^{j m \pi/2} \right. \\ \text{mode} \quad \left. + g_m(R_0, R) \frac{R}{jm} \frac{2N_f i_f}{m \pi} j e^{j m \pi/2} \right] \end{array} \right.$$

$$\frac{g_m(R, R)}{m^3} \quad \text{odd function of } m$$

$$\frac{f_m(R, R_0) \sin m \theta_0 \cos m \theta}{m^3} \quad \text{even function of } m$$

$$\left\{ \begin{array}{l} B_{rda} = 4\mu_0 N_f l R_0 \sum_{m=-\infty}^{+\infty} \frac{f_m(R, R_0) 2N_f i_f \sin m \theta_0 \cos m \theta}{m^3 \pi \theta_0} \\ \text{mode} \\ = \frac{8\mu_0 N_f^2 l R_0}{\pi \theta_0} \cdot \sum_{m=-\infty}^{+\infty} \frac{f_m(R, R_0) \sin m \theta_0 \cos m \theta}{m^3} \quad i_f \\ = L_f i_f \end{array} \right.$$

$$v_f = i_f R_f + L_f \frac{di_f}{dt}$$