

Lecture Notes 17, 4/24/03

I. Magnetic Diffusion in Moving Media

$$\underline{J} = \sigma(\underline{\underline{E}} + \bar{\underline{\omega}} \times \bar{\underline{B}}) = \sigma(\underline{\underline{E}} + \bar{\underline{\omega}} \times \mu \bar{\underline{H}})$$

$$\nabla \times \bar{\underline{E}} = -\mu \frac{\partial \bar{\underline{H}}}{\partial t}$$

$$\nabla \times \bar{\underline{H}} = \underline{J} = \sigma(\underline{\underline{E}} + \bar{\underline{\omega}} \times \mu \bar{\underline{H}})$$

$$\nabla \cdot \bar{\underline{H}} = 0$$

$$\nabla \times (\nabla \times \bar{\underline{H}}) = \nabla(\cancel{\nabla \cdot \bar{\underline{H}}}) - \nabla^2 \bar{\underline{H}} = \sigma(\nabla \times \bar{\underline{E}}) + \sigma \nabla \times (\bar{\underline{\omega}} \times \bar{\underline{H}})$$

$$\nabla \times (\bar{\underline{\omega}} \times \bar{\underline{H}}) = \bar{\underline{\omega}}(\cancel{\nabla \cdot \bar{\underline{H}}}) - \bar{\underline{H}}(\cancel{\nabla \cdot \bar{\underline{\omega}}}) + (\bar{\underline{H}} \cdot \cancel{\bar{\underline{\omega}}}) \bar{\underline{\omega}} - \bar{\underline{\omega}} \cdot \nabla \bar{\underline{H}}$$

($\bar{\underline{\omega}}$ is spatially constant)

$$-\nabla^2 \bar{\underline{H}} = -\sigma \mu \frac{\partial \bar{\underline{H}}}{\partial t} - \sigma \mu (\bar{\underline{\omega}} \cdot \nabla) \bar{\underline{H}}$$

$$\nabla^2 \bar{\underline{H}} = \sigma \mu \left[\frac{\partial \bar{\underline{H}}}{\partial t} + (\bar{\underline{\omega}} \cdot \nabla) \bar{\underline{H}} \right]$$

II. Boundary Conditions for Thin Sheets and Shells

A. Governing Equations of Thickness Δ

$$\underline{K} = \Delta \sigma \underline{E}_t = \sigma_s \underline{E}_t ; \quad \sigma_s = \Delta \sigma$$

$$\bar{\underline{K}} = \bar{\underline{J}} \Delta = \sigma \Delta (\bar{\underline{E}} + \bar{\underline{\omega}} \times \mu \bar{\underline{H}})$$

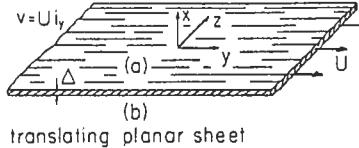
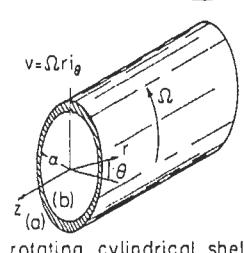
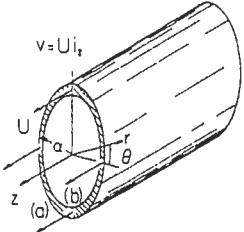
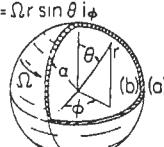
$$\nabla \times \bar{\underline{K}} = \sigma_s (\underbrace{\nabla \times \bar{\underline{E}}}_{-\mu \frac{\partial \bar{\underline{H}}}{\partial t}} + \nabla \times (\bar{\underline{\omega}} \times \mu \bar{\underline{H}}))$$

$$\nabla \times \bar{\underline{K}} = -\sigma_s \mu \left[\frac{\partial \bar{\underline{H}}}{\partial t} + (\bar{\underline{\omega}} \cdot \nabla) \bar{\underline{H}} \right]$$

$$\frac{\nabla_{\Sigma} \cdot \bar{\underline{K}}}{\bar{n} \times [\bar{\underline{H}}]} = \bar{\underline{K}}$$

Table, 6.3.1, Fig. 6.4.1, Fig. 6.4.2, Fig. 6.4.3, and 6.4.4
 from Electromagnetic Fields and Energy by Hermann A.
 Haus and James R. Melcher. Used with permission.

Table 6.3.1. Boundary conditions on conducting moving sheets and shells.
Normal flux density, B_n , is continuous and $\sigma_s \equiv \Delta\sigma$.

Configuration	Boundary condition
 translating planar sheet	$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) [H_y] = -\sigma_s \frac{\partial}{\partial y} \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial y} \right) B_x \quad (a)$
 rotating cylindrical shell	$\left(\frac{1}{\alpha^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right) [H_\theta] = -\frac{\sigma_s}{\alpha} \frac{\partial}{\partial \theta} \left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \theta} \right) B_r \quad (b)$
 translating cylindrical shell	$\left(\frac{1}{\alpha^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right) [H_z] = -\sigma_s \frac{\partial}{\partial z} \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial z} \right) B_r \quad (c)$
 rotating spherical shell	<p>Either</p> $\left(\frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} \sin \theta + \frac{\partial^2}{\partial \phi^2} \right) [H_\phi] = -\sigma_s \alpha \sin \theta \frac{\partial}{\partial \phi} \left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right) B_r \quad (d)$ <p>or</p> $\left(\frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} \sin \theta + \frac{\partial^2}{\partial \phi^2} \right) [H_\theta] = -\sigma_s \alpha \frac{\partial}{\partial \theta} [\sin^2 \theta \left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right) B_r] \quad (e)$

B. Translating Planar Sheet

$$B_n = B_x, \bar{v} = U \bar{i}_y, \bar{K} = K_y \bar{i}_y + K_z \bar{i}_z$$

$$\nabla \times \bar{K} = \bar{i}_x \left(\frac{\partial K_z}{\partial y} - \frac{\partial K_y}{\partial z} \right) = -\sigma_s \mu \left[\frac{\partial H_x}{\partial t} + U \frac{\partial H_x}{\partial y} \right]$$

$$\frac{\partial K_y}{\partial y} + \frac{\partial K_z}{\partial z} = 0$$

$$K_y = H_z(x=0_-) - H_z(x=0_+)$$

$$K_z = H_y(x=0_+) - H_y(x=0_-)$$

III. Magnetic Induction Motor

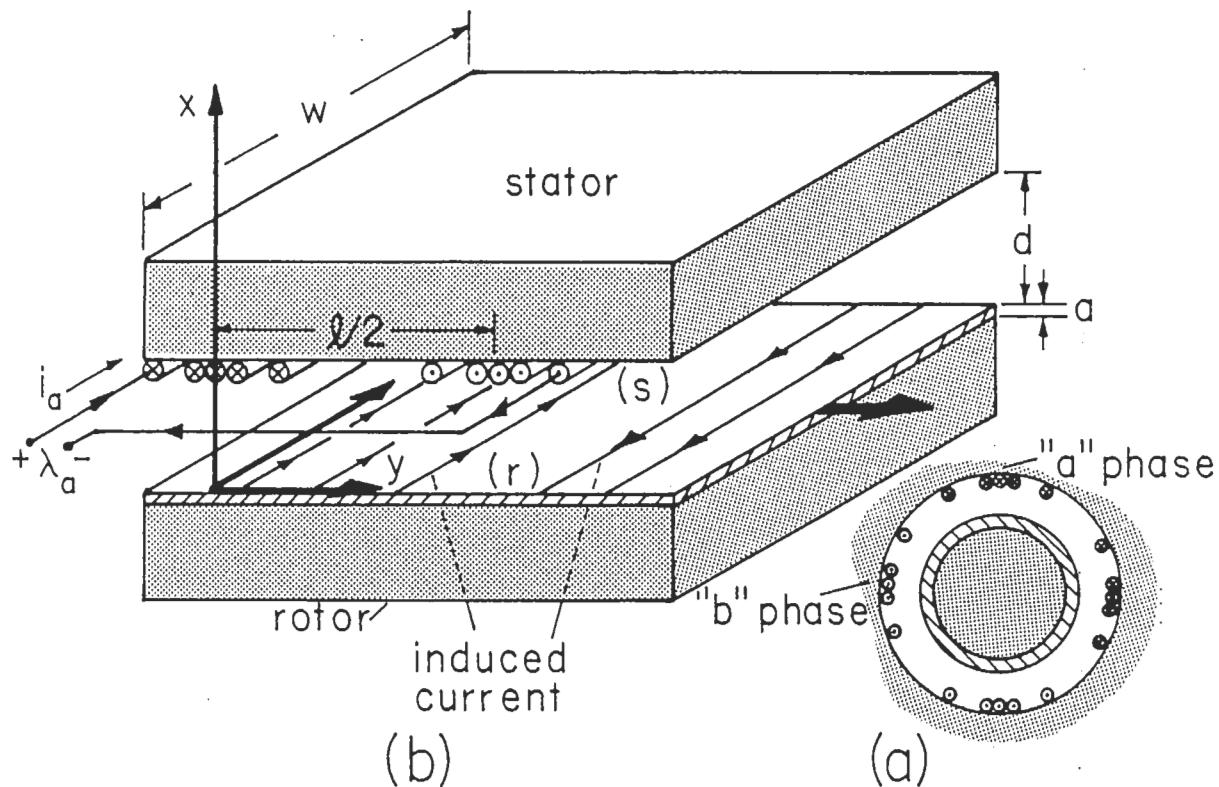
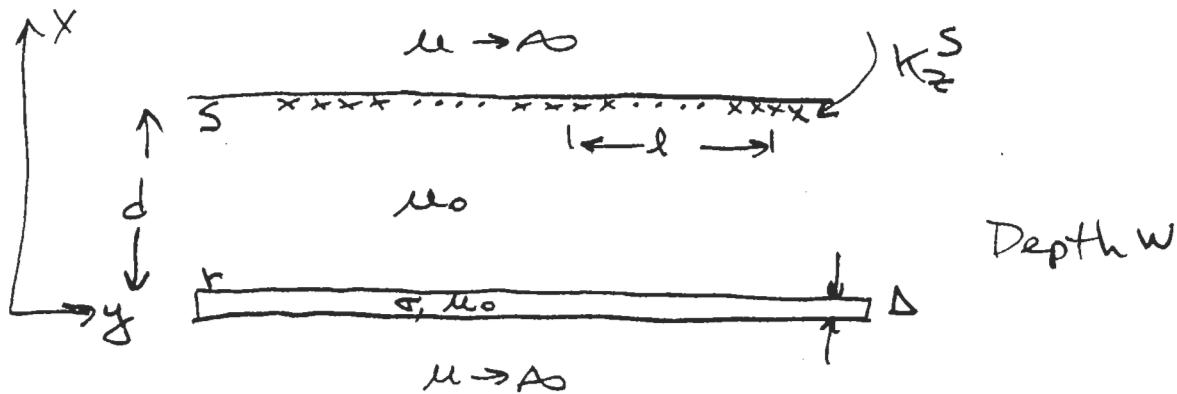


Fig. 6.4.1. (a) Cross section of rotating induction machine with thin-sheet conductor on rotor. (b) Developed model for (a) with air gap d and sheet conductor of thickness a . One of two phases on the stator is shown.



A. Sources

$$K_z^S = \operatorname{Re} \left[\hat{i}_a e^{j\omega t} N_a \cos k_y + \hat{i}_b e^{j\omega t} N_b \cos(k_y - \lambda/4) \right]$$

$$\cos k_y = \frac{e^{jky} + e^{-jky}}{2}$$

$$K_z^S = \operatorname{Re} \left[\hat{K}_+^S e^{j(\omega t - ky)} + \hat{K}_-^S e^{j(\omega t + ky)} \right]$$

$$\hat{K}_{\pm}^S = \frac{1}{2} (N_a \hat{i}_a + N_b \hat{i}_b e^{\pm jk\lambda/4})$$

B. Magnetic Fields: e^{-jky} variation (+ wave)

$$\vec{A} = A_z(x, y) \hat{z}$$

$$\begin{bmatrix} \hat{A}^S \\ \hat{A}^r \end{bmatrix} = \frac{\mu_0}{k} \begin{bmatrix} -\operatorname{coth}kd & \frac{1}{\sinh kd} \\ -\frac{1}{\sinh kd} & \operatorname{coth}kd \end{bmatrix} \begin{bmatrix} \hat{H}_y^S \\ \hat{H}_y^r \end{bmatrix}$$

$$\hat{H}_y^r = -\hat{K}_+^S$$

$$\frac{\partial \hat{K}_+^S}{\partial y} = -\nabla_S \mu_0 \left[\frac{\partial \hat{H}_x}{\partial t} + \frac{\omega}{k} \frac{\partial \hat{H}_x}{\partial y} \right] = \frac{\partial \hat{H}_y^r}{\partial y}$$

$$-jk \hat{H}_{yx}^r = -\nabla_S \mu_0 j (\omega - kv) \hat{H}_{xx}^r$$

$$\mu_0 \hat{H}_x = \frac{\partial \hat{A}_z}{\partial y} \Rightarrow \mu_0 \hat{H}_{xx}^r = -jk \hat{A}_{zz}^r$$

$$-jk \hat{H}_{yx}^r = -\nabla_S j (\omega - kv) (-jk \hat{A}_{zz}^r)$$

$$\hat{H}_{yy}^r = -j \nabla_S (\omega - kv) \hat{A}_{zz}^r$$

$$\begin{aligned}\hat{A}_{z+}^r &= \frac{\mu_0}{l} \left[-\frac{\hat{H}_{y+}^S}{\sinh(lk)} + \coth(lk) \hat{H}_{y+}^r \right] \\ -\frac{\hat{H}_{y+}^r}{j\sqrt{S}(\omega - lk)} &= \frac{\mu_0}{l} \left[\frac{\hat{K}_+^S}{\sinh(lk)} + \coth(lk) \hat{H}_{y+}^r \right] \\ -\hat{H}_{y+}^r \left[\frac{\mu_0}{l} \coth(lk) + \frac{1}{j\sqrt{S}(\omega - lk)} \right] &= \frac{\mu_0 \hat{K}_+^S}{l \sinh(lk)} \\ \hat{H}_{y+}^r &= \frac{-\mu_0 \hat{K}_+^S}{l \sinh(lk) \left[\frac{\mu_0}{l} \coth(lk) + \frac{1}{j\sqrt{S}(\omega - lk)} \right]}\end{aligned}$$

$$S_+ = \frac{\mu_0 \sqrt{S}}{l} (\omega - lk)$$

$$\begin{aligned}\hat{H}_{y+}^r &= \frac{-\hat{K}_+^S}{l \sinh(lk) \left[\coth(lk) - \frac{j}{\mu_0 \sqrt{S} (\omega - lk)} \right]} \\ &= \frac{-\hat{K}_+^S l}{l \sinh(lk) \left[\coth(lk) - \frac{j}{S_+} \right]} \\ &= \frac{-\hat{K}_+^S S_+}{l \sinh(lk) \left[S_+ \coth(lk) - j \right]} \\ &= \frac{-\hat{K}_+^S S_+ \left[S_+ \coth(lk) + j \right]}{l \sinh(lk) \left[S_+^2 \coth^2(lk) + 1 \right]}\end{aligned}$$

(6)

C. $e^{j\theta y}$ variations (- wave)

$$k \rightarrow -k$$

$$\hat{K}_+^S \rightarrow \hat{K}_-^S$$

$$\hat{H}_{y-}^r = - \frac{\hat{K}_-^S S_- [+ S_- \coth kld + j]}{\sinh kld [S_-^2 \coth^2 kld + 1]}$$

$$S_- = \frac{\mu_0 Js}{le} (\omega + leV)$$

D. Time-Average Force

$$\langle f_y \rangle_t = \frac{\pi}{2} lw \frac{1}{2} \operatorname{Re} [\mu_0 \hat{H}_{x+}^r \hat{H}_{y+}^{r*} + \mu_0 \hat{H}_{x-}^r \hat{H}_{y-}^{r*}]$$

$p = \# \text{ of poles} = \# \text{ of half-wavelengths}$

$\frac{pl}{2} = \text{total rotor length}$

$$\mu_0 \hat{H}_{x\pm}^r = \mp j le \hat{A}_{z\pm}^r$$

$$\hat{A}_{z\pm}^r = \frac{\mu_0}{le} \left[\frac{\hat{K}_\pm^S}{\sinh kld} + \coth kld \hat{H}_{y\pm}^r \right]$$

$$\langle f_y \rangle_t = \frac{plw}{4} \operatorname{Re} [-j le \hat{A}_{z+}^r \hat{H}_{y+}^{r*} + j le \hat{A}_{z-}^r \hat{H}_{y-}^{r*}]$$

$$= \frac{plw}{4} \left\{ \operatorname{Re}(-j) \left[\frac{\mu_0}{le} \right] \left[\frac{\hat{K}_+^S}{\sinh kld} + \coth kld \hat{H}_{y+}^r \right] \hat{H}_{y+}^{r*} \right. \\ \left. + j \frac{\mu_0}{le} \left[\frac{\hat{K}_-^S}{\sinh kld} + \coth kld \hat{H}_{y-}^r \right] \hat{H}_{y-}^{r*} \right\}$$

$$\begin{aligned}
 \langle f_y \rangle_t &= \frac{plw \mu_0}{4 \text{sinh}^2} Re j \left[-\hat{K}_+^S \hat{H}_{y+}^* + \hat{K}_-^S \hat{H}_{y-}^* \right] \\
 &= \frac{plw \mu_0 Re j}{4 \text{sinh}^2} \left[\frac{|\hat{K}_+^S|^2 S_+ [S_+ \coth^2 j - j]}{\text{sinh}^2 [S_+^2 \coth^2 j + 1]} - \frac{|\hat{K}_-^S|^2 S_- [S_- \coth^2 j]}{\text{sinh}^2 [S_-^2 \coth^2 j + 1]} \right] \\
 &= \frac{plw \mu_0}{4 \text{sinh}^2} \left[\frac{|\hat{K}_+^S|^2 S_+}{[S_+^2 \coth^2 j + 1]} - \frac{|\hat{K}_-^S|^2 S_-}{[S_-^2 \coth^2 j + 1]} \right]
 \end{aligned}$$

E. Balanced two Phase Currents ($N_a = N_b \equiv N$)

$$\hat{i}_b = \hat{i}_a e^{-j\pi/2} \Rightarrow \hat{K}_+^S = N \hat{i}_a (1+i) = N \hat{i}_a$$

$$k_f = 2\pi$$

$$\hat{K}_-^S = \frac{N \hat{i}_a}{2} (1 + e^{j\pi}) = 0$$

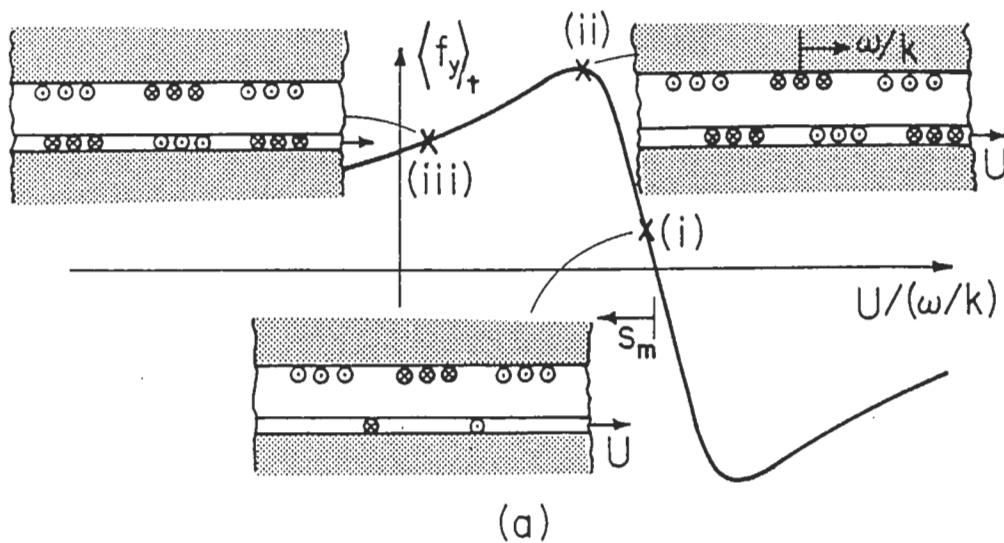
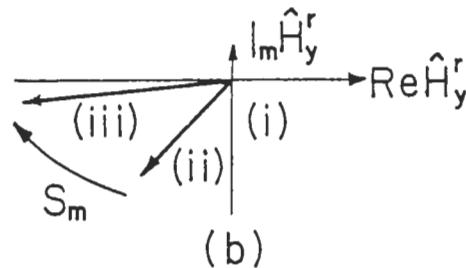


Fig. 6.4.2. (a) Time-average force for induction machine of Fig. 6.4.1 with balanced two-phase excitation. Abscissa is material velocity relative to wave phase velocity. The slip is $s_m \equiv S_{mt}/(\mu_0 \sigma_s \omega/k)$. Insets show spatial phase of stator and rotor currents at a given instant. (b) Phasor \hat{H}_y^r , showing effect of increasing S_m on the phase and amplitude. Operating points (i) + (iii) are shown in (a). In nomenclature of lumped parameter induction machines, (i) is resistance dominated operation while (iii) is reactance dominated.



$$\langle f_y \rangle_t = \frac{p l w \omega_0}{4 \tanh^2 d} \frac{|\hat{K}_+|^2 S_+}{S_+^2 \coth^2 d + 1}$$

$$\langle f_y \rangle_{t \rightarrow \infty} \Rightarrow \frac{d}{S_+} \left[\frac{S_+}{S_+^2 \coth^2 d + 1} \right] = \frac{1}{S_+^2 \coth^2 d + 1} \frac{-S_+ (2 \tanh^2 d)}{[S_+^2 \coth^2 d + 1]} = 0$$

$$2 S_+^2 \coth^2 d = S_+^2 \coth^2 d + 1$$

$$S_+ = \frac{1}{\coth d} = \tanh d$$

E. Electrical Terminal Relations

$$\Phi_\lambda(y) = \omega [A^s(y') - A^s(y' + l/2)]$$

$$= \omega \operatorname{Re} \left[\hat{A}_+^s e^{j(\omega t - hy')} + \hat{A}_-^s e^{j(\omega t + hy')} - \hat{A}_+^s e^{-j\pi} e^{j(\omega t - hy')} - \hat{A}_-^s e^{j\pi} e^{j(\omega t + hy')} \right]$$

$$= 2 \omega \operatorname{Re} [\hat{A}_+^s e^{-jhy'} + \hat{A}_-^s e^{jhy'}] e^{j\omega t}$$

$$\lambda_a = \int_{-\frac{\omega}{4}}^{+\frac{\omega}{4}} \Phi_\lambda(y') N_a \cos hy' dy' = \omega N_a \operatorname{Re} \int_{-\frac{\omega}{4}}^{+\frac{\omega}{4}} (\hat{A}_+^s e^{-jhy'} + \hat{A}_-^s e^{jhy'}) (e^{jhy'} + e^{-jhy'}) dy'$$

$$= \omega N_a \operatorname{Re} \left\{ e^{j\omega t} \left[\hat{A}_+^s \frac{\omega}{2} + \frac{\hat{A}_+^s e^{-2jhy'}}{-2jh} \right]_{-\frac{\omega}{4}}^{+\frac{\omega}{4}} + \frac{\hat{A}_-^s e^{2jhy'}}{2jh} \left[\hat{A}_-^s \frac{\omega}{2} \right]_{-\frac{\omega}{4}}^{+\frac{\omega}{4}} \right\}$$

$$\hat{\lambda}_a = \omega N_a R e e^{j\omega t} [\hat{A}_+^s + \hat{A}_-^s] \frac{l}{2}$$

$$\hat{A}_+^s = \frac{\mu_0}{l} \left[-\coth \text{ld} \hat{H}_{y+}^s + \frac{1}{\text{snhd}} \hat{H}_{y+}^r \right]$$

$$= \frac{\mu_0}{l} \left[\coth \text{ld} \hat{R}_+^s + \frac{\hat{H}_y^r}{\text{snhd}} \right]$$

$$\hat{A}_-^s = \frac{\mu_0}{l} \left[\coth \text{ld} \hat{R}_-^s + \frac{\hat{H}_y^-}{\text{snhd}} \right]$$

$$\hat{\lambda}_a = \frac{\omega N_a R e}{2l} e^{j\omega t} \left[\coth \text{ld} (\hat{R}_+^s + \hat{R}_-^s) + \frac{(\hat{H}_y^r + \hat{H}_y^-)}{\text{snhd}} \right]$$

$$\begin{aligned} N_a &\rightarrow -jN_b, \quad \hat{R}_+^s + \hat{R}_-^s \rightarrow \hat{R}_+^s - \hat{R}_-^s \\ \hat{H}_y^r + \hat{H}_y^- &\rightarrow \hat{H}_y^r - \hat{H}_y^- \end{aligned}$$

$$\hat{\lambda}_b = \frac{\omega N_b \mu_0 R e}{2l} [-j e^{j\omega t} \left[\coth \text{ld} (\hat{R}_+^s - \hat{R}_-^s) + \frac{(\hat{H}_y^r - \hat{H}_y^-)}{\text{snhd}} \right]]$$

F. Balanced Two Phase Equivalent Circuit
 $\hat{R}_-^S = 0$

$$\begin{aligned} \hat{V}_a &= j\omega \hat{I}_a = j\omega \frac{\mu_0 w N l}{2d} \left[\coth(d) \hat{R}_+^S - \frac{\hat{R}_+^S S_+ [S_+ \coth(d) + j]}{\sinh^2 d [S_+^2 \coth^2 d + 1]} \right] \\ &= \frac{j\omega \mu_0 w N^2 l}{2d} \hat{I}_a \left[\coth(d) - \frac{S_+ (j + S_+ \coth(d))}{\sinh^2 d (1 + S_+^2 \coth^2 d)} \right] \end{aligned}$$

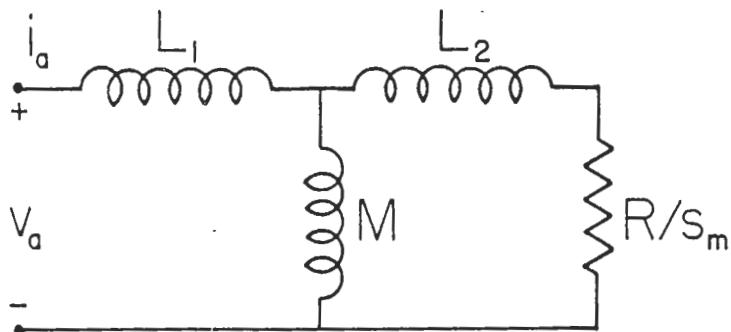


Fig. 6.4.3. Equivalent circuit for balanced operation of induction machine.

$$L_1 = L_2 = \frac{\omega N^2 l^2 \mu_0}{4\pi} \tanh\left(\frac{\pi d}{l}\right)$$

$$M = \frac{\omega N^2 l^2 \mu_0}{4\pi \sinh\left(\frac{2\pi d}{l}\right)} ; R = \frac{l \omega N^2}{2\sigma_s}$$

$$S_m = \left(1 - \frac{R}{\omega}\right) = \frac{S_+}{\mu_0 \sigma_s w / l e}$$

5. Single Phase Machine

$$i_b = 0 \Rightarrow K_+^s = K_-^s = \frac{1}{2} N_a i_s$$

$$\langle f_g \rangle_t = \frac{plw\mu_0}{4\sinh^2 ld} \left| \frac{1}{2} N_a i_s \right|^2 \left[\frac{S_+}{S_+^2 \coth^2 ld + 1} - \frac{S_-}{S_-^2 \coth^2 ld + 1} \right]$$

$$S_+ = \frac{\mu_0 \sigma s}{l} (w - ld)$$

$$S_- = \frac{\mu_0 \sigma s}{l} (w + ld)$$

$$\frac{d \langle f_g \rangle_t}{d w} \Big|_{w=0} = \frac{plw\mu_0}{4\sinh^2 ld} \frac{1}{4} N_a^2 \left| \frac{i_s}{i_s} \right|^2 \left\{ \frac{-1}{S_+^2 \coth^2 ld + 1} + \frac{2S_+^2 \coth^2 ld}{(S_+^2 \coth^2 ld + 1)^2} \right. \\ \left. - \frac{1}{S_-^2 \coth^2 ld + 1} + \frac{2S_-^2 \coth^2 ld}{(S_-^2 \coth^2 ld + 1)^2} \right\} \frac{\mu_0 \sigma s}{l}$$

$$R_m = S_+ \Big|_{w=0} = S_- \Big|_{w=0} = \frac{\mu_0 \sigma s w}{l}$$

$$\frac{d \langle f_g \rangle_t}{d w} = \frac{plw\mu_0}{4\sinh^2 ld} \frac{1}{4} N_a^2 \left| \frac{i_s}{i_s} \right|^2 \left[\frac{R_m^2 \coth^2 ld}{(1 + R_m^2 \coth^2 ld)^2} - 1 \right] 2$$

$$\frac{d \langle f_g \rangle_t}{d w} > 0 \text{ if } R_m > \tanh ld$$

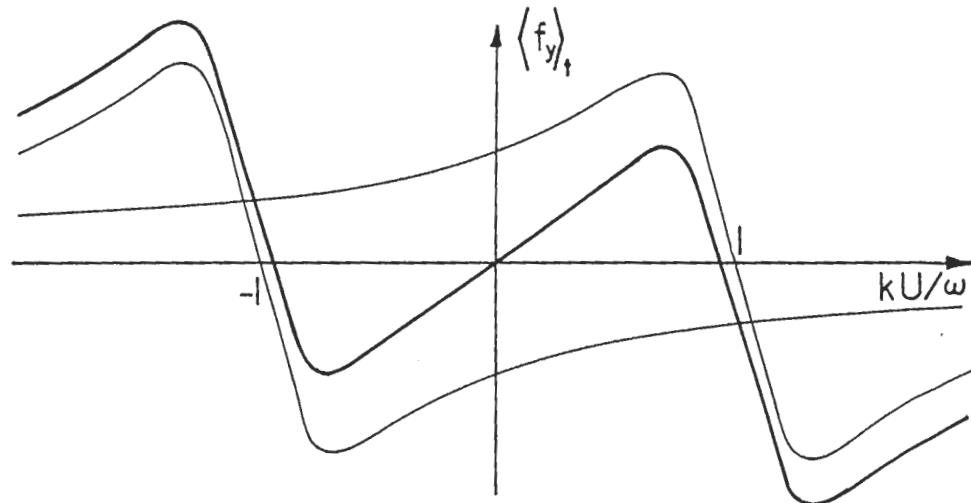


Fig. 6.4.4. Time-average force for single-phase induction machine as function of material velocity normalized to wave velocity. Total force is superposition of forces due to forward and backward wave components.

5. Tachometer

$$\hat{i}_a = 0 \Rightarrow \hat{R}_+^S = -\frac{j}{2} \hat{i}_b N_b, \quad \hat{R}_-^S = \frac{j}{2} \hat{i}_b N_b$$

$$\hat{v}_a = -j \omega wl N_a N_b \mu_0 \hat{i}_b \left[\frac{S_+}{1 + j S_+ \coth ld} - \frac{S_-}{1 + j S_- \coth ld} \right]$$

Prob. 6.4.2 For the circuit, loop equations are

$$\begin{bmatrix} j\omega(L_1+M) & -j\omega M \\ -j\omega M & j\omega(L_2+M)+\frac{R}{\rho_m} \end{bmatrix} \begin{bmatrix} \hat{i}_a \\ \hat{i}_b \end{bmatrix} = \begin{bmatrix} \hat{v}_a \\ 0 \end{bmatrix} \quad (1)$$

Thus,

$$\hat{i}_a = \frac{\hat{v}_a [j\omega(L_2+M)+\frac{R}{\rho_m}]}{j\omega(L_1+M)[j\omega(L_2+M)+\frac{R}{\rho_m}] + \omega^2 M^2} \quad (2)$$

and written in the form of Eq. 6.4.17, this becomes

$$\hat{v}_a = \left\{ j\omega(L_1+M) - j\omega \rho_m \frac{[j\omega M^2 R + \omega^2 M^2 (L_2+M) \rho_m]}{R^2 \left[1 + \omega^2 \frac{(L_2+M)^2 \rho_m^2}{R^2} \right]} \right\} \quad (3)$$

where comparison with Eq. 6.4.17 shows that

$$\frac{\rho_m}{R} \omega(L_2+M) = S_m \coth \frac{Rd}{2} \quad (4)$$

$$L_1+M = \frac{w \ell N_a^2 \mu_0}{2 \pi} \coth \frac{Rd}{2} \quad (5)$$

$$S_m \omega M^2 / R = S_m w \ell N_a^2 \mu_0 / 2 R \sinh^2 \frac{Rd}{2} \quad (6)$$

These three conditions do not uniquely specify the unknowns. But, add to them the condition that $L_1=L_2$ and it follows from Eq. 6 that

$$\frac{\rho_m}{R} = \frac{S_m}{\sinh^2 \frac{Rd}{2}} \frac{w \ell^2 N_a^2 \mu_0}{4 \pi \omega M^2} \quad (7)$$

so that Eq. 4 becomes an expression that can be solved for M

$$M = \frac{w N_a^2 \mu_0 \ell^2}{4 \pi \sinh \frac{Rd}{2}} \quad (8)$$

and Eq. 5 then gives

$$L_1 \equiv L_2 = \frac{w \ell^2 N_a^2 \mu_0}{4 \pi} \left[\coth \frac{Rd}{2} - \frac{1}{\sinh \frac{Rd}{2}} \right] = \frac{w \ell^2 N_a^2 \mu_0}{4 \pi} \tanh \left(\frac{Rd}{2} \right) \quad (9)$$

Finally, a return to Eq. 7 gives

$$\frac{\rho_m}{R} = \frac{S_m}{\omega} \frac{4 \pi}{w \ell^2 N_a^2 \mu_0} \quad (10)$$

These parameters check with those from the figure.