

(1)

6.641 Lecture Notes #20

5/13/03

Method of Characteristics for Moving Strings

1. String Wave Equation with no external forces for a stationary string

$$\frac{\partial^2 \xi}{\partial t^2} = v_p^2 \frac{\partial^2 \xi}{\partial x^2} \quad v_p^2 = \sqrt{T/m}$$

2. String wave equation for connecting string in x direction at speed U

$$\frac{\partial^2 \xi}{\partial t'^2} = v_p^2 \frac{\partial^2 \xi}{\partial x^2}$$

$$\frac{\partial \xi}{\partial t'} = \frac{\partial \xi}{\partial t} + U \frac{\partial \xi}{\partial x}$$

$$\frac{\partial^2 \xi}{\partial t'^2} = \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \left(\frac{\partial \xi}{\partial t} + U \frac{\partial \xi}{\partial x} \right)$$

$$= \frac{\partial^2 \xi}{\partial t^2} + 2U \frac{\partial^2 \xi}{\partial x \partial t} + U^2 \frac{\partial^2 \xi}{\partial x^2} = v_p^2 \frac{\partial^2 \xi}{\partial x^2}$$

3. Define $v = \frac{\partial \xi}{\partial t}$, $S = T \frac{\partial \xi}{\partial x}$

$$\frac{\partial^2 \xi}{\partial t^2} + 2U \frac{\partial^2 \xi}{\partial x \partial t} + U^2 \frac{\partial^2 \xi}{\partial x^2} = v_p^2 \frac{\partial^2 \xi}{\partial x^2}$$

$$\frac{\partial v}{\partial t} + 2U \frac{\partial v}{\partial x} + \frac{U^2}{T} \frac{\partial S}{\partial x} = \frac{v_p^2}{T} \frac{\partial S}{\partial x} = \frac{1}{m} \frac{\partial S}{\partial x}$$

$$\frac{\partial v}{\partial t} + 2U \frac{\partial v}{\partial x} + \frac{\partial S}{\partial x} \left(\frac{U^2}{T} - \frac{1}{m} \right) = 0$$

$$\frac{\partial v}{\partial t} + 2U \frac{\partial v}{\partial x} + \frac{\partial S}{\partial x} \frac{1}{T} (U^2 - v_p^2) = 0$$

$$\frac{\partial S}{\partial t} = T \frac{\partial v}{\partial x}$$

(2)

$$\begin{bmatrix} 1 & 2U & 0 & \frac{1}{T}(U^2 - U_p^2) \\ 0 & -T & 1 & 0 \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \end{bmatrix} \begin{bmatrix} \frac{\partial U}{\partial T} \\ \frac{\partial U}{\partial x} \\ \frac{\partial S}{\partial T} \\ \frac{\partial S}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ dU \\ dS \end{bmatrix}$$

4. 1st characteristic equation

$$\det \begin{bmatrix} 1 & 2U & 0 & \frac{1}{T}(U^2 - U_p^2) \\ 0 & -T & 1 & 0 \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \end{bmatrix} = 0$$

$$= \det \begin{bmatrix} -T & 1 & 0 \\ dx & 0 & 0 \\ 0 & dt & dx \end{bmatrix} + (dt)_{x(dt)} \begin{bmatrix} 2U & 0 & \frac{1}{T}(U^2 - U_p^2) \\ -T & 1 & 0 \\ 0 & dt & dx \end{bmatrix}$$

$$= -dx(dx) + dt [dx(2U) - dt(U^2 - U_p^2)] = 0$$

$$= -\left(\frac{dx}{dt}\right)^2 + \frac{dx}{dt}(2U) - (U^2 - U_p^2) = 0$$

$$\left(\frac{dx}{dt}\right)^2 - 2U \frac{dx}{dt} + U^2 - U_p^2 = 0$$

$$\frac{dx}{dt} = 2U \pm \sqrt{4U^2 - 4(U^2 - U_p^2)} = U \pm U_p$$

$$\frac{dx}{dt} = U + U_p \text{ (fast wave)} ; \quad \frac{dx}{dt} = U - U_p \text{ (slow wave)}$$

(3)

5. IInd characteristic equation

$$\det \begin{bmatrix} 1 & 0 & 0 & \frac{1}{T}(U^2 - U_P^2) \\ 0 & 0 & 1 & 0 \\ dt & dw & 0 & 0 \\ 0 & dS & dt & dx \end{bmatrix} = 0$$

$$= \begin{bmatrix} 1 & 0 & \frac{1}{T}(U^2 - U_P^2) \\ dt & dw & 0 \\ 0 & dS & dx \end{bmatrix}$$

$$= - [dw dx] - dt \left[-\frac{1}{T}(U^2 - U_P^2) dS \right] = 0$$

$$= -dw \frac{dx}{dt} + \frac{1}{T}(U^2 - U_P^2) dS = 0$$

fast wave solution: $\frac{dx}{dt} = U + U_P$

$$-dw(U + U_P) + \frac{1}{T}(U^2 - U_P^2) dS = 0$$

$$-dw + \frac{(U - U_P)}{T} dS = 0$$

$$w - \frac{5}{T}(U - U_P) = C_{\text{fast}} \quad (\text{Constant})$$

Slow wave solution: $\frac{dx}{dt} = U - U_P$

$$-dw(U - U_P) + \frac{1}{T}(U^2 - U_P^2) dS = 0$$

$$dw - \frac{1}{T}(U + U_P) dS = 0$$

(4)

$$v - \frac{(U + v_p)}{T} S = C_{\text{slow}} \quad (\text{constant})$$

6. Solutions at intersection of fast and slow waves

$$v = \frac{C_{\text{fast}} + C_{\text{slow}}}{2}$$

$$\frac{S}{J_{\text{int}}} = \frac{C_{\text{fast}} - C_{\text{slow}}}{2}$$

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5/13/03

(4)

$$v - \frac{(v+v_p)}{T} s = C_{\text{slow}} \quad (\text{constant})$$

6. Solutions at intersection of fast and slow waves

$$v = \frac{1}{2v_p} [C_{\text{fast}}(v+v_p) - C_{\text{slow}}(v-v_p)]$$

$$\frac{s}{\sqrt{T}} = \frac{C_{\text{fast}} - C_{\text{slow}}}{2}$$