

Massachusetts Institute of Technology
 Department of Electrical Engineering and Computer Science
 6.641 Electromagnetic Fields, Forces, and Motion
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Lecture Notes 15, 4/8/03

$$F_x = \nabla \cdot \bar{\tau}_x \quad , \quad \bar{\tau}_x = T_{xx} \bar{i}_x + T_{xy} \bar{i}_y + T_{xz} \bar{i}_z$$

$$F_y = \nabla \cdot \bar{\tau}_y \quad , \quad \bar{\tau}_y = T_{yx} \bar{i}_x + T_{yy} \bar{i}_y + T_{yz} \bar{i}_z$$

$$F_z = \nabla \cdot \bar{\tau}_z \quad , \quad \bar{\tau}_z = T_{zx} \bar{i}_x + T_{zy} \bar{i}_y + T_{zz} \bar{i}_z$$

$$\bar{\tau} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}$$

$$f_x = \int_V F_x dV = \int_V \nabla \cdot \bar{\tau}_x dV = \oint_S \bar{\tau}_x \cdot \bar{n} d\alpha = \oint_S [T_{xx} n_x + T_{xy} n_y + T_{xz} n_z] dS$$

$$\bar{\tau}_x \cdot \bar{n} = T_{xx} n_x + T_{xy} n_y + T_{xz} n_z = T_{xn} n_n$$

$$\bar{\tau}_y \cdot \bar{n} = T_{yx} n_x + T_{yy} n_y + T_{yz} n_z = T_{yn} n_n$$

$$\bar{\tau}_z \cdot \bar{n} = T_{zx} n_x + T_{zy} n_y + T_{zz} n_z = T_{zn} n_n$$

$$f_i = \int_V \nabla \cdot \bar{\tau}_i dV = \oint_S \bar{\tau}_i \cdot \bar{n} dV = \oint_S T_{ij} n_j dS = \int_V F_i dV$$

$$\bar{F}_i = \nabla \cdot \bar{\tau}_i = \frac{\partial T_{ix}}{\partial x} + \frac{\partial T_{iy}}{\partial y} + \frac{\partial T_{iz}}{\partial z}$$

$$= \frac{\partial T_{ij}}{\partial x_j}$$

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A.EQS Stress Tensor

$$\bar{F} = \rho_f \bar{E} - \frac{1}{2} \bar{E} \cdot \bar{E} \nabla \epsilon + \nabla \left(\frac{1}{2} \bar{E} \cdot \bar{E} \frac{\partial \epsilon}{\partial p} p \right)$$

$$= \nabla \cdot (\epsilon \bar{E}) \bar{E} - \frac{1}{2} (\bar{E} \cdot \bar{E}) \nabla \epsilon + \nabla \left(\frac{1}{2} \bar{E} \cdot \bar{E} \frac{\partial \epsilon}{\partial p} p \right)$$

$$F_i = \frac{\partial (\epsilon E_j)}{\partial x_j} E_i - \frac{1}{2} E_k E_k \frac{\partial \epsilon}{\partial x_i} + \frac{\partial}{\partial x_i} \left(\frac{1}{2} E_k E_k \frac{\partial \epsilon}{\partial p} p \right)$$

$$\nabla \times \bar{E} = 0 \Rightarrow \frac{\partial E_i}{\partial x_j} = \frac{\partial E_j}{\partial x_i}$$

$$F_i = \frac{\partial}{\partial x_j} (\epsilon E_j E_i) - \epsilon E_j \frac{\partial E_i}{\partial x_j} - \frac{1}{2} E_k E_k \frac{\partial \epsilon}{\partial x_i} + \frac{\partial}{\partial x_i} \left(\frac{1}{2} E_k E_k \frac{\partial \epsilon}{\partial p} p \right)$$

$$F_i = \frac{\partial}{\partial x_j} (\epsilon E_i E_j) - \epsilon E_j \frac{\partial E_i}{\partial x_j} - \frac{1}{2} E_k E_k \frac{\partial \epsilon}{\partial x_i} + \underbrace{\epsilon \frac{\partial}{\partial x_i} \left(\frac{1}{2} E_j E_j \right)}_{\delta_{ij}} + \frac{\partial}{\partial x_i} \left(\frac{1}{2} E_k E_k \frac{\partial \epsilon}{\partial p} p \right)$$

$$F_i = \frac{\partial}{\partial x_j} (\epsilon E_i E_j) - \frac{\partial}{\partial x_i} \left[\frac{1}{2} \epsilon E_k E_k - \frac{1}{2} p \frac{\partial \epsilon}{\partial p} E_k E_k \right]$$

$$\frac{\partial}{\partial x_j} = \delta_{ij} \frac{\partial}{\partial x_i}$$

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad \text{Kronecker Delta}$$

$$F_i = \frac{\partial}{\partial x_j} \left[\epsilon E_i E_j - \frac{1}{2} \delta_{ij} E_k E_k \left(\epsilon - p \frac{\partial \epsilon}{\partial p} \right) \right] = \frac{\partial}{\partial x_j} (T_{ij})$$

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$$T_{ij} = \epsilon E_i E_j - \frac{1}{2} S_{ij} E_k E_k (\epsilon - \rho \frac{\partial \epsilon}{\partial \rho})$$

B.MQS Stress Tensor

$$\begin{aligned}\bar{F} &= \bar{J} \times \bar{B} - \frac{1}{2} \bar{H} \cdot \bar{H} \nabla u + \nabla \left(\frac{1}{2} \rho \frac{\partial u}{\partial \rho} \bar{H} \cdot \bar{H} \right) \\ &= (\nabla \times \bar{H}) \times (\mu \bar{H}) - \frac{1}{2} \bar{H} \cdot \bar{H} \nabla u + \nabla \left(\frac{1}{2} \rho \frac{\partial u}{\partial \rho} \bar{H} \cdot \bar{H} \right)\end{aligned}$$

$$(\nabla \times \bar{H}) \times \bar{H} = (\bar{H} \cdot \nabla) \bar{H} - \frac{1}{2} \nabla (\bar{H} \cdot \bar{H})$$

$$\bar{F} = u \left[(\bar{H} \cdot \nabla) \bar{H} - \frac{1}{2} \nabla (\bar{H} \cdot \bar{H}) \right] - \frac{1}{2} \bar{H} \cdot \bar{H} \nabla u + \nabla \left(\frac{1}{2} \rho \frac{\partial u}{\partial \rho} \bar{H} \cdot \bar{H} \right)$$

$$\begin{aligned}F_i &= u \left[H_j \frac{\partial}{\partial x_j} H_i - \frac{1}{2} \frac{\partial}{\partial x_i} (H_k H_k) \right] - \frac{1}{2} H_k H_k \frac{\partial u}{\partial x_i} + \frac{\partial}{\partial x_i} \left(\frac{1}{2} \rho \frac{\partial u}{\partial \rho} H_k H_k \right) \\ &= \underbrace{\frac{\partial}{\partial x_j} (u H_i H_j)}_{\nabla \cdot \bar{B} = 0} - \underbrace{H_i \frac{\partial}{\partial x_j} (u H_j)}_{\frac{\partial}{\partial x_j}} - \underbrace{\frac{u}{2} \frac{\partial}{\partial x_i} H_k H_k}_{\frac{\partial}{\partial x_i} \left(\frac{1}{2} u H_k H_k \right)} - \underbrace{\frac{1}{2} H_k H_k \frac{\partial u}{\partial x_i}}_{\frac{\partial}{\partial x_i} \left(\frac{1}{2} u H_k H_k \right)} \\ &\quad + \frac{\partial}{\partial x_i} \left(\frac{1}{2} \rho \frac{\partial u}{\partial \rho} H_k H_k \right)\end{aligned}$$

$$F_i = \frac{\partial}{\partial x_j} (u H_i H_j) - \frac{\partial}{\partial x_i} \left(\frac{1}{2} u H_k H_k - \rho \frac{\partial u}{\partial \rho} H_k H_k \right)$$

$$= \frac{\partial}{\partial x_j} \left[u H_i H_j - \frac{1}{2} S_{ij} H_k H_k (u - \rho \frac{\partial u}{\partial \rho}) \right] = \frac{\partial}{\partial x_j} T_{ij}$$

$$T_{ij} = u H_i H_j - \frac{1}{2} S_{ij} H_k H_k (u - \rho \frac{\partial u}{\partial \rho})$$