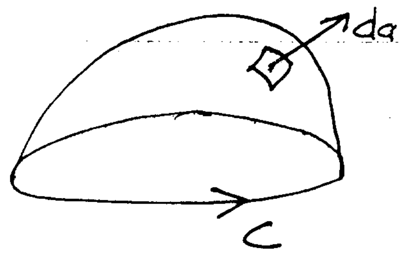
Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.641 Electromagnetic Fields, Forces, and Motion
Spring 2003

Lecture Notes 1, 2/4/03

I. Maxwell's Equations in Integral Form in Free Space

1. Faraday's Law

$$\underbrace{\oint_C \overline{E} \cdot \overline{ds}}_{\text{Circulation of } \overline{E}} = -\frac{d}{dt} \underbrace{\int_S \mu_0 \overline{H} \cdot \overline{da}}_{\text{magnetic flux}}$$



$\mu_0 = 4\pi \times 10^{-7}$ henries/m [magnetic permeability of free space]

EQS form: $\oint_C \overline{E} \cdot \overline{ds} = 0$ (Kirchoff's Voltage Law, conservative electric field)

MQS circuit form: $v = L\frac{di}{dt}$ (Inductor)

2. Ampère's Law (with displacement current)

$$\underbrace{\oint_C \overline{H} \cdot \overline{ds}}_{\text{Circulation of } \overline{H}} = \underbrace{\int_S \overline{J} \cdot \overline{da}}_{\text{Conduction Current}} + \underbrace{\frac{d}{dt} \int_S \epsilon_0 \overline{E} \cdot \overline{da}}_{\text{Displacement Current}}$$

MQS form: $\oint_C \overline{H}\cdot\overline{ds} = \int_S \overline{J}\cdot\overline{da}$

EQS circuit form: $i = C\frac{dv}{dt}$ (capacitor)

3. Gauss' Law for Electric Field

$$\oint_S \epsilon_0 \overline{E}\cdot\overline{da} = \int_V \rho\, dV$$

$\epsilon_0 \approx \frac{10^{-9}}{36\pi} \approx 8.854\times10^{-12}$ farads/m

$c = \frac{1}{\sqrt{\epsilon_0\mu_0}}$ (Speed of electromagnetic waves in free space)

4. Gauss' Law for Magnetic Field

$$\oint_S \mu_0 \overline{H}\cdot\overline{da} = 0$$

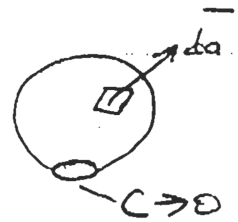
In free space: $\overline{B} = \mu_0\overline{H}$

magnetic flux density

magnetic field intensity

5. Conservation of Charge

Take Ampère's law with displacement current and let contour $C \to 0$



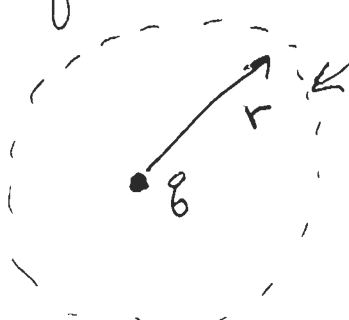
$$\lim_{C \to 0} \oint_C \overline{H} \cdot d\overline{s} = 0 = \oint_S \overline{J} \cdot d\overline{a} + \frac{d}{dt} \underbrace{\oint_S \epsilon_0 \overline{E} \cdot d\overline{a}}_{\int_V \rho \, dV}$$

$$\underbrace{\oint_S \overline{J} \cdot d\overline{a}}_{\substack{\text{Total current} \\ \text{leaving volume} \\ \text{through surface}}} + \frac{d}{dt} \underbrace{\int_V \rho \, dV}_{\substack{\text{total charge} \\ \text{inside volume}}} = 0$$

6. Lorentz Force Law

$$\overline{f} = q(\overline{E} + \overline{v} \times \mu_0 \overline{H})$$

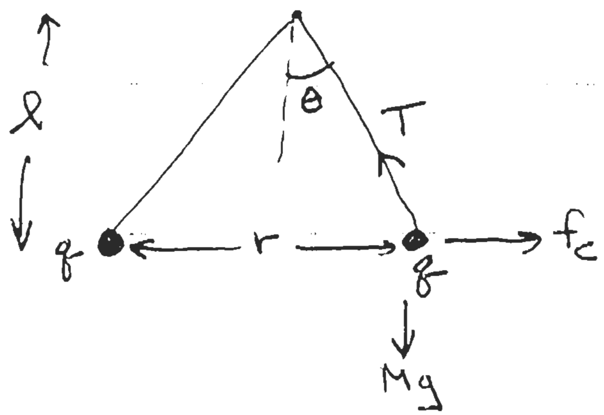
II. Electric Field from Point Charge



Mathematical Gaussian Spherical Surface of radius r

$$\oint_S \epsilon_0 \overline{E} \cdot d\overline{a} = \epsilon_0 E_r 4\pi r^2 = q$$

$$E_r = \frac{q}{4\pi\epsilon_0 r^2}$$



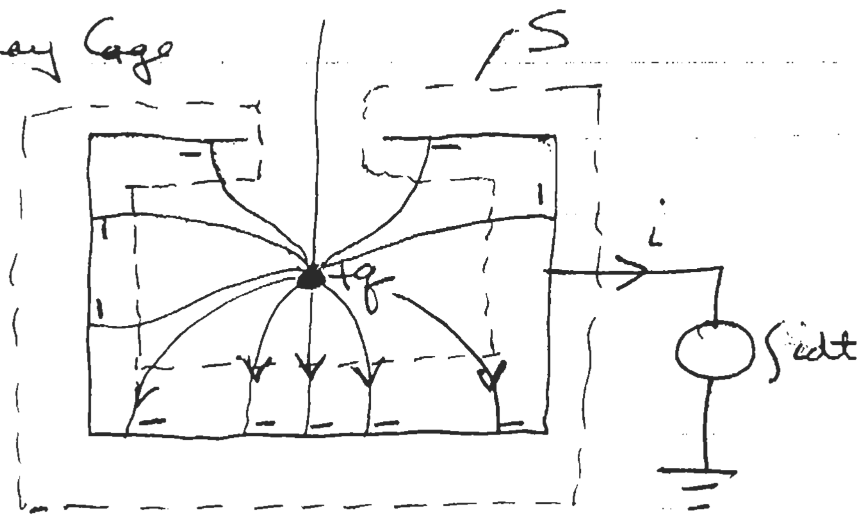
$$T\sin\theta = f_c = \frac{q^2}{4\pi\epsilon_0 r^2}$$

$$T\cos\theta = Mg$$

$$\tan\theta = \frac{q^2}{4\pi\epsilon_0 r^2 Mg} = \frac{r}{2l}$$

$$q = \left[\frac{2\pi\epsilon_0 r^3 Mg}{l}\right]^{1/2}$$

III. Faraday Cage



$$\oint_S \bar{J}\cdot\bar{da} = i = -\frac{d}{dt}\int \rho\, dV = -\frac{d}{dt}(-q) = \frac{dq}{dt}$$

$$\int i\,dt = q$$

I. Divergence Theorem

1. Divergence Operation

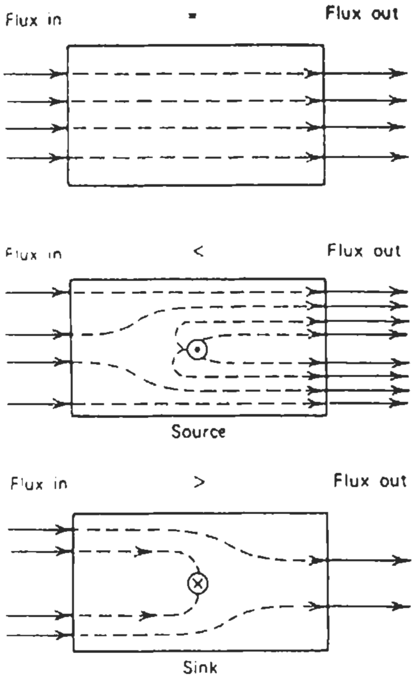


Figure 1-13 The net flux through a closed surface tells us whether there is a source or sink within an enclosed volume.

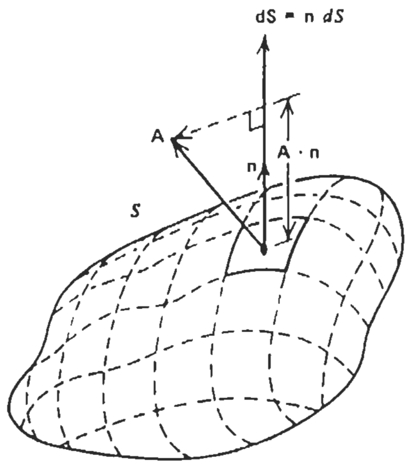


Figure 1-14 The flux of a vector **A** through the closed surface S is given by the surface integral of the component of **A** perpendicular to the surface S. The differential vector surface area element **dS** is in the direction of the unit normal **n**.

$$\oint_S \bar{A}\cdot\overline{dS} = \int_V \operatorname{div}(\bar{A})\,dV$$

$$\operatorname{div}\bar{A} = \lim_{\Delta V\to 0}\frac{\oint_S \bar{A}\cdot\overline{dS}}{\Delta V}$$

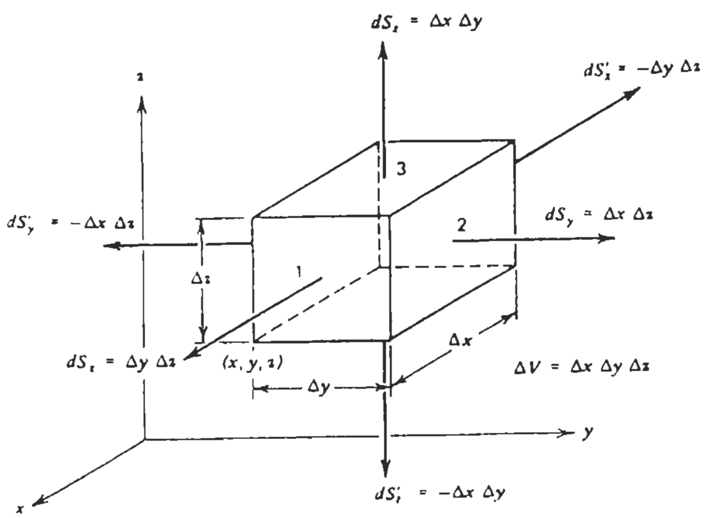


Figure 1-15 Infinitesimal rectangular volume used to define the divergence of a vector.

$$\Phi = \int_1 A_x(x,y,z)\,dy\,dz - \int_{1'} A_x(x-\Delta x,y,z)\,dy\,dz$$

$$+ \int_2 A_y(x,y+\Delta y,z)\,dx\,dz - \int_{2'} A_y(x,y,z)\,dx\,dz$$

$$+ \int_3 A_z(x,y,z+\Delta z)\,dx\,dy - \int_{3'} A_z(x,y,z)\,dx\,dy$$

$$\Phi \approx \Delta x \Delta y \Delta z \left\{ \frac{[A_x(x,y,z) - A_x(x-\Delta x, y, z)]}{\Delta x} + \frac{[A_y(x, y+\Delta y, z) - A_y(x,y,z)]}{\Delta y} + \frac{[A_z(x,y,z+\Delta z) - A_z(x,y,z)]}{\Delta z} \right\}$$

$$\approx \Delta V \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right]$$

$$\operatorname{div} \bar{A} = \lim_{\Delta V \to 0} \frac{\oint_S \bar{A} \cdot d\bar{s}}{\Delta V} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Del Operator: $\nabla = \bar{i}_x \dfrac{\partial}{\partial x} + \bar{i}_y \dfrac{\partial}{\partial y} + \bar{i}_z \dfrac{\partial}{\partial z}$

$$\operatorname{div} \bar{A} = \nabla \cdot \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

2. Gauss' Integral Theorem

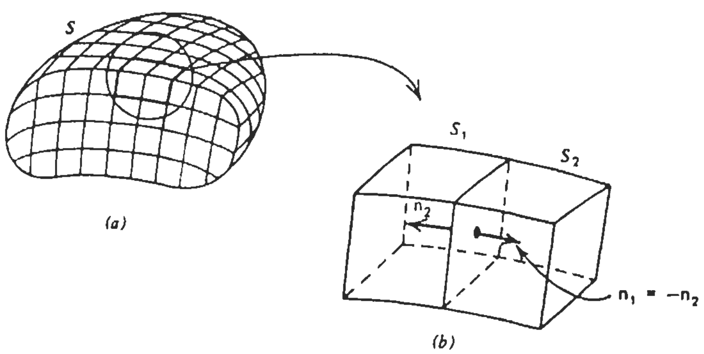


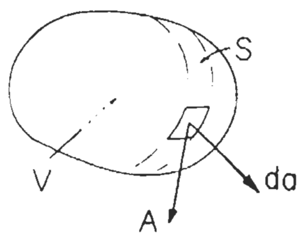
Figure 1-17 Nonzero contributions to the flux of a vector are only obtained across those surfaces that bound the outside of a volume. (a) Within the volume the flux leaving one incremental volume just enters the adjacent volume where (*b*) the outgoing normals to the common surface separating the volumes are in opposite directions.

$$\oint_S \bar{A} \cdot d\bar{S} = \sum_{\substack{i=1 \\ N \to \infty}}^{N} \oint_{dS_i} \bar{A} \cdot d\bar{S}_i$$

$$= \lim_{\substack{N \to \infty \\ \Delta V_n \to 0}} \sum_{i=1}^{N} (\nabla \cdot \bar{A}) \Delta V_i$$

$$= \int_V \nabla \cdot \bar{A} \, dV$$

$$\int_V \nabla \cdot \mathbf{A} dV = \oint_S \mathbf{A} \cdot d\mathbf{a}$$



3. Gauss' Law In Differential Form

$$\oint_S \epsilon_0 \overline{E} \cdot \overline{da} = \int_V \nabla \cdot (\epsilon_0 \overline{E}) dV = \int_V \rho dV$$

$$\nabla \cdot (\epsilon_0 \overline{E}) = \rho$$

$$\oint_S \mu_0 \overline{H} \cdot \overline{da} = \int_V \nabla \cdot (\mu_0 \overline{H}) dV = 0$$

$$\nabla \cdot (\mu_0 \overline{H}) = 0$$

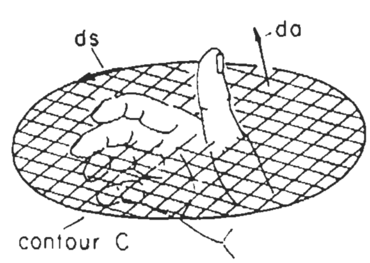
II. Stokes' Theorem

1. Curl Operation

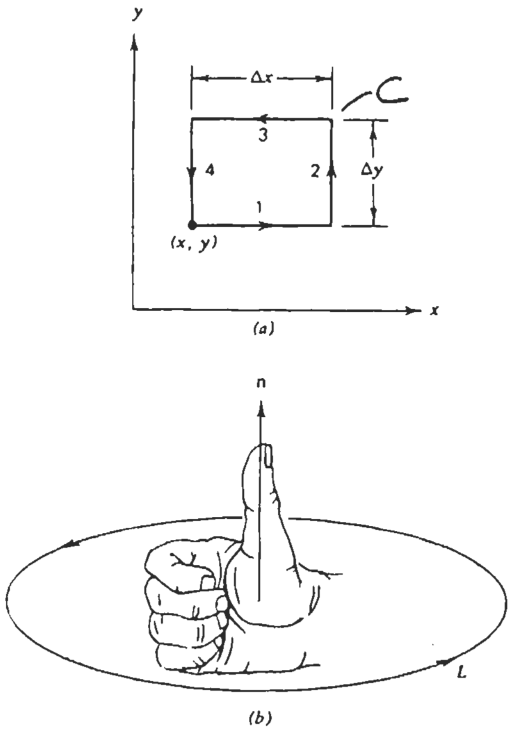
$$\oint_C \overline{A} \cdot \overline{ds} = \int_S \mathrm{curl}(\overline{A}) \cdot \overline{da}$$

$$\mathrm{curl}(\overline{A})_n = \lim_{da \to 0} \frac{\oint_C \overline{A} \cdot \overline{ds}}{da_n}$$

STOKES'



$$\int_S \nabla \times \mathbf{A} \cdot d\mathbf{a} = \oint_C \mathbf{A} \cdot d\mathbf{s}$$



$$\oint_C \bar{A}\cdot d\bar{s} = \underset{1}{\int_x^{x+\Delta x} A_x(x,y)\,dx} + \underset{2}{\int_y^{y+\Delta y} A_y(x+\Delta x, y)\,dy}$$

$$+ \underset{3}{\int_{x+\Delta x}^{x} A_x(x,y+\Delta y)\,dx} + \underset{4}{\int_{y+\Delta y}^{y} A_y(x,y)\,dy}$$

$$= \Delta x \Delta y \left[\frac{[A_x(x,y) - A_x(x,y+\Delta y)]}{\Delta y} + \frac{[A_y(x+\Delta x,y) - A_y(x,y)]}{\Delta x} \right]$$

$$= da_z \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right]$$

Figure 1-19 (*a*) Infinitesimal rectangular contour used to define the circulation. (*b*) The right-hand rule determines the positive direction perpendicular to a contour.

$$\text{curl}\,(\bar{A})_z = \frac{\oint \bar{A}\cdot d\bar{s}}{da_z} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

By symmetry

$$\text{curl}\,(\bar{A})_y = \frac{\oint \bar{A}\cdot d\bar{s}}{da_y} = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$$

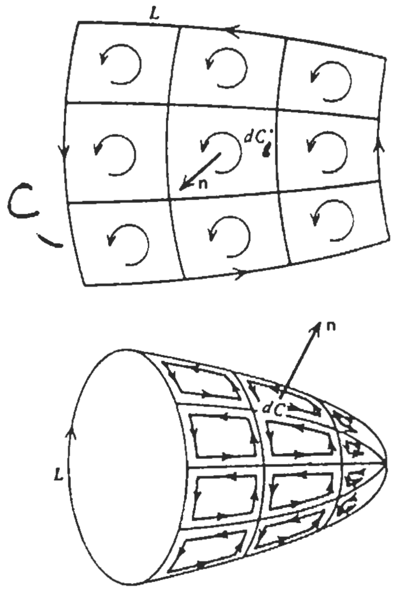
$$\text{curl}\,(\bar{A})_x = \frac{\oint \bar{A}\cdot d\bar{s}}{da_x} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

$$\text{curl}\,\bar{A} = \bar{i}_x \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] + \bar{i}_y \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] + \bar{i}_z \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

$$= \det \begin{bmatrix} \bar{i}_x & \bar{i}_y & \bar{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{bmatrix}$$

$$= \nabla \times \bar{A}$$

2. Stokes' Integral Theorem



$$\lim_{N\to\infty} \sum_{i=1}^{N} \oint_{dC_i} \overline{A}\cdot\overline{ds_i} = \oint_C \overline{A}\cdot\overline{ds}$$

$$= \sum_{i=1}^{N\to\infty} (\nabla\times\overline{A})\cdot\overline{da_i}$$

$$= \int_S (\nabla\times\overline{A})\cdot\overline{da}$$

Figure 1-23 Many incremental line contours distributed over any surface, have nonzero contribution to the circulation only along those parts of the surface on the boundary contour L.

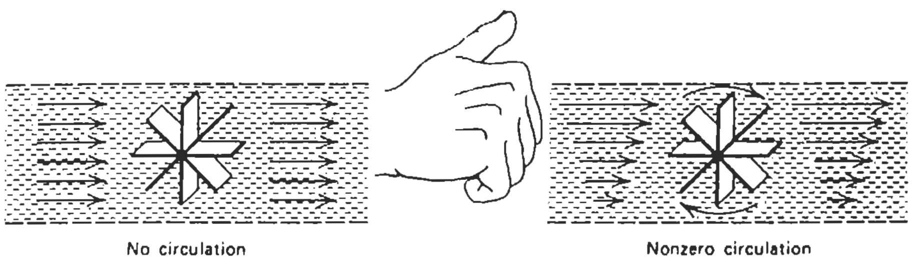


Figure 1-20 A fluid with a velocity field that has a curl tends to turn the paddle wheel. The curl component found is in the same direction as the thumb when the fingers of the right hand are curled in the direction of rotation.

3. Faraday's Law In Differential Form

$$\oint_C \overline{E}\cdot\overline{ds} = \int_S (\nabla\times\overline{E})\cdot\overline{da} = -\frac{d}{dt}\int_S \mu_0\overline{H}\cdot\overline{da}$$

$$\nabla\times\overline{E} = -\mu_0\frac{\partial\overline{H}}{\partial t}$$

4. Ampère's Law In Differential Form

$$\oint_C \bar{H}\cdot d\bar{s} = \int_S \nabla\times \bar{H}\cdot d\bar{a} = \int_S \bar{J}\cdot d\bar{a} + \frac{d}{dt}\int_S \epsilon_0 \bar{E}\cdot d\bar{a}$$

$$\nabla\times\bar{H} = \bar{J} + \epsilon_0\frac{\partial \bar{E}}{\partial t}$$

III. Applications to Maxwell's Equations

1. Vector Identity

$$\lim_{C\to 0}\oint_C \bar{A}\cdot d\bar{s} = 0 = \oint_S (\nabla\times\bar{A})\cdot d\bar{a} = \int_V \nabla\cdot(\nabla\times\bar{A})\,dV$$

$$\nabla\cdot(\nabla\times\bar{A}) = 0$$

2. Charge Conservation

$$\nabla\cdot\left\{\nabla\times\bar{H} = \bar{J} + \epsilon_0\frac{\partial\bar{E}}{\partial t}\right\}$$

$$0 = \nabla\cdot\left[\bar{J} + \epsilon_0\frac{\partial\bar{E}}{\partial t}\right]$$

$$0 = \nabla\cdot\bar{J} + \frac{\partial\rho}{\partial t}$$

3. Magnetic Field

$$\nabla\cdot\left\{\nabla\times\bar{E} = -\mu_0\frac{\partial\bar{H}}{\partial t}\right\}$$

$$0 = -\frac{\partial}{\partial t}\left[\nabla\cdot\mu_0\bar{H}\right] \Rightarrow \nabla\cdot(\mu_0\bar{H}) = 0$$

IV. Boundary Conditions

1. Gauss' Continuity Condition

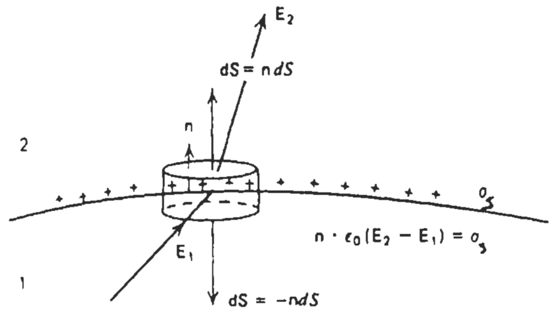


Figure 2-19 Gauss's law applied to a differential sized pill-box surface enclosing some surface charge shows that the normal component of $\varepsilon_0 \mathbf{E}$ is discontinuous in the surface charge density.

$$\oint_S \epsilon_0 \bar{E} \cdot d\bar{a} = \int_S \sigma_s dS \Rightarrow \epsilon_0 (E_{2n} - E_{1n}) dS = \sigma_s dS$$

$$\epsilon_0 (E_{2n} - E_{1n}) = \sigma_s \Rightarrow \bar{n} \cdot \left[\epsilon_0 (\bar{E}_2 - \bar{E}_1)\right] = \sigma_s$$

2. Continuity of Tangential $\bar{E}$

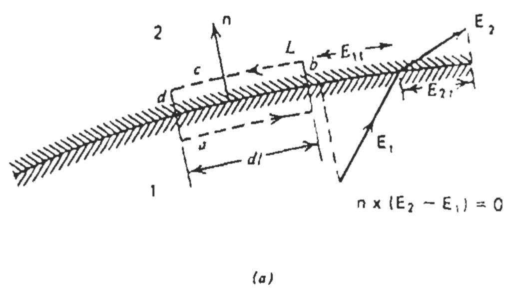


Figure 3-12 (a) Stokes' law applied to a line integral about an interface of discontinuity shows that the tangential component of electric field is continuous across the boundary.

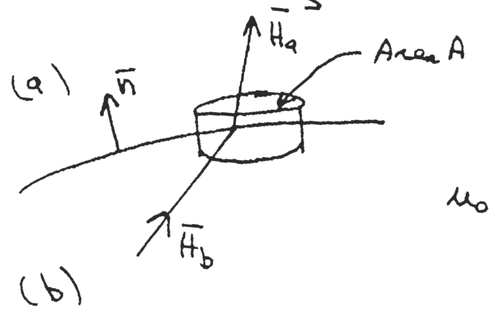
$$\oint_C \bar{E} \cdot d\bar{s} = (E_{1t} - E_{2t}) dl = 0 \Rightarrow E_{1t} - E_{2t} = 0$$

$$\bar{n} \times (\bar{E}_1 - \bar{E}_2) = 0$$

Equivalent to $\Phi_1 = \Phi_2$ along boundary

3. Normal H

$$\nabla\cdot\mu_0\bar{H}=0 \Rightarrow \oint_S \mu_0\bar{H}\cdot d\bar{a}=0$$



$$\mu_0(H_{an}-H_{bn})A=0$$

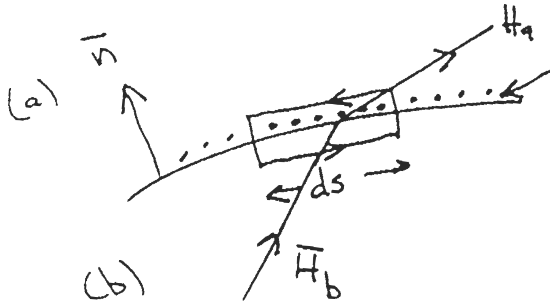
$$H_{an}=H_{bn}$$

$$\bar{n}\cdot[\bar{H}_a-\bar{H}_b]=0$$

Boundary Conditions

4. Tangential $\bar{H}$

$$\nabla\times\bar{H}=\bar{J} \Rightarrow \oint_C \bar{H}\cdot d\bar{s}=\int_S \bar{J}\cdot d\bar{a}$$



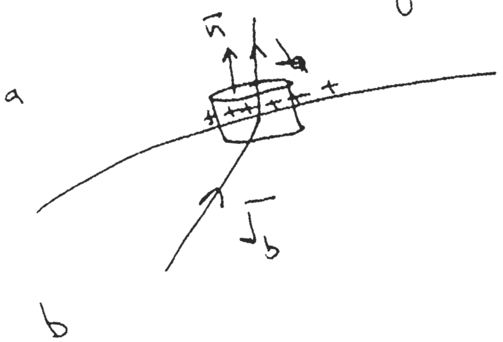
surface current K out of page (amp/meter)

$$H_{bt}\cancel{ds}-H_{at}\cancel{ds}=K\cancel{ds}$$

$$H_{bt}-H_{at}=K$$

$$\bar{n}\times[\bar{H}_a-\bar{H}_b]=\bar{K}$$

5. Conservation of Charge Boundary Condition



$$\nabla\cdot\bar{J}_u+\frac{\partial\rho_u}{\partial t}=0$$

$$\oint_S \bar{J}_u\cdot d\bar{a}+\frac{d}{dt}\int_V \rho_u\,dV=0$$

$$\bar{n}\cdot[\bar{J}_a-\bar{J}_b]+\frac{d}{dt}\sigma_u=0$$