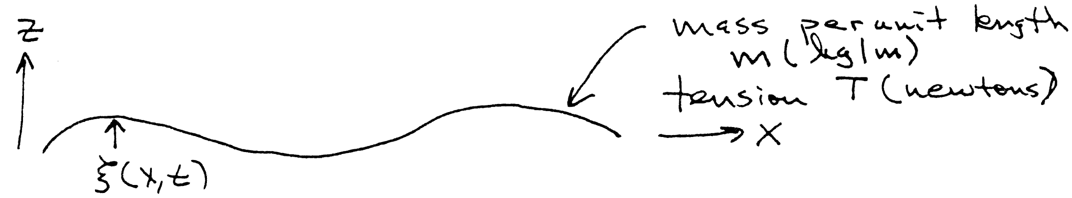
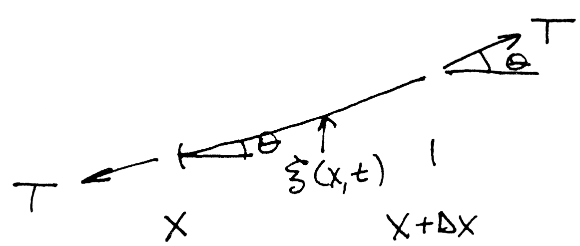
Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.641 Electromagnetic Fields, Forces, and Motion
Spring 2003

Lecture Notes 18, 5/6/03

Waves and Instabilities In Elastic Media

I. Transverse Motions of Wires Under Tension





force per unit length (nt/m)

$$m\cancel{\Delta x}\frac{\partial^2 \xi}{\partial t^2} = \frac{T_z(x+\Delta x) - T_z(x)}{\Delta x} + F_{ext}\cancel{\Delta x}$$

$$m\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial T_z}{\partial x} + F_{ext}$$

$$T_z = T\sin\theta \approx T\tan\theta \approx T\left[\frac{\xi(x+\Delta x) - \xi(x)}{\Delta x}\right]$$

$$\approx T\frac{\partial \xi}{\partial x}$$

$$m\frac{\partial^2 \xi}{\partial t^2} = T\frac{\partial^2 \xi}{\partial x^2} + F_{ext}$$

II. Transverse Motions of Membranes

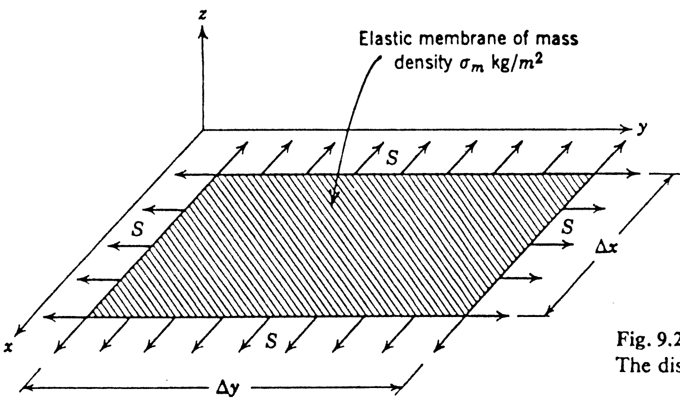


Fig. 9.2.1 A plane-elastic membrane in equilibrium subject to a tension S N/m along its edges.

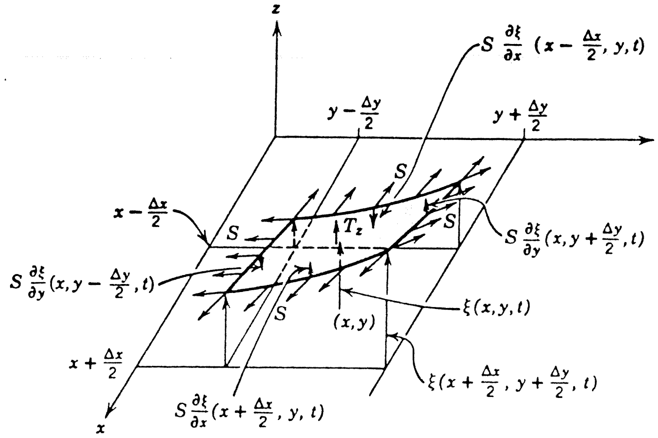


Fig. 9.2.2 Section of membrane having area $(\Delta x\, \Delta y)$ and subject to the uniform tension S. The displacement at the center of the section (x, y) is $\xi(x, y, t)$.

$$\sigma_m \Delta x \Delta y \frac{\partial^2 \xi}{\partial t^2} = S\Delta x\left[\frac{\partial \xi}{\partial y}\left(x, y+\frac{\Delta y}{2}, t\right) - \frac{\partial \xi}{\partial y}\left(x, y-\frac{\Delta y}{2}, t\right)\right]$$

membrane mass per unit area (kg/m^2)

$$+ S\Delta y\left[\frac{\partial \xi}{\partial x}\left(x+\frac{\Delta x}{2}, y, t\right) - \frac{\partial \xi}{\partial x}\left(x-\frac{\Delta x}{2}, y, t\right)\right]$$

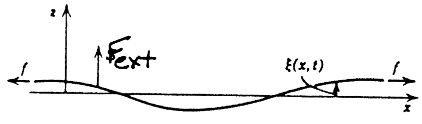
$$+ T_z \Delta x \Delta y$$

membrane tension (newtons/m)

external force per unit area (newtons/m^2)

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S\left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2}\right) + T_z$$

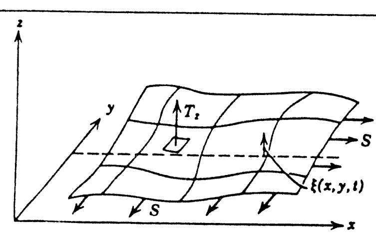
Wire or "String"



$$m\frac{\partial^2 \xi}{\partial t^2} = T\frac{\partial^2 \xi}{\partial x^2} + F_{ext}$$

ξ—transverse displacement
m—mass/unit length
T—tension (constant force)
F_{ext}—transverse force/unit length

Membrane



$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S\left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2}\right) + T_z$$

ξ—transverse displacement
σ_m—surface mass density
S—tension in y- and z-directions (constant force per unit length)
T_z—z-directed force per unit area

I. Non-Dispersive Waves on a String ($F_{ext} = 0$)

Dispersion Equation

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{T}{m}\frac{\partial^2 \xi}{\partial x^2}$$

$$v_s = \sqrt{T/m}$$

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2} \qquad \text{(wave equation)}$$

$$\xi = \mathrm{Re}\, \hat{\xi} e^{j(\omega t - kx)}$$

$$-\omega^2 \hat{\xi} = -k^2 v_s^2 \hat{\xi} \Rightarrow \omega^2 = k^2 v_s^2 \Rightarrow \omega = \pm k v_s$$

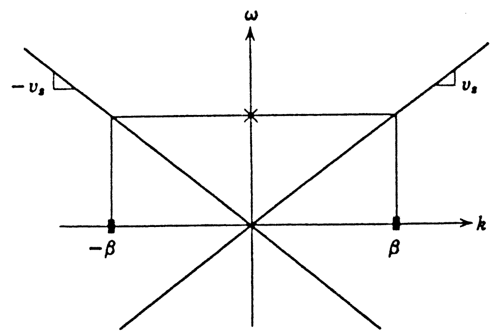
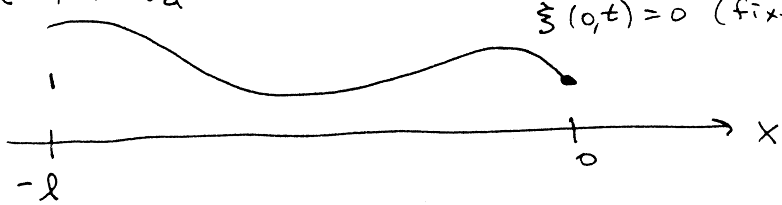


Fig. 10.1.1 Dispersion equation for waves on the simple string.

B. Driven and Transient Response

$$\xi(-l,t) = \xi_d \sin \omega_d t = \mathrm{Re}\,(-j\xi_d e^{j\omega_d t})$$

$$\xi(0,t) = 0 \quad \text{(fixed end)}$$



$$k = \pm \frac{\omega_d}{v_s}$$

$$\xi(x,t) = \mathrm{Re}\left[\left(\hat{\xi}_1 e^{-j\frac{\omega_d}{v_s}x} + \hat{\xi}_2 e^{+j\frac{\omega_d}{v_s}x}\right)e^{j\omega_d t}\right]$$

$$\xi(0,t) = 0 = \mathrm{Re}\left[(\hat{\xi}_1 + \hat{\xi}_2)e^{j\omega_d t}\right] \Rightarrow \hat{\xi}_1 = -\hat{\xi}_2$$

$$\hat{\xi}(-l,t) = -j\xi_d = \hat{\xi}_1 e^{\frac{j\omega_d l}{v_s}} + \hat{\xi}_2 e^{-j\frac{\omega_d l}{v_s}} = \hat{\xi}_1\left(e^{j\frac{\omega_d l}{v_s}} - e^{-j\omega_d l/v_s}\right)$$

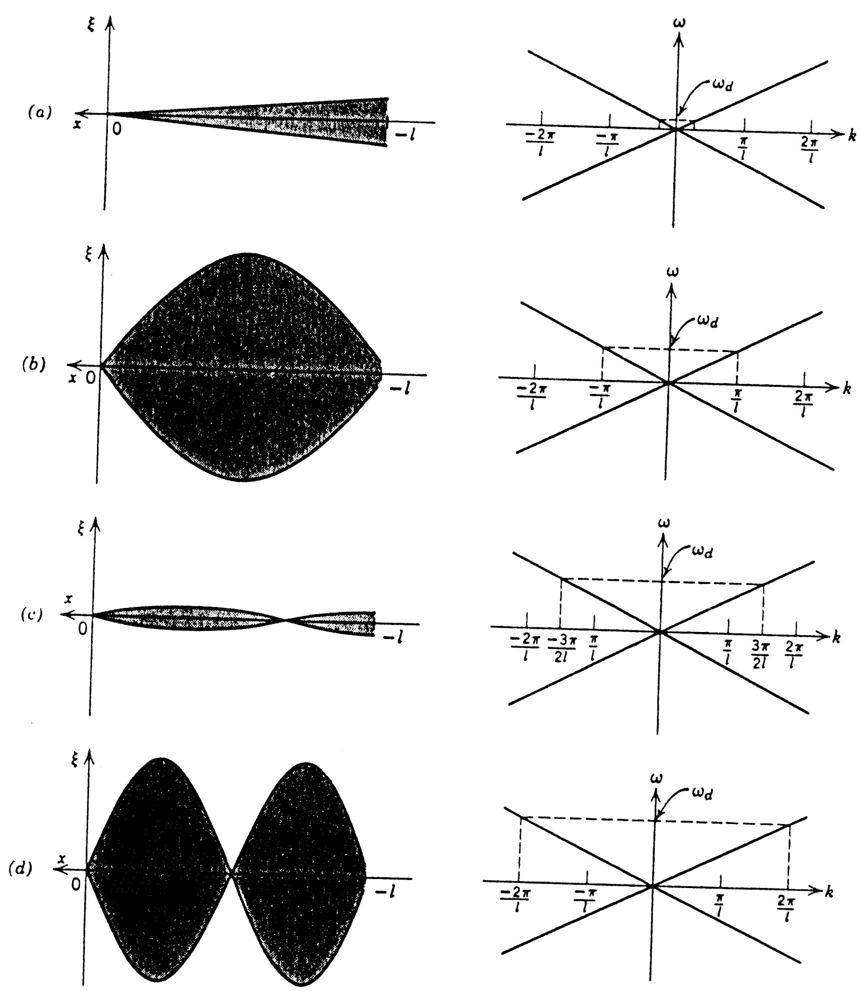
$$= 2j\hat{\xi}_1 \sin\frac{\omega_d l}{v_s}$$

$$-j\hat{\xi}_d = 2j\sin\frac{\omega_d l}{v_s}\,\hat{\xi}_1 \Rightarrow \hat{\xi}_1 = \frac{-\xi_d}{2\sin\frac{\omega_d l}{v_s}}$$

$$\hat{\xi}(x) = \frac{-\xi_d}{2\sin\frac{\omega_d l}{v_s}}\left(e^{-j\frac{\omega_d x}{v_s}} - e^{j\frac{\omega_d x}{v_s}}\right) = \frac{+\xi_d}{\cancel{2}\sin\frac{\omega_d l}{v_s}}\left(\cancel{2}j\sin\frac{\omega_d x}{v_s}\right)$$

$$= \frac{j\sin\frac{\omega_d x}{v_s}\,\xi_d}{\sin\frac{\omega_d l}{v_s}}$$

$$\xi(x,t) = \mathrm{Re}\,\hat{\xi}(x)e^{j\omega_d t} = \mathrm{Re}\,\frac{j\sin\frac{\omega_d x}{v_s}}{\sin\frac{\omega_d l}{v_s}}\xi_d e^{j\omega_d t} = \frac{-\xi_d\sin\frac{\omega_d x}{v_s}\sin\omega_d t}{\sin\frac{\omega_d l}{v_s}}$$



Resonance:

$$\sin\frac{\omega_d l}{v_s} = 0$$

$$\frac{\omega_d l}{v_s} = n\pi$$

Fig. 9.2.6 Sketch of experiment in which a taut spring is fixed at the left end and deflected sinusoidally at the right end. (*a*) Deflections in the quasi-static limit at which the frequency is low compared with the reciprocal of the time required for a disturbance to propagate from one end of the spring to the other; (*b*) to (*d*) deflection as frequency is varied from value at which $k = \pi/l$ to $k = 2\pi/l$. The excitation amplitude is kept the same in going from (*b*) to (*d*). Actual experiment can be seen in film, "Complex Waves I" produced by Education Development Center for National Committee on Electrical Engineering Films.

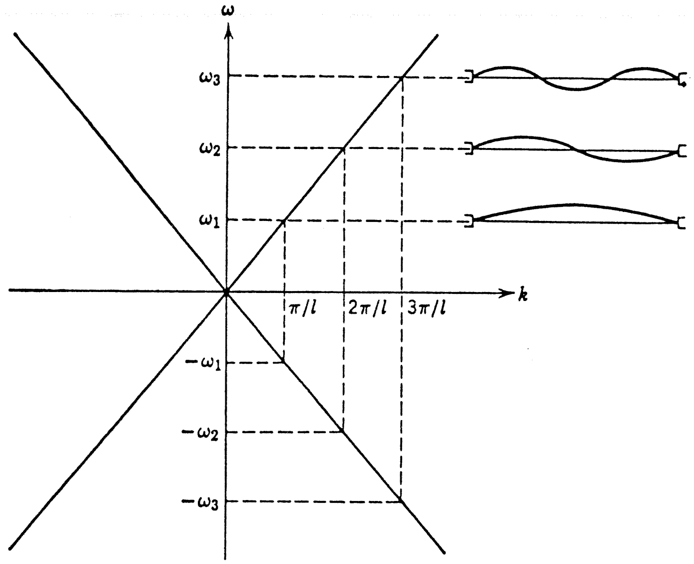


Fig. 9.2.7 Allowed wavenumbers (eigenvalues) $k = k_n$ as they are related to the eigenfrequencies ω_n by the dispersion equation.

IV Cut-off or Evanescent Waves

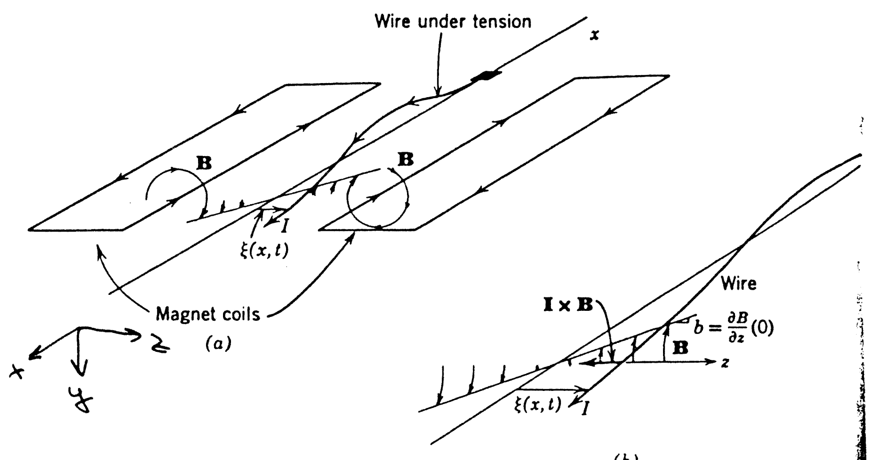


Fig. 10.1.2 (*a*) A conducting wire is stretched along the x-axis and is free to undergo transverse motions in the horizontal plane. Magnet coils produce a field **B** which is zero along the x-axis; (*b*) the wire carries a current I so that deflections from the x-axis result in a force that tends to restore the wire to its equilibrium position.

$$m\frac{\partial^2 \xi}{\partial t^2} = T\frac{\partial^2 \xi}{\partial x^2} + F_{ext}$$

$$F_{ext} = (\overline{I}\times\overline{B})\cdot\overline{i_z} = I\overline{i_x}\times\left(\underbrace{\frac{\partial B_y}{\partial z}}_{-b}\xi\,\overline{i_y}\right)\cdot\overline{i_z}$$

$$= -Ib\xi$$

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{T}{m}\frac{\partial^2 \xi}{\partial x^2} - \frac{Ib\xi}{m}$$

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2} - \omega_c^2 \xi \; ; \quad v_s^2 = \frac{T}{m}, \quad \omega_c^2 = \frac{Tb}{m}$$

$$\xi = \mathrm{Re}\, \hat{\xi} e^{j(\omega t - kx)}$$

$$+\omega^2 = +k^2 v_s^2 + \omega_c^2 \Rightarrow k = \pm \sqrt{\omega^2 - \omega_c^2}/v_s$$

$$\xi(0,t) = 0, \quad \xi(-l,t) = \xi_d \sin \omega_d t$$

$$k = \pm |k_r|, \quad \omega_d > \omega_c \quad \left(|k_r| = \sqrt{\omega_d^2 - \omega_c^2}/v_s \right)$$

$$k = \pm j|k_i|, \quad \omega_d < \omega_c \quad \left(|k_i| = \sqrt{\omega_c^2 - \omega_d^2}/v_s \right)$$

$\omega_d > \omega_c$

$$\xi = -\xi_d \frac{\sin |k_r| x}{\sin |k_r| l} \sin \omega_d t$$

$\omega_d < \omega_c$

$$\xi = -\xi_d \frac{\sinh |k_i| x}{\sinh |k_i| l} \sin \omega_d t$$

Waves and Instabilities in Stationary Media

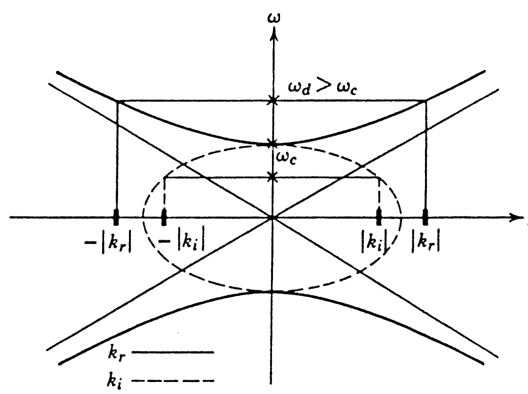


Fig. 10.1.3 Dispersion relation for the wire subject to a restoring force distributed along its length (for the case shown in Fig. 10.1.2). Complex values of k are shown as functions of real values of ω.

Dynamics of Electromechanical Continua

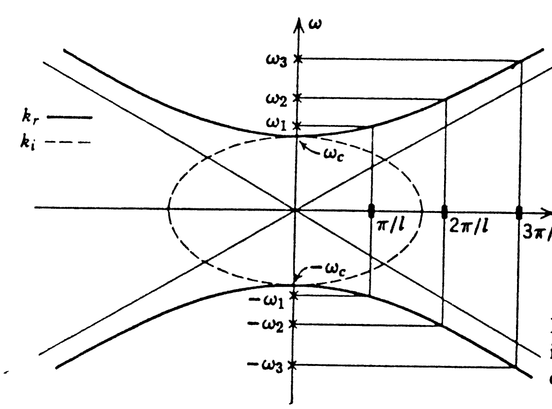


Fig. 10.1.6 A dispersion equation for waves on the wire in Fig. 10.1.2 showing the relationship between the eigenfrequencies ω_n and the eigenvalues $k = n\pi/l$.

resonance $(|k_r| l = n\pi) \Rightarrow \dfrac{\omega^2 - \omega_c^2}{v_s^2} = \left(\dfrac{n\pi}{l}\right)^2 \Rightarrow \omega = \left[\omega_c^2 + \left(\dfrac{n\pi}{l}\right)^2 v_s^2\right]^{1/2}$

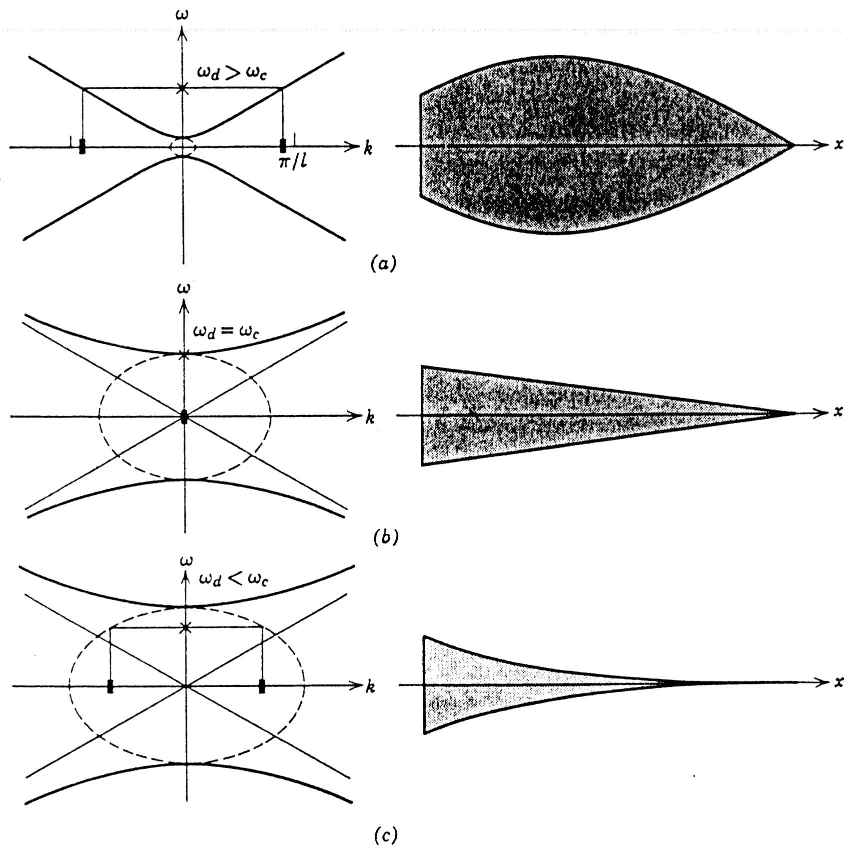
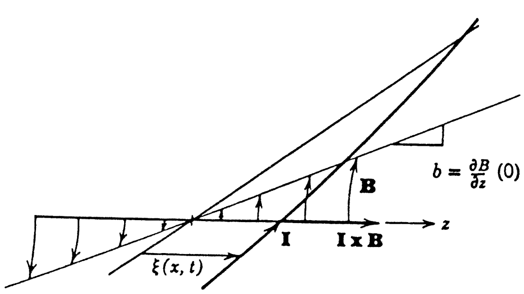


Fig. 10.1.4 Envelope of wire deflection in magnetic field. The wire is fixed at the right end and driven at a fixed sinusoidal frequency at the left end. The ω-k plots show the effect of the current I on the dispersion equation. The current I (or cutoff frequency ω_c) is being raised so that (a), $I \approx 0$, (b) I is sufficient just to cut off the propagation ($\omega_d = \omega_c$), and (c) the waves are evanescent, $\omega_d < \omega_c$. This experiment can be seen in the film "Complex Waves I," produced for the National Committee on Electrical Engineering films by Education Development Center, Newton, Mass.

V. Absolute or Nonconvective Instability



$F_{ext} = +Ib\xi$

Fig. 10.1.9 Wire carrying current I in a magnetic field that is zero along the axis $\xi = 0$. Current is reversed from the situation shown in Fig. 10.1.2.

$$m\frac{\partial^2 \xi}{\partial t^2} = \frac{T}{m}\frac{\partial^2 \xi}{\partial x^2} + \frac{Ib}{m}\xi$$

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2} + \omega_c^2$$

$$\xi = \mathrm{Re}\, \hat{\xi} e^{j(\omega t - kx)}$$

$$-\omega^2 = -k^2 v_s^2 + \omega_c^2 \Rightarrow \omega^2 = k^2 v_s^2 - \omega_c^2$$

Waves and Instabilities in Stationary Media

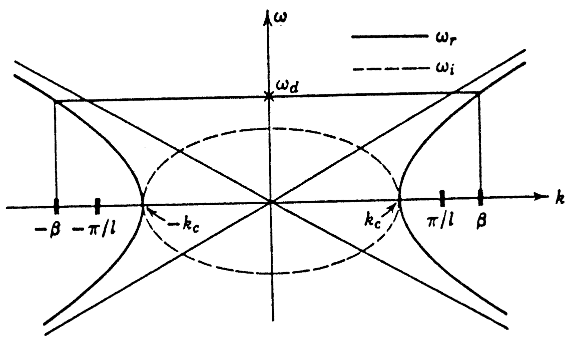


Fig. 10.1.10 Plot of the dispersion equation for physical situation shown in Fig. 10.1.2 with the current reversed, as in Fig. 10.1.9. Complex values of ω are shown for real values of k.

Take undriven spring : $\xi(-l, t) = 0, \quad \xi(0, t) = 0$

$$\xi = \mathrm{Re}\, \hat{\xi}(x) e^{j\omega t}$$

$$\hat{\xi}(x) = \xi_1 e^{-jkx} + \xi_2 e^{+jkx}$$

$$k = \frac{\sqrt{\omega^2 + \omega_c^2}}{v_s}$$

$$\hat{\xi}(x=0) = 0 = \xi_1 + \xi_2$$

$$\hat{\xi}(x=-l) = 0 = \xi_1 e^{jkl} + \xi_2 e^{-jkl} = \xi_1 (e^{jkl} - e^{-jkl})$$

$$= 2j \xi_1 \sin kl$$

$$k \cancel{l} = \frac{n\pi}{l}$$

$$\omega^2 = \left(\frac{n\pi}{l}\right)^2 - \omega_c^2, \quad \text{If } \left(\frac{n\pi}{l}\right) < \omega_c, \quad \omega = \pm j|\omega_i|$$

Negative imaginary roots are absolutely unstable $\therefore e^{|\omega_i| t}$

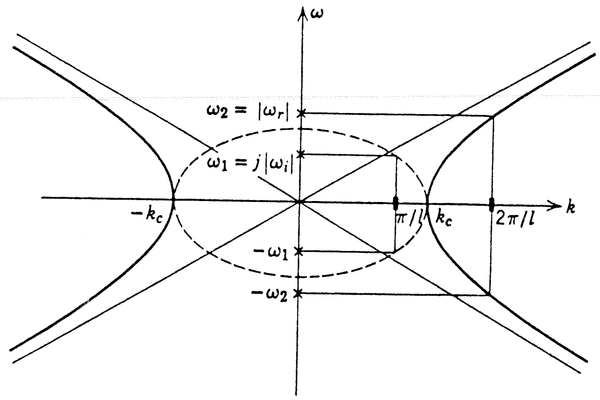


Fig. 10.1.11 The dispersion equation for the system of Fig. 10.1.2 with current as shown in Fig. 10.1.9. Complex values of ω are shown for real values of k. The allowed values of k give rise to the eigenfrequencies as shown.

VI. Electric Field Levitation of Membrane

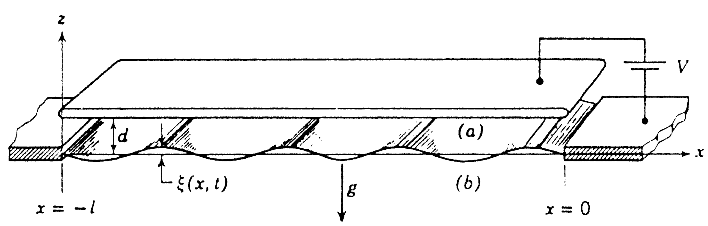


Fig. 10.1.14 Conducting elastic membrane held horizontal in a gravitational field by an electrostatic force.

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \frac{\partial^2 \xi}{\partial x^2} - \sigma_m g + T_z^e$$

$$\bar{E} = \frac{-V}{d-\xi} \bar{i}_z$$

$$T_z^e = (T_{zj}^a - T_{zj}^b) n_j = T_{zz}^a = \frac{1}{2} \epsilon_0 E_z^2 = \frac{1}{2} \epsilon_0 \left(\frac{V}{d-\xi} \right)^2$$

$$= \frac{1}{2} \frac{\epsilon_0 V^2}{d^2} \left(\frac{1}{1-(\xi/d)} \right)^2$$

$$\approx \frac{1}{2} \frac{\epsilon_0 V^2}{d^2} (1 + 2\xi/d)$$

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \frac{\partial^2 \xi}{\partial x^2} - \sigma_m g + \frac{1}{2} \frac{\epsilon_0 V^2}{d^2} + \frac{\epsilon_0 V^2}{d^3} \xi$$

Equilibrium: $\xi = 0 \Rightarrow \sigma_m g = \frac{1}{2}\frac{\epsilon_0 V^2}{d^2}$

Perturbations: $\cancel{\sigma_m}\frac{\partial^2 \xi}{\partial t^2} = \frac{S}{\sigma_m}\frac{\partial^2 \xi}{\partial x^2} + \frac{\epsilon_0 V^2 \xi}{d^3 \sigma_m}$

$v_s^2 = \frac{S}{\sigma_m}$, $\frac{\epsilon_0 V^2}{d^3 \sigma_m} = \omega_c^2$

$\xi = \operatorname{Re} \hat{\xi} e^{j(\omega t - kx)}$

$-\omega^2 = -k^2 v_s^2 + \omega_c^2$

$\xi(0,t) = \xi(-l,t) = 0 \Rightarrow k = \frac{n\pi}{l}$

$\omega^2 = \left(\frac{n\pi}{l} v_s\right)^2 - \omega_c^2$

Stable if: $\omega_c^2 = \frac{\epsilon_0 V^2}{\sigma_m d^3} < \left(\frac{n\pi}{l} v_s\right)^2 \Rightarrow \frac{\epsilon_0 V^2}{\sigma_m d^3 v_s^2} = \frac{\epsilon_0 V^2}{d^3 S} < \left(\frac{n\pi}{l}\right)$

First Unstable mode: $n = 1$

$$\underbrace{\frac{\epsilon_0 V^2}{d^3 S}} < \left(\frac{\pi}{l}\right)^2$$

$$\frac{2\sigma_m g}{dS} < \left(\frac{\pi}{l}\right)^2 \Rightarrow \sigma_m < \frac{\left(\frac{\pi}{l}\right)^2 S d}{2g}$$