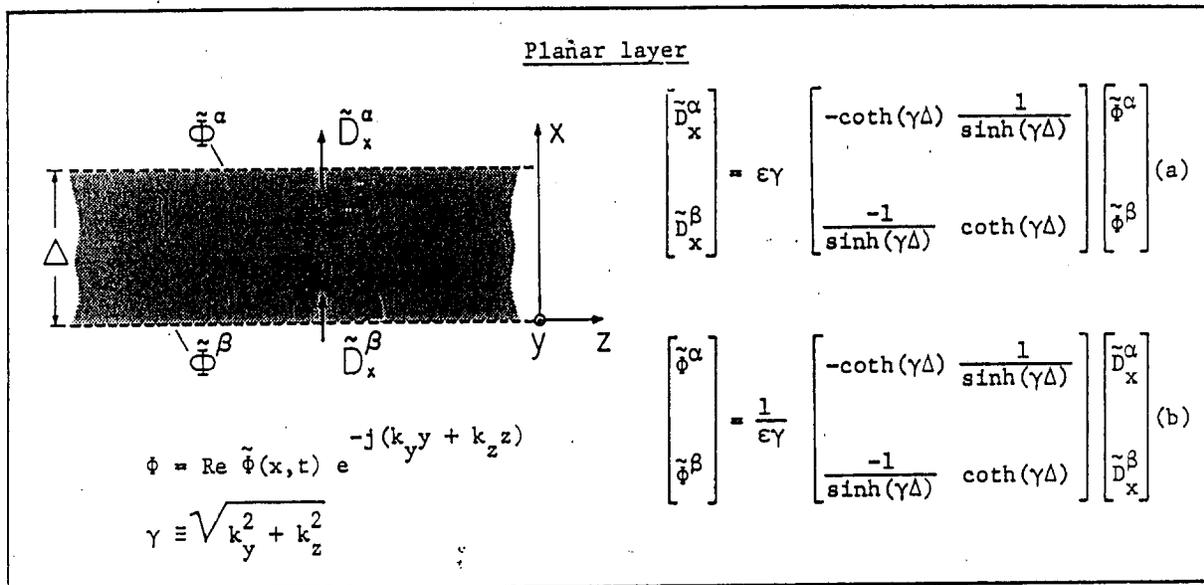


Section (2.16) Flux-Potential Relations for Laplacian Fields

I. Planar Layers

A. Electric Fields

Table 2.16.1. Flux-potential transfer relations for planar layer in terms of electric potential and normal displacement (ϕ, D_x) . To obtain magnetic relations, substitute $(\phi, D_x, \epsilon) \rightarrow (\Psi, B_x, \mu)$.



$$\nabla \cdot \bar{E} = 0$$

$$\nabla \times \bar{E} = 0 \Rightarrow \bar{E} = -\nabla \Phi$$

$$\nabla^2 \Phi = 0$$

$$\Phi(x, y, z, t) = \text{Re} \left[\hat{\Phi}(x, t) e^{-j(k_y y + k_z z)} \right]$$

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\frac{d^2 \hat{\Phi}}{dx^2} - (k_y^2 + k_z^2) \hat{\Phi} = 0 \quad ; \quad \gamma = \sqrt{k_y^2 + k_z^2}$$

$$\tilde{\Phi} = \tilde{\Phi}_1 \sinh \gamma x + \tilde{\Phi}_2 \cosh \gamma x$$

$$\tilde{\Phi}(x=\Delta) = \tilde{\Phi}^\alpha ; \tilde{\Phi}(x=0) = \tilde{\Phi}^\beta$$

$$\tilde{\Phi}(x) = \frac{\tilde{\Phi}^\alpha \sinh \gamma x - \tilde{\Phi}^\beta \sinh \gamma (x-\Delta)}{\sinh \gamma \Delta}$$

$$D_x = -\epsilon \frac{\partial \Phi}{\partial x} \Rightarrow \tilde{D}_x = \frac{-\epsilon \gamma [\tilde{\Phi}^\alpha \cosh \gamma x - \tilde{\Phi}^\beta \cosh \gamma (x-\Delta)]}{\sinh \gamma \Delta}$$

$$\tilde{D}_x^\alpha = \tilde{D}_x(x=\Delta) = \frac{-\epsilon \gamma}{\sinh \gamma \Delta} [\tilde{\Phi}^\alpha \cosh \gamma \Delta - \tilde{\Phi}^\beta]$$

$$\tilde{D}_x^\beta = \tilde{D}_x(x=0) = \frac{-\epsilon \gamma}{\sinh \gamma \Delta} [\tilde{\Phi}^\alpha - \tilde{\Phi}^\beta \cosh \gamma \Delta]$$

B. Magnetic Fields

$$\nabla \times \vec{H} = 0 \Rightarrow \vec{H} = -\nabla \Psi$$

$$\nabla \cdot \vec{H} = 0 \Rightarrow \nabla^2 \Psi = 0 = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2}$$

$$\Psi(x, y, z, t) = \text{Re} [\tilde{\Psi}(x, t) e^{-j(k_y y + k_z z)}]$$

$$\frac{d^2 \tilde{\Psi}}{dx^2} - \gamma^2 \tilde{\Psi} = 0 ; \quad \gamma = \sqrt{k_y^2 + k_z^2}$$

$$\tilde{\Psi}(x) = \frac{\tilde{\Psi}^\alpha \sinh \gamma x - \tilde{\Psi}^\beta \sinh \gamma (x-\Delta)}{\sinh \gamma \Delta}$$

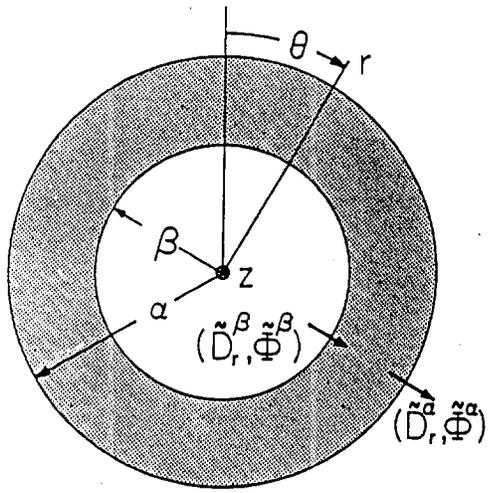
$$B_x = -\mu \frac{\partial \tilde{\Psi}}{\partial x} \Rightarrow \tilde{B}_x(x) = \frac{-\mu \gamma [\tilde{\Psi}^\alpha \cosh \gamma x - \tilde{\Psi}^\beta \cosh \gamma (x-\Delta)]}{\sinh \gamma \Delta}$$

$$\tilde{B}_x^\alpha = \tilde{B}_x(x=\Delta) = \frac{-\mu \gamma}{\sinh \gamma \Delta} [\tilde{\Psi}^\alpha \cosh \gamma \Delta - \tilde{\Psi}^\beta]$$

$$\tilde{B}_x^\beta = \tilde{B}_x(x=0) = \frac{-\mu \gamma}{\sinh \gamma \Delta} [\tilde{\Psi}^\alpha - \tilde{\Psi}^\beta \cosh \gamma \Delta]$$

II. Cylindrical Annulus

Table 2.16.2. Flux-potential relations for cylindrical annulus in terms of electric potential and normal displacement (ϕ, D_r). To obtain magnetic relations, substitute $(\phi, D_r, \epsilon) \rightarrow (\psi, B_r, \mu)$.



$$\phi = \text{Re } \tilde{\phi}(r, t) e^{-j(m\theta + kz)}$$

$$\nabla^2 \Phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Phi(r, \theta, z, t) = \text{Re} [\tilde{\Phi}(r, t) e^{-j(m\theta + kz)}]$$

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{d\tilde{\Phi}}{dr} \right) - \frac{m^2}{r^2} \tilde{\Phi} - k^2 \tilde{\Phi} = 0$$

1. $m=0, k=0$ solutions

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{d\tilde{\Phi}}{dr} \right) = 0 \Rightarrow r \frac{d\tilde{\Phi}}{dr} = C \Rightarrow \tilde{\Phi} = C \ln r + D$$

$$\tilde{\Phi}(r) = \frac{\tilde{\Phi}^\alpha \ln \frac{r}{\beta} - \tilde{\Phi}^\beta \ln \frac{r}{\alpha}}{\ln \frac{\alpha}{\beta}}$$

$$D_r = -\epsilon \frac{\partial \tilde{\Phi}}{\partial r} \Rightarrow D_r = \epsilon \frac{\tilde{\Phi}^\alpha - \tilde{\Phi}^\beta}{r \ln \frac{\alpha}{\beta}}$$

$$D_r^\alpha = \epsilon \frac{\tilde{\Phi}^\alpha - \tilde{\Phi}^\beta}{\alpha \ln \frac{\alpha}{\beta}}, \quad D_r^\beta = \epsilon \frac{\tilde{\Phi}^\alpha - \tilde{\Phi}^\beta}{\beta \ln \frac{\alpha}{\beta}}$$

$$\begin{bmatrix} D_r^\alpha \\ D_r^\beta \end{bmatrix} = \frac{-\epsilon}{\ln \frac{\alpha}{\beta}} \begin{bmatrix} \frac{1}{\alpha} & -\frac{1}{\alpha} \\ \frac{1}{\beta} & -\frac{1}{\beta} \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix}$$

2. $k=0, m \neq 0$

$$r \frac{d}{dr} \left(r \frac{d\tilde{\Phi}}{dr} \right) - m^2 \tilde{\Phi} = 0 \Rightarrow \tilde{\Phi} = A_1 r^m + A_2 r^{-m}$$

$$\tilde{\Phi} = \tilde{\Phi}^\alpha \frac{\left[\left(\frac{\beta}{r} \right)^m - \left(\frac{r}{\beta} \right)^m \right]}{\left[\left(\frac{\beta}{\alpha} \right)^m - \left(\frac{\alpha}{\beta} \right)^m \right]} + \tilde{\Phi}^\beta \frac{\left[\left(\frac{r}{\alpha} \right)^m - \left(\frac{\alpha}{r} \right)^m \right]}{\left[\left(\frac{\beta}{\alpha} \right)^m - \left(\frac{\alpha}{\beta} \right)^m \right]}$$

3. $k \neq 0, m \neq 0$

$$\tilde{\Phi} = A_1 I_m(kr) + A_2 K_m(kr) \quad [\text{Modified Bessel Functions}]$$

$$I_m(r \rightarrow \infty) \rightarrow \infty, \quad I_m(r \rightarrow 0) \rightarrow \text{finite}$$

$$K_m(r \rightarrow \infty) \rightarrow 0, \quad K_m(r \rightarrow 0) \rightarrow \infty$$

$$I_m(jkr) = j^m I_m(kr), \quad H_m(jkr) = \frac{2}{\pi} j^{-(m+1)} K_m(kr)$$

$$\hat{\Phi}(r) = \left\{ \hat{\Phi}^\alpha [H_m(jk\beta)J_m(jkr) - J_m(jk\beta)H_m(jkr)] + \hat{\Phi}^\beta [J_m(jk\alpha)H_m(jkr) - H_m(jk\alpha)J_m(jkr)] \right\}$$

$$[H_m(jk\beta)J_m(jk\alpha) - J_m(jk\beta)H_m(jk\alpha)]$$

$$\begin{bmatrix} \tilde{D}_r^\alpha \\ \tilde{D}_r^\beta \end{bmatrix} = \epsilon \begin{bmatrix} f_m(\beta, \alpha) & g_m(\alpha, \beta) \\ g_m(\beta, \alpha) & f_m(\alpha, \beta) \end{bmatrix} \begin{bmatrix} \tilde{\phi}^\alpha \\ \tilde{\phi}^\beta \end{bmatrix} \quad (a)$$

$k = 0, m = 0$

$f_0(x, y) = \frac{1}{y} \ln\left(\frac{x}{y}\right); g_0(x, y) = \frac{1}{x} \ln\left(\frac{x}{y}\right)$

$k = 0, m = 1, 2, \dots$

$$f_m(x, y) = \frac{m}{y} \frac{[\left(\frac{x}{y}\right)^m + \left(\frac{y}{x}\right)^m]}{[\left(\frac{x}{y}\right)^m - \left(\frac{y}{x}\right)^m]}$$

$$g_m(x, y) = \frac{2m}{x} \frac{1}{[\left(\frac{x}{y}\right)^m - \left(\frac{y}{x}\right)^m]}$$

$k \neq 0, m = 0, 1, 2, \dots^*$

$$f_m(x, y) = \frac{jk[H_m(jkx)J'_m(jky) - J'_m(jkx)H_m(jky)]}{[J'_m(jkx)H'_m(jky) - J'_m(jky)H'_m(jkx)]}$$

$$g_m(x, y) = \frac{-2j}{\pi x [J'_m(jkx)H'_m(jky) - J'_m(jky)H'_m(jkx)]}$$

$$f_m(x, y) = \frac{k[K'_m(kx)I'_m(ky) - I'_m(kx)K'_m(ky)]}{[I'_m(kx)K'_m(ky) - I'_m(ky)K'_m(kx)]}$$

$$g_m(x, y) = \frac{1}{x [I'_m(kx)K'_m(ky) - I'_m(ky)K'_m(kx)]}$$

$$\begin{bmatrix} \tilde{\phi}^\alpha \\ \tilde{\phi}^\beta \end{bmatrix} = \frac{1}{\epsilon} \begin{bmatrix} F_m(\beta, \alpha) & G_m(\alpha, \beta) \\ G_m(\beta, \alpha) & F_m(\alpha, \beta) \end{bmatrix} \begin{bmatrix} \tilde{D}_r^\alpha \\ \tilde{D}_r^\beta \end{bmatrix} \quad (b)$$

$k = 0, m = 0$

No inverse

$k = 0, m = 1, 2, \dots$

$$F_m(x, y) = \frac{y}{m} \frac{[\left(\frac{x}{y}\right)^m + \left(\frac{y}{x}\right)^m]}{[\left(\frac{x}{y}\right)^m - \left(\frac{y}{x}\right)^m]}$$

$$G_m(x, y) = \frac{2y}{m} \frac{1}{[\left(\frac{x}{y}\right)^m - \left(\frac{y}{x}\right)^m]}$$

$k \neq 0, m = 0, 1, 2, \dots^*$

$$F_m(x, y) = \frac{1}{jk} \frac{[J'_m(jkx)H'_m(jky) - H'_m(jkx)J'_m(jky)]}{[J'_m(jky)H'_m(jkx) - J'_m(jkx)H'_m(jky)]}$$

$$G_m(x, y) = \frac{-2}{j\pi k(kx) [J'_m(jky)H'_m(jkx) - J'_m(jkx)H'_m(jky)]}$$

$$F_m(x, y) = \frac{1}{k} \frac{[I'_m(kx)K'_m(ky) - K'_m(kx)I'_m(ky)]}{[I'_m(ky)K'_m(kx) - I'_m(kx)K'_m(ky)]}$$

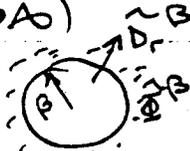
$$G_m(x, y) = \frac{1}{k(kx) [I'_m(ky)K'_m(kx) - I'_m(kx)K'_m(ky)]}$$

Drop ($\beta \rightarrow 0$)



$\tilde{D}_r^\alpha = \epsilon f_m(0, \alpha) \tilde{\Phi}^\alpha; f_m(0, \alpha) = -\frac{k I'_m(k\alpha)}{I_m(k\alpha)}$

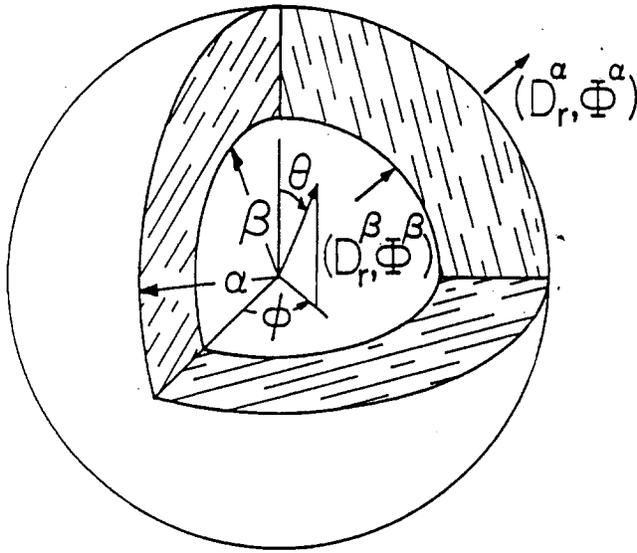
Bubble ($\alpha \rightarrow 0$)



$\tilde{D}_r^\beta = \epsilon f_m(\infty, \beta) \tilde{\Phi}^\beta; f_m(\infty, \beta) = \frac{-k K'_m(k\beta)}{K_m(k\beta)}$

III. Spherical Shell

Table 2.16.3. Flux-potential transfer relations for spherical shell in terms of electric potential and normal displacement (Φ, D_r). To obtain magnetic relations, substitute (Φ, D_r, ϵ) \rightarrow (Ψ, B_r, μ).



$$\phi = \text{Re } \tilde{\phi}(r, t) P_n^m(\cos \theta) e^{-jm\phi}$$

$$P_n^m = (1-x^2)^{m/2} \frac{d^m P_n}{dx^m}$$

$$P_0 = 1, P_1 = x, P_2 = \frac{1}{2}(3x^2 - 1)$$

$$P_3 = \frac{1}{2}(5x^3 - 3x)$$

$$P_4 = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

| m | P_0^m | P_1^m | $P_1^m \cos m\phi$ | P_2^m | $P_2^m \cos m\phi$ | P_3^m | $P_3^m \cos m\phi$ | | | | | | | | | | | | | | | | | | | | |
|---|---------|---------------|---|-------------------|--|------------------------------------|---|--|---|---|--|--|---|---|---|---|--|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | $\cos \theta$ | <table border="1"><tr><td>+</td></tr><tr><td>-</td></tr></table> | + | - | $\frac{1}{2}(3 \cos^2 \theta - 1)$ | <table border="1"><tr><td>+</td></tr><tr><td>-</td></tr><tr><td>+</td></tr></table> | + | - | + | $\frac{1}{2}(5 \cos^3 \theta - 3 \cos \theta)$ | <table border="1"><tr><td>+</td></tr><tr><td>-</td></tr><tr><td>+</td></tr><tr><td>-</td></tr></table> | + | - | + | - | | | | | | | | | | | |
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| - | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 1 | 0 | $\sin \theta$ | <table border="1"><tr><td>+</td><td>-</td><td>+</td></tr></table> | + | - | + | $3 \sin \theta \cos \theta$ | <table border="1"><tr><td>+</td><td>-</td><td>+</td></tr><tr><td>-</td><td>+</td><td>-</td></tr></table> | + | - | + | - | + | - | $\frac{3}{2} \sin \theta (5 \cos^2 \theta - 1)$ | <table border="1"><tr><td>+</td><td>-</td><td>+</td></tr><tr><td>-</td><td>+</td><td>-</td></tr><tr><td>+</td><td>-</td><td>+</td></tr></table> | + | - | + | - | + | - | + | - | + | | |
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| 2 | 0 | 0 | | $3 \sin^2 \theta$ | <table border="1"><tr><td>+</td><td>-</td><td>+</td><td>-</td><td>+</td></tr><tr><td>-</td><td>+</td><td>-</td><td>+</td><td>-</td></tr></table> | + | - | + | - | + | - | + | - | + | - | $15 \sin^2 \theta \cos \theta$ | <table border="1"><tr><td>+</td><td>-</td><td>+</td><td>-</td><td>+</td></tr><tr><td>-</td><td>+</td><td>-</td><td>+</td><td>-</td></tr></table> | + | - | + | - | + | - | + | - | + | - |
| + | - | + | - | + | | | | | | | | | | | | | | | | | | | | | | | |
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| 3 | 0 | 0 | | 0 | | $15 \sin^3 \theta$ | <table border="1"><tr><td>+</td><td>-</td><td>+</td><td>-</td><td>+</td><td>-</td><td>+</td></tr></table> | + | - | + | - | + | - | + | | | | | | | | | | | | | |
| + | - | + | - | + | - | + | | | | | | | | | | | | | | | | | | | | | |

$$\begin{bmatrix} \tilde{D}_r^\alpha \\ \tilde{D}_r^\beta \end{bmatrix} = \epsilon \begin{bmatrix} f_n(\beta, \alpha) & g_n(\alpha, \beta) \\ g_n(\beta, \alpha) & f_n(\alpha, \beta) \end{bmatrix} \begin{bmatrix} \tilde{\phi}^\alpha \\ \tilde{\phi}^\beta \end{bmatrix} \quad (a)$$

$$f_n(x, y) = \frac{[n(\frac{y}{x})^n + (n+1)(\frac{x}{y})^{n+1}]}{[x(\frac{x}{y})^n - y(\frac{y}{x})^n]}$$

$$g_n(x, y) = \frac{(2n+1)}{x^2 [\frac{1}{y}(\frac{x}{y})^n - \frac{1}{x}(\frac{y}{x})^n]}$$

$$\begin{bmatrix} \tilde{\phi}^\alpha \\ \tilde{\phi}^\beta \end{bmatrix} = \frac{1}{\epsilon} \begin{bmatrix} F_n(\beta, \alpha) & G_n(\alpha, \beta) \\ G_n(\beta, \alpha) & F_n(\alpha, \beta) \end{bmatrix} \begin{bmatrix} \tilde{D}_r^\alpha \\ \tilde{D}_r^\beta \end{bmatrix} \quad (b)$$

$$F_n(x, y) = \frac{y}{x} \frac{[\frac{1}{n}(\frac{y}{x})^n + \frac{1}{n+1}(\frac{x}{y})^{n+1}]}{[\frac{1}{y}(\frac{x}{y})^n - \frac{1}{x}(\frac{y}{x})^n]}$$

$$G_n(x, y) = \frac{y}{x} \frac{(2n+1)}{n(n+1)} \frac{1}{[\frac{1}{y}(\frac{x}{y})^n - \frac{1}{x}(\frac{y}{x})^n]}$$

$$\Phi(r, \theta, \phi) = \text{Re}[\tilde{\Phi}(r) \Theta(\theta) e^{-jm\phi}]$$

$$\nabla^2 \Phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

$$\frac{1}{\sin \theta \Theta} \frac{d}{d\theta} \left[\sin \theta \frac{d\Theta}{d\theta} \right] - \frac{m^2}{\sin^2 \theta} = -k^2$$

$$\frac{1}{\tilde{\Phi}} \frac{d}{dr} \left(r^2 \frac{d\tilde{\Phi}}{dr} \right) = k^2$$

$$u = \cos \theta, \quad \sqrt{1-u^2} = \sin \theta$$

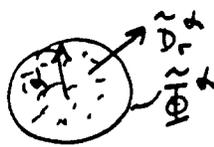
$$(1-u^2) \frac{d^2 \Theta}{du^2} - 2u \frac{d\Theta}{du} + \left(k^2 - \frac{m^2}{1-u^2} \right) \Theta = 0$$

$$k^2 = n(n+1), \quad n \text{ an integer}$$

$$\Theta = P_n^m(u)$$

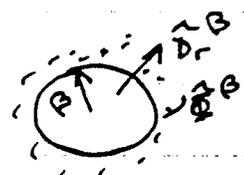
$$\tilde{\Phi} = \frac{\tilde{\Phi}^{\alpha} \left[\left(\frac{r}{\beta} \right)^n - \left(\frac{\beta}{r} \right)^{n+1} \right]}{\left[\left(\frac{\alpha}{\beta} \right)^n - \left(\frac{\beta}{\alpha} \right)^{n+1} \right]} + \frac{\tilde{\Phi}^{\beta} \left[\left(\frac{r}{\alpha} \right)^n - \left(\frac{\alpha}{r} \right)^{n+1} \right]}{\left[\left(\frac{\beta}{\alpha} \right)^n - \left(\frac{\alpha}{\beta} \right)^{n+1} \right]}$$

Drop
 $\beta \rightarrow 0$



$$D_r^{\alpha} = -\frac{\epsilon n}{\alpha} \tilde{\Phi}^{\alpha}$$

Bubble
 $\alpha \rightarrow \infty$



$$D_r^{\beta} = +\frac{\epsilon(n+1)}{\beta} \tilde{\Phi}^{\beta}$$