18.385 Problem. Rodolfo R. Rosales.[1] (December 5, 2002).

• Balancing a broom.

Statement:

Consider the problem of balancing a broom upright, by placing it on a surface that moves up and down in some prescribed manner. Specifically:

Assume a rough flat horizontal surface, which oscillates up and down following some prescribed law (that is, at any time the surface can be described by the equation $y = Y(t)$, where y is the vertical coordinate, and Y is some oscillatory function). On this surface we place a broom, in upright position, with the sweeping side pointing up.[2] **Question: Can we prescribe Y in such a way that the broom remains upright — i.e.: the position is stable?**

In order to answer the question, consider the following idealized situation:

A) Replace the broom by a mass m, placed at the upper end of a (massless) rigid rod of length L. Let the displacement of the rod from the vertical position be given by the angle θ, with $\theta = 0$ corresponding to the rod standing vertical, and the mass on the upper end.

B) The bottom of the rod is attached to a hinge that allows it to rotate in a plane. Thus the motion of the rod is restricted to occur on a plane.

C) Assume that friction can be neglected.

D) The hinge to which the rod is attached oscillates up and down, with position $x = 0$ and $y = Y(t)$ — x is the horizontal coordinate on the plane where the rod moves. The mass is then at $x = L\,\sin(\theta)$ and $y = Y + L\,\cos(\theta)$ — we measure angles clockwise from the top.

Now, do the following:

(1)

Use Newton's laws to derive the equation of motion for the mass m. You should obtain a second order ODE for the angle θ, with coefficients depending on the parameters g (the acceleration of gravity) and the length of the rod L — in addition to the forcing function $Y = Y(t)$.

Hint: Only two forces act on the mass m, namely: gravity and a force $F = F(t)$ along the rod. The force F has just the right strength to keep the (rigid) rod at constant length L — this is enough to determine F, though you do not need to calculate it.

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[2] Because the surface is rough, the contact point of the broom with the surface will not move relative to the surface.

(2)

You should notice that adding a constant velocity to the hinge motion (that is: $Y \to Y + v\,t$, where v is a constant) does not change the equation of motion. Why should this be so? What physical principle is involved?

(3)

Write down the (linearized) equations for small perturbations of the equilibrium position ($\theta = 0$) that we wish stabilized. **Stability occurs if and only if $Y = Y(t)$ can be selected so that the solutions of this linear equation do not grow in time** — strictly speaking we should also consider the possible effects of nonlinearity, but we will ignore this issue here.

(4)

You should notice that it is possible to stabilize $\theta = 0$ by taking $Y = -a\,t^2$, where $a > 0$ is a constant acceleration. How large does a have to be for this to happen? Give a justification of this result based on physical reasoning, without involving any equations (this is something you should have been able to predict before you wrote a single equation).

(5)

Of course, the "solution" found in **(4)** is not very satisfactory, since Y grows without bound in it. Consider now oscillatory forcing functions of the form:

$$Y = \ell \cos(\omega\, t)\,, \tag{1}$$

where $\ell > 0$ and $\omega > 0$ are constants (with dimensions of length and time^{-1}, respectively).

The objective is to find conditions on (ℓ, ω) that guarantee stability. (2)

The next steps will lead you through this process, but first: **Nondimensionalize the (linearized) stability equation.** In doing so it is convenient to use the time scale provided by the forcing to nondimensionalize time — i.e.: let the nondimensional time be $\boxed{\tau = \omega\, t\,.}$

This step should lead you to an equation describing the evolution of the angle θ (valid for small angles), involving two nondimensional parameters. One of them, $\boxed{\epsilon = \ell/L,}$ measures the amplitude of the oscillations in terms of the length of the rod. The other measures the time scale of the forcing (as given by $1/\omega$) in terms of the time scale of the gravitational instability — a function of g and L. **Call this second parameter** $\boxed{\mu}$ — note that in the equation only μ^2 appears, not μ itself.

(6)

Find the stability range for μ as a function of ϵ, for the values $0 < \epsilon \leq 0.6$ — it is enough to pick a few values of ϵ, say $\epsilon = 0.1$, 0.2, 0.3, 0.4, 0.5, 0.6, and then to compute the stability range for each of them.

Note/hint: This step will require not just analysis, but some numerical computation. So as not to be forced to explore all possible values of μ when looking for the stability ranges (numerically an impossible task), you should notice that the analysis for $\epsilon = 0$ can be done exactly — and should provide you with a good hint as to where to look.

(7)

Write the period $p = \dfrac{2\pi}{\omega}$ of the forcing, in terms of the nondimensional parameter μ, and the parameters g and L. The results of **part (6)** should provide you with the period ranges (for a given oscillation amplitude) where stability occurs. Use this information to provide a rough explanation of why it is relatively easy to balance a broom on the palm of your hand (using the strategy outlined in this problem — try it), and why you will not be able to balance a pencil.

(8)

For $0 \leq \epsilon \ll 1$ and $0 \leq \mu \ll 1$ you should be able to obtain analytical approximations for the stable ranges. Do so, and compare your results with those of **part (6)**.

Hint: Floquet theory provides a function (the Floquet Trace $\alpha = \alpha(\mu, \epsilon)$) that characterizes linearized stability — stability if and only if $|\alpha| \leq 1$. Compute this function for μ and ϵ small.

THE END.