

• **Coastline Fractal.**

Statement: _____

In this problem we construct a fractal that is a *very idealized* caricature of what a coastline looks like. The construction proceeds by iteration of a basic process, which we describe next.

We start with a simple curve, Γ_0 , and apply to it a simple process, that yields a new curve Γ_1 . This new curve is made up of several parts, each of which is a scaled down copy of Γ_0 . The same simple process is then applied to each of these parts, yielding Γ_2 . Then we iterate, to obtain in this fashion a series of curves Γ_n , for $n = 0, 1, 2, 3 \dots$. The fractal is then the limit of this process: $\Gamma = \lim_{n \rightarrow \infty} \Gamma_n$ — provided the limit exists.

For the “coastline fractal” we **start by picking an angle $0 < \theta < \pi$, and a length $R_0 > 0$** . Then the first curve is:

$$\Gamma_0 = \text{Circular arc of radius } R_0, \text{ subtending an angle } \theta. \tag{1}$$

Next **divide Γ_0 into three equal sub-arcs, each subtending an angle $\theta/3$, and replace each of these pieces by a properly scaled version of Γ_0 . This yields Γ_1** . The process is then repeated on each of the three pieces making up Γ_1 , so as to obtain Γ_2 , and so on ad infinitum. The first two steps in this construction are illustrated in figure 1.

The issue of whether or not the limit $\lim_{n \rightarrow \infty} \Gamma_n$ exists is easy to settle. Consider an arbitrary radial line within the circle sector associated with Γ_0 , and the intersection of this line with Γ_n . It should be clear that this intersection is unique. Let d_n be the distance of this intersection from the origin of the radial line. Then $\{d_n\}$ is an increasing, bounded sequence — so it has a limit. This limit defines a point along the radial line. The set of all these points is the fractal Γ .

Now do the following:

(1) _____

For each $n = 0, 1, 2, 3 \dots$, calculate the length ℓ_n of the curve Γ_n . What is the “length” of Γ ?

(2) _____

Calculate the fractal dimension (self-similar or box) of Γ .

¹MIT, Department of Mathematics, room 2-337, Cambridge, MA 02139.

Coastline Fractal construction

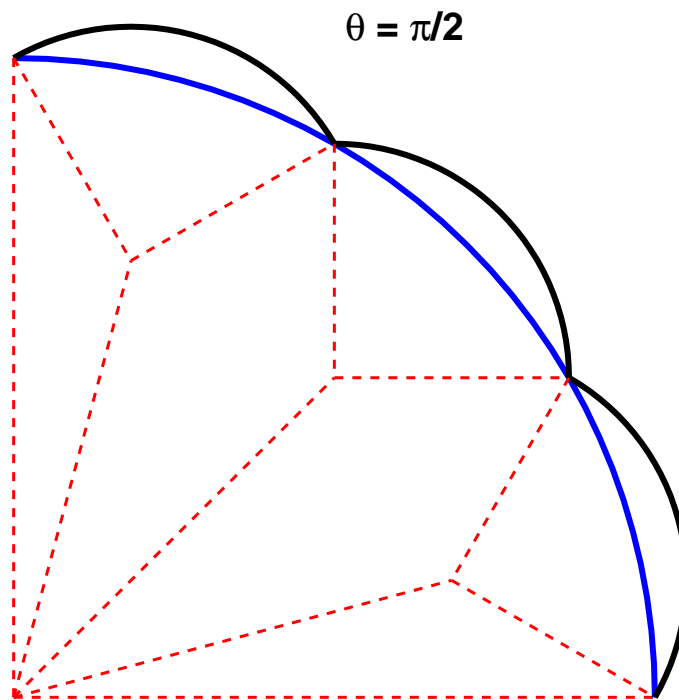


Figure 1: This figure illustrates the first two steps in the construction of the coastline fractal. That is, the curve Γ_0 (solid blue) and the curve Γ_1 (solid black). The dashed red lines indicate various radial lines useful in the construction.

Hint:

The first thing you will need to calculate is the “scaling” factor between Γ_0 and each of the three parts that make up Γ_1 . With this scaling factor $0 < S_c = S_c(\theta) < 1$, everything else follows.

Notes:

Real coastlines are not this simple, of course. At the very least the number of parts into which each sector is divided should not be a constant (3 here), nor should the parts be equal in size, nor should they all subtend the same angle θ . But further: the sectors need not be exactly circular — though, this is probably not a terrible approximation.

THE END.