• Variable Length Pendulum.

Statement:

Consider a pendulum (in a plane), whose arm length $L > 0$ changes in time (i.e.: $L = L(t)$). To make matters more precise:

(a) Let the **hinge** for the pendulum be at origin in the plane: $x = y = 0$.

(b) Let the mass M for the pendulum be at $x = L \sin\theta$ and $y = -L\cos\theta$, where θ is the angle measured (counter-clockwise) from the down-rest position of the pendulum.

(c) Let g be the acceleration of gravity, and assume that frictional forces can be neglected.

(d) Assume that the mass of the pendulum arm can be neglected.

Now **do the following**

A Using Newton's laws, **derive the equations for the pendulum.**

Hint: There are two forces acting on the mass M:

- The force of gravity (of magnitude Mg, pointing downwards).
- A force (of magnitude $F = F(t)$) acting along the arm of the pendulum.

The force F is not known a-priori, but it must have the exact magnitude to keep the distance from the mass to the pendulum hinge at the length $L = L(t)$. This is enough to determine this force.

B Consider the following situation: you have a mass tied up at the end of a string. The string goes through a small hole somewhere — say, the hole at the end of a fishing rod. Now, pull steadily on the string, shortening the string length from the hole to the mass (do not move the hole while this happens). You should observe that, quite often, you end up with the mass going around the "fishing rod", wrapping the string there. **Explain this behavior using the equations derived in A.** *(Note that real life is neither 2-D, nor frictionless: the equations tend to over-predict what happens).*

[1]MIT, Department of Mathematics, room 2-337, Cambridge, MA 02139.

C Study the stability of the $\theta = 0$ equilibrium position for the pendulum. Linearize the equations near this solution, and obtain an equation of the form

$$\frac{d^2\varphi}{dt^2} + V(t)\varphi = 0\,, \tag{1}$$

where $\varphi = L\theta$ and $V = V(t)$ is some "potential" obtained from L and its derivatives.

D Argue that, if L is sinusoidal, with small amplitude variations, then one can take

$$V = \Omega^2(1 + \epsilon\cos(\omega t))\,, \tag{2}$$

in (1), where ϵ is small. Then (1) becomes Mathieu's equation.

E Take $\Omega = 1$ in Mathieu's equation and use Floquet theory to study the stability of the pendulum. That is, calculate (numerically) the trace of the Floquet matrix as a function of ϵ and ω (say, for $0 \le \epsilon \le 0.3$ and $0.5 \le \omega \le 5$). Note that the period to use in the calculation is $2\pi/\omega$ — i.e.: the period of $V = V(t)$ — and that instability corresponds to $\alpha = \mathsf{trace}/2$ having magnitude bigger than one.

Alternatively: you can take $\omega = 1$, and then vary Ω and ϵ.

Answers:

Answer to Part A: Derivation of the equations.

Using Newton's law, we can write — for the position of the mass M — the equations

$$\left.\begin{array}{rcl} M\,\ddot{x} &=& -F\,\sin(\theta)\,, \\ M\,\ddot{y} &=& +F\,\cos(\theta) - M\,g\,, \end{array}\right\} \tag{3}$$

where $F = F(t)$ is the (unknown at this stage) force along the pendulum arm (we use the convention that $F > 0$ corresponds to tension on the pendulum arm). We also have that:

$$\left.\begin{array}{rcl} x &=& +L\,\sin(\theta)\,, \\ y &=& -L\,\cos(\theta)\,, \end{array}\right\} \tag{4}$$

where $L = L(t)$ is the (given) variable length of the pendulum. At this stage it is convenient to **introduce complex notation** (since it simplifies the algebra considerably), with $\boxed{z = x + i\,y.}$ Then equations (3) and (4) take the form:

$$M\,\ddot{z} = i\,F\,e^{i\,\theta} - i\,M\,g\,, \quad \text{with} \quad z = -i\,L\,e^{i\,\theta}\,. \tag{5}$$

From the second equation in (5) we obtain: $\boxed{\ddot{z} = (\ddot{\theta}\, L + 2\, \dot{\theta}\, \dot{L} - i\, \ddot{L} + i\, (\dot{\theta})^2\, L)\, e^{i\,\theta}}$ (intermediate step: $\dot{z} = (\dot{\theta}\, L - i\, \dot{L})\, e^{i\,\theta}$). Thus:

$$M\, (\ddot{\theta}\, L + 2\, \dot{\theta}\, \dot{L}) + i\, M\, ((\dot{\theta})^2\, L - \ddot{L}) = i\, F - i\, M\, g\, e^{-i\,\theta}\,. \tag{6}$$

From the imaginary part of this equation we obtain a formula for F, namely:

$$F = M\, g\, \cos(\theta) + M\, ((\dot{\theta})^2\, L - \ddot{L})\,. \tag{7}$$

The real part gives the equation of motion for θ, namely:

$$\ddot{\theta}\, L + 2\, \dot{\theta}\, \dot{L} + g\, \sin(\theta) = 0\,. \tag{8}$$

Alternatively, in terms of $\boxed{\varphi = L\,\theta,}$ this last equation can be written in the form

$$\ddot{\varphi} - \frac{\ddot{L}}{L}\,\varphi + g\, \sin(\frac{\varphi}{L}) = 0\,. \tag{9}$$

Remark 1 Derivation of the equations using Lagrangian Mechanics.
Equation (8) is straightforward to derive using Lagrangians, since:

$$\begin{aligned}
\mathcal{L} &= \text{Lagrangian} \\
&= \text{Kinetic Energy} - \text{Potential Energy} \\
&= \frac{M}{2}(\dot{x}^2 + \dot{y}^2) - g\, M\, y \\
&= \frac{M}{2}(\dot{\theta}^2\, \dot{L}^2 + \dot{L}^2) + g\, M\, L\, \cos(\theta)\,.
\end{aligned} \tag{10}$$

Then the Euler-Lagrange equation for $\mathcal{L}$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0\,, \tag{11}$$

is precisely equation (8).

Answer to Part B: Steady pulling on the pendulum mass.

In this case $\boxed{L = L_0\,(1 - \omega\, t),}$ where $L_0 > 0$ and $\omega > 0$ are constants. Then equation (9) becomes

$$\ddot{\varphi} + g\, \sin(\frac{\varphi}{L}) = 0\,. \tag{12}$$

Let us write this equation in nondimensional form, with $\phi = \dfrac{\varphi}{L_0}$ and $\tau = \omega\, t$. Then

$$\frac{d^2\,\phi}{d\tau^2} + \gamma \sin(\frac{\phi}{1-\tau}) = 0\,, \quad \text{where} \quad \gamma = \frac{g}{L_0\,\omega^2} > 0 \tag{13}$$

is a nondimensional parameter. Near $\tau = 1$, the general solution to this equation behaves like

$$\phi = \phi_0 + \phi_1\,(1-\tau) + \gamma \int_\tau^1 (\tau - s)\, \sin\left(\frac{\phi_0}{1-s} + \phi_1\right)\, ds + \ldots \tag{14}$$

where ϕ_0 and ϕ_1 are constants. Since (generally) $\phi_0 \neq 0$, it follows that

$$\theta = \frac{\varphi}{L} = \frac{L_0\,\phi}{L} \approx \frac{\phi_0}{1-\omega\, t} \quad \text{as} \quad t \to \frac{1}{\omega}\,. \tag{15}$$

Thus θ grows unboundedly as the string is pulled. Of course: the mathematical model breaks down way before $\theta = \infty$ can occur, but it does give an explanation for the observed behavior.

Answer to Part C: Linearized stability equations.

Near equilibrium, both θ and $\varphi = L\,\theta$ are small.[2] Using (9) and linearizing, we obtain:

$$\ddot{\varphi} + V(t)\,\varphi = 0\,, \quad \text{where} \quad V = \frac{g - \ddot{L}}{L}\,. \tag{16}$$

Answer to Part D: Mathieu's equation.

We now take $L = L(t)$ sinusoidal, of the form $\boxed{L = L_0\,(1 + \delta\, \cos(\omega\, t)),}$ where $L_0 > 0$, $\omega > 0$, and δ are constants, with δ small. Then

$$V = \frac{g - \ddot{L}}{L} = \frac{1}{1 + \delta\, \cos(\omega\, t)}\, \left(\Omega^2 + \delta\, \omega^2\, \cos(\omega\, t)\right) \approx \Omega^2\,(1 + \epsilon\, \cos(\omega\, t)) \tag{17}$$

where $\boxed{\Omega = \sqrt{g/L_0},}$ $\boxed{\epsilon = \delta\, \omega^2/\Omega^2,}$ and we have used the fact that δ is small.

Let us now nondimensionalize the equations, using $\boxed{\phi = \varphi/L_0}$ and $\boxed{\tau = \omega t.}$ Then

$$\frac{d^2\,\phi}{d\,\tau^2} + \left(\mu^2 + \delta\, \cos(\tau)\right)\, \phi = 0\,, \tag{18}$$

where $\boxed{\mu = \Omega/\omega}$ **is the ratio of the angular frequencies (pendulum to forcing).**

[2] Assume that L is oscillatory and stays away from zero. Thus the singular behavior studied in part (B) is avoided.

Answer to Part E: Floquet analysis and stability.

The stability question (are the solutions of equation (18) bounded, or do they grow?) can be decided using **Floquet Theory.** For this purpose first we introduce the **Floquet Matrix**, defined by:

$$\mathcal{F}_{\mathcal{M}} = \mathcal{F}_{\mathcal{M}}(\mu, \delta) = \begin{bmatrix} \phi_1(\tau = 2\pi) & \phi_2(\tau = 2\pi) \\ \dfrac{d\,\phi_1}{d\tau}(\tau = 2\pi) & \dfrac{d\,\phi_2}{d\tau}(\tau = 2\pi) \end{bmatrix}, \tag{19}$$

where $\phi_1$and ϕ_2 are the solutions of (18) defined by the initial conditions (at $\tau = 0$)

$$\phi_1 = 1\,, \quad \frac{d\,\phi_1}{d\tau} = 0\,, \qquad \text{and} \qquad \phi_2 = 0\,, \quad \frac{d\,\phi_2}{d\tau} = 1\,, \tag{20}$$

and 2π is the period of the coefficients in equation (18). The **Floquet Trace** is then given by

$$\mathcal{F}_{\mathcal{T}} = \mathcal{F}_{\mathcal{T}}(\mu, \delta) = \frac{1}{2}\,\mathsf{Trace}(\mathcal{F}_{\mathcal{M}}) = \frac{1}{2}\left(\phi_1(\tau = 2\pi) + \frac{d\,\phi_2}{d\tau}(\tau = 2\pi)\right). \tag{21}$$

The conditions for stability/instability are then

$$|\mathcal{F}_{\mathcal{T}}| \leq 1 \;\; \text{(stability)} \qquad \text{and} \qquad |\mathcal{F}_{\mathcal{T}}| > 1 \;\; \text{(instability)}\,. \tag{22}$$

One of the questions we would like to answer is: **can the pendulum be de-stabilized by selecting the frequency and amplitude of the forcing appropriately?** Of course, in general $\mathcal{F}_{\mathcal{T}}$ can only be computed numerically. However, we note that for $\delta = 0$ an analytic solution is possible (since then equation (18) is just the linear harmonic oscillator). In this case:

$$\mathcal{F}_{\mathcal{T}}(\mu, 0) = \cos(2\pi\mu)\,, \tag{23}$$

so $\mathcal{F}_{\mathcal{T}}(n/2, 0) = (-1)^n$ for n a natural number. Thus, **for δ small, we should explore near $\mu = n/2$ to find ranges where the pendulum is destabilized by the forcing** ($\mathcal{F}_{\mathcal{T}}$ is a continuous function of its arguments). Since $\mu = n/2$ yields $\omega = 2\,\Omega/n$, the unstable parameter values occur in situations where the forcing frequency is a subharmonic of twice the the unperturbed pendulum frequency. *Why this is so can be easily understood in terms of resonances.* **For δ small:**

- At leading order (0-th), the solutions to equation (18) is a sinusoidal of angular frequency μ.
- At 1-st order, the term $\cos(\tau)\,\phi$ in the equation creates the frequencies $\mu - 1$ and $\mu + 1$.
- At 2-nd order, the frequencies $\mu + n$, with $-2 \leq n \leq 2$, appear.

- In general, at m-th order, the frequencies $\mu + n$, with $-m \leq n \leq m$, appear.
- A **resonance will occur** if $\mu + n = \pm\mu$, for some n. That is, if $\boxed{\mu = n/2.}$
- The larger n is, the further up the expansion the resonance occurs. Thus, the instabilities that occur for larger values of n should be weaker. The figures below confirm this expectation: both the ranges where instability occurs, and the deviations there above absolute value one of $\mathcal{F}_\mathcal{T}$, decrease very fast as n grows. *Finally: note that a large value of μ corresponds to very slow forcing. It is natural to expect instabilities in this regime to be very hard to produce!*

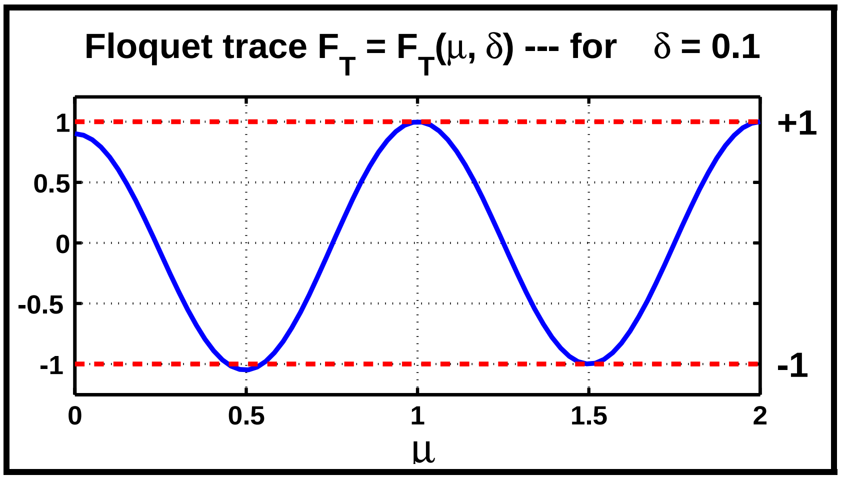

Figure 1:
Floquet Trace $\mathcal{F}_\mathcal{T}$, for $\delta = 0.1$ **and** $0 \leq \mu \leq 2$.

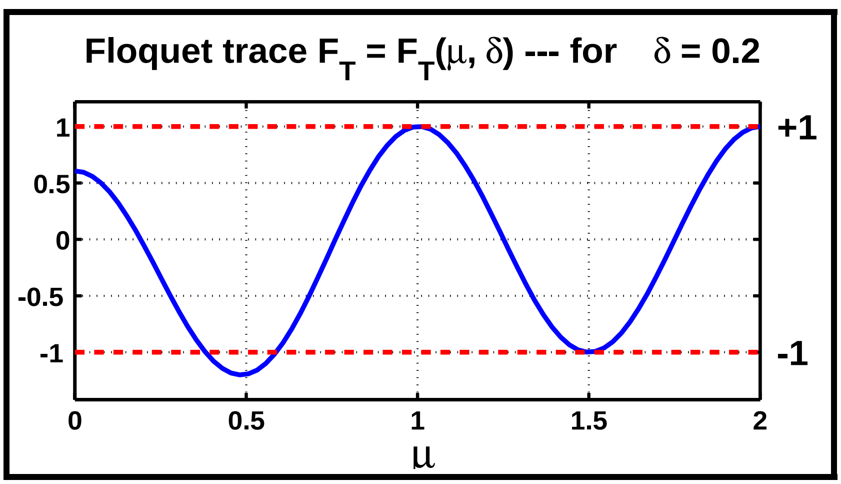

Figure 2:
Floquet Trace $\mathcal{F}_\mathcal{T}$, for $\delta = 0.2$ **and** $0 \leq \mu \leq 2$.

Description of the Figures:

The figures in this problem illustrate the behavior of the Floquet Trace $\mathcal{F}_\mathcal{T}(\mu, \delta)$, as a function of μ, for a sequence of increasingly larger values of (small) δ. We note how windows of instability arise

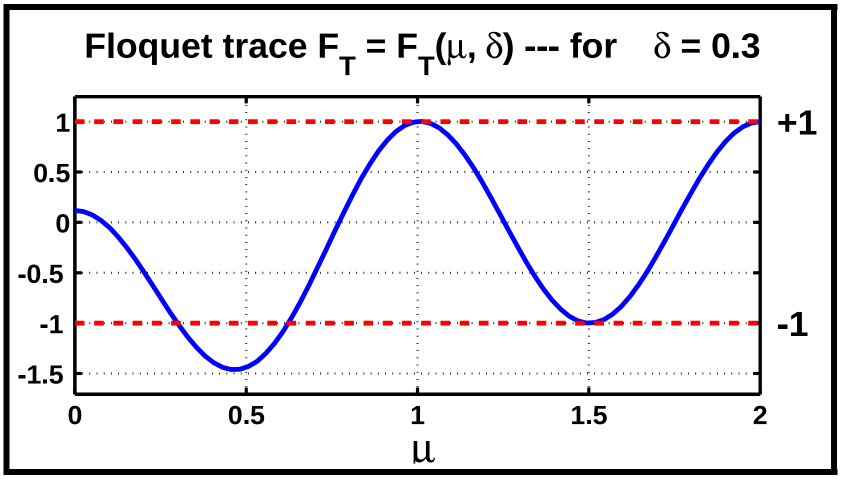

Figure 3:

Floquet Trace $\mathcal{F}_T$, for $\delta = 0.3$ and $0 \leq \mu \leq 2$.

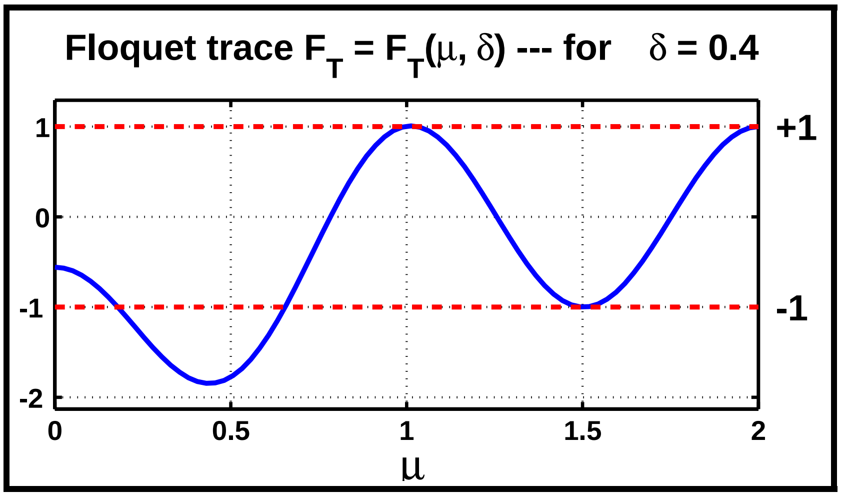

Figure 4:

Floquet Trace $\mathcal{F}_T$, for $\delta = 0.4$ and $0 \leq \mu \leq 2$.

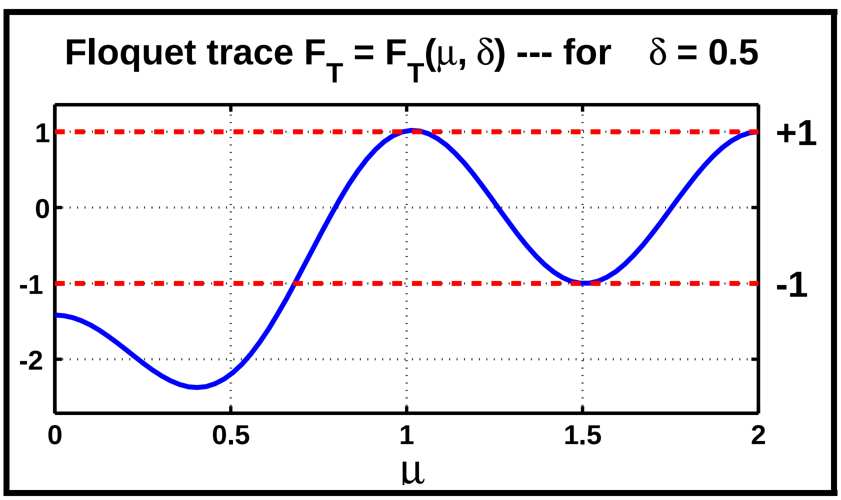

Figure 5:

Floquet Trace $\mathcal{F}_T$, for $\delta = 0.5$ and $0 \leq \mu \leq 2$.

near each of the critical values of μ (i.e.: $\mu = n/2$), and grow in width as δ grows. We also note that, for a given δ, the windows widths decrease very fast as n gets larger.

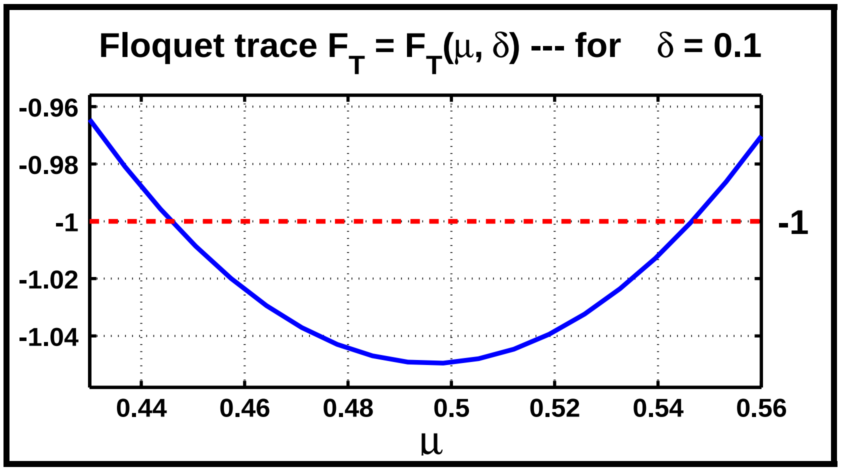

Figure 6:

Floquet Trace $\mathcal{F}_{\mathcal{T}}$, for $\delta = 0.1$ and $\mu \approx 0.5$.

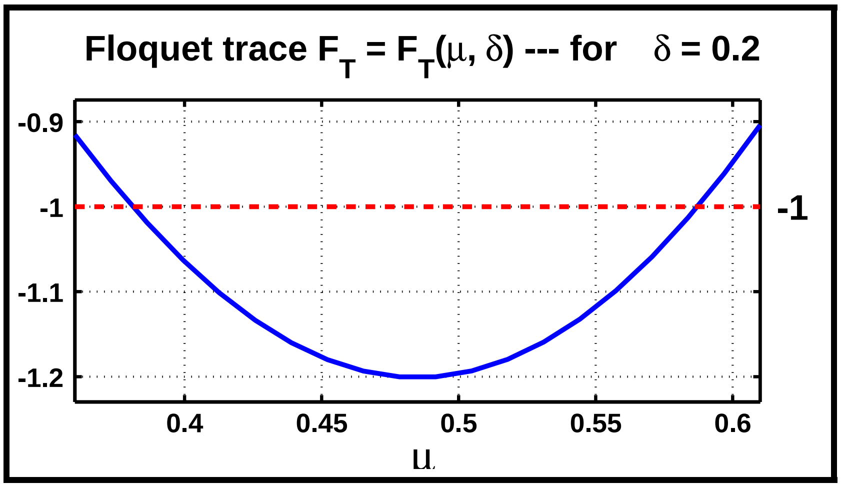

Figure 7:

Floquet Trace $\mathcal{F}_{\mathcal{T}}$, for $\delta = 0.2$ and $\mu \approx 0.5$.

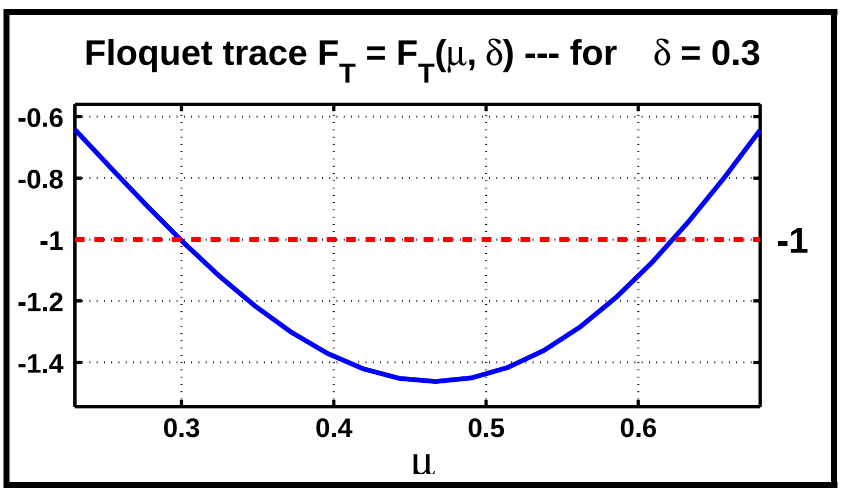

Figure 8:

Floquet Trace $\mathcal{F}_{\mathcal{T}}$, for $\delta = 0.3$ and $\mu \approx 0.5$.

First consider the plots of the Floquet Trace $\mathcal{F}_{\mathcal{T}}$ — as a function in the range $0 \leq \mu \leq 2$ — for the values $\delta = 0.1$, 0.2, 0.3, 0.4 and 0.5 (see Figures 1 through 5). On this scale the instability window near $\mu = 0.5$ is clearly visible for $\delta \geq 0.1$, while the other windows (near

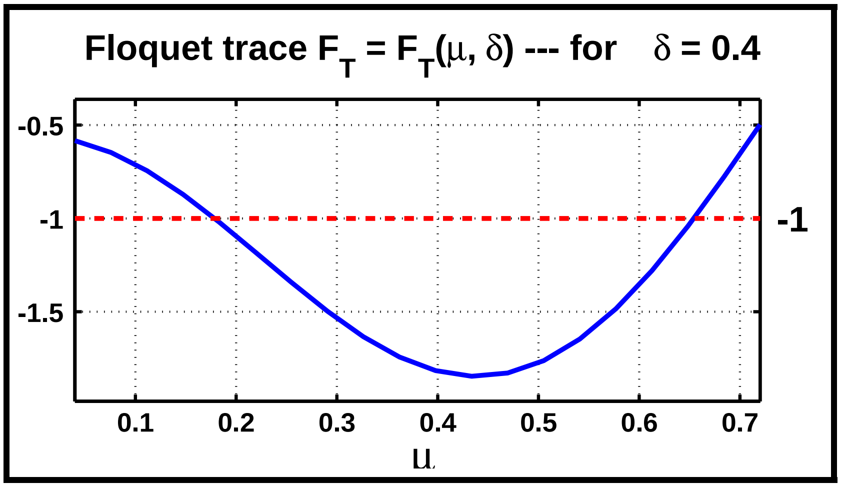

Figure 9:
Floquet Trace $\mathcal{F}_T$, for $\delta = 0.4$ and $\mu \approx 0.5$.

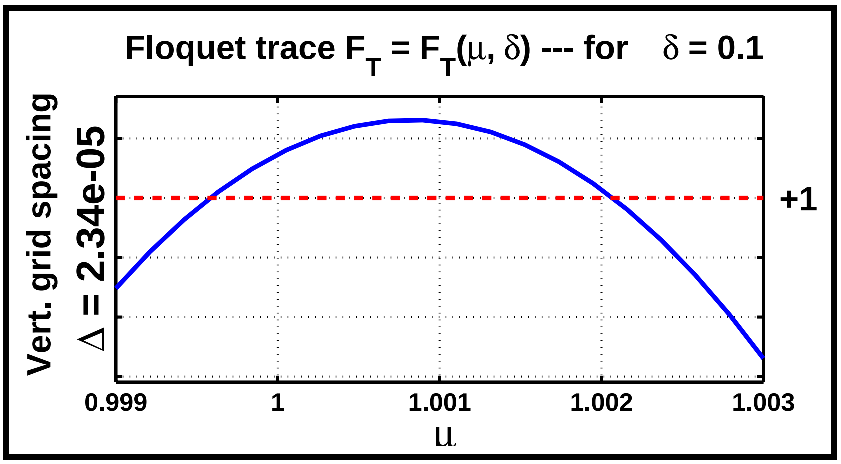

Figure 10:
Floquet Trace $\mathcal{F}_T$, for $\delta = 0.1$ and $\mu \approx 1.0$.

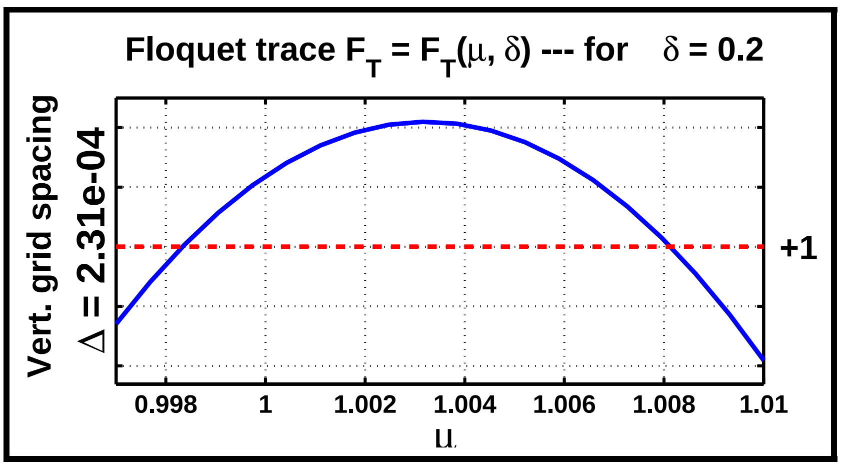

Figure 11:
Floquet Trace $\mathcal{F}_T$, for $\delta = 0.2$ and $\mu \approx 1.0$.

$\mu = 1,\ 1.5,$ and 2) are too small to be seen.[3] In particular, note that by $\delta = 0.5$ the instability window near $\mu = 0.5$ has grown so much that there is no longer a stable range for μ small — note

[3]These windows can be seen in the plots involving small ranges of μ; see Figures 6 through 18.

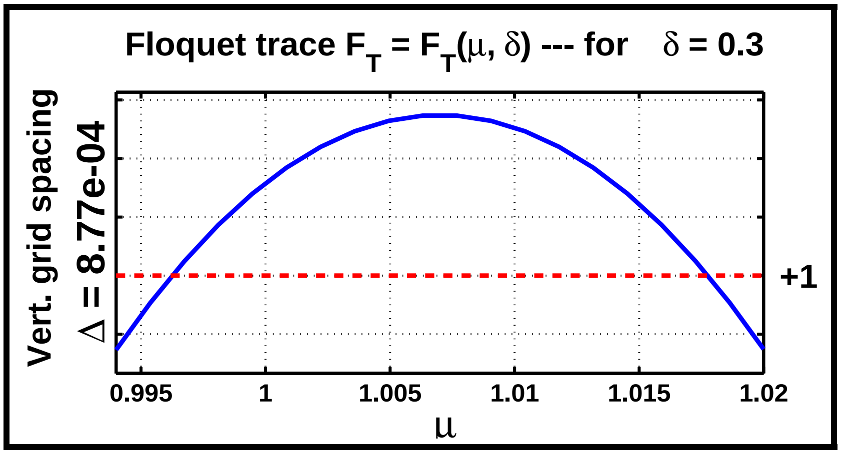

Figure 12: **Floquet Trace $\mathcal{F}_T$, for $\delta = 0.3$ and $\mu \approx 1.0$.**

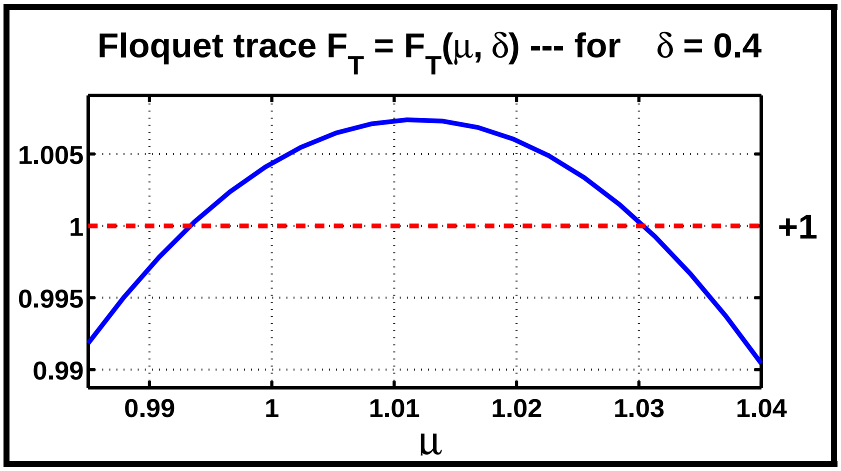

Figure 13: **Floquet Trace $\mathcal{F}_T$, for $\delta = 0.4$ and $\mu \approx 1.0$.**

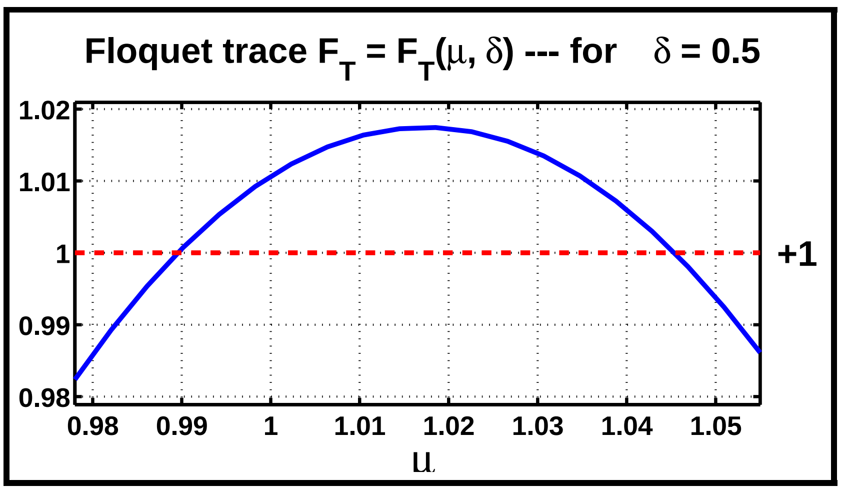

Figure 14: **Floquet Trace $\mathcal{F}_T$, for $\delta = 0.5$ and $\mu \approx 1.0$.**

that μ small corresponds to a forcing frequency that is much faster than the natural pendulum frequency. A fairly large forcing amplitude is required to de-stabilize the equilibrium position under such conditions.

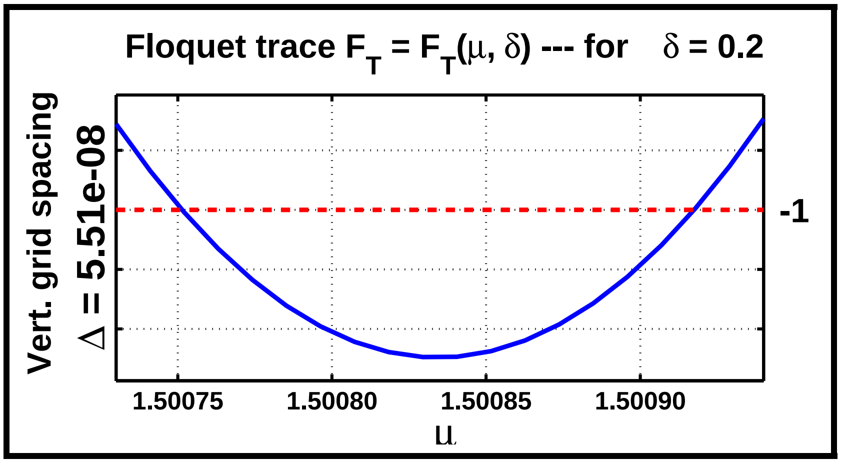

Figure 15:
Floquet Trace $\mathcal{F}_T$, for $\delta = 0.2$ and $\mu \approx 1.5$.

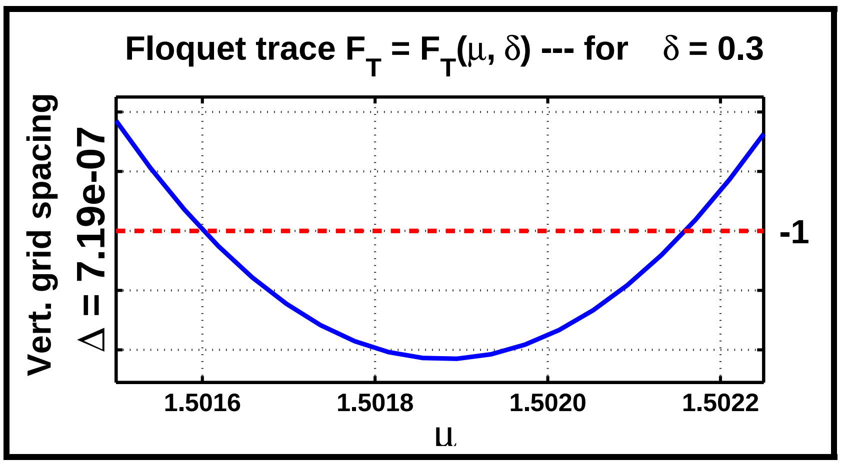

Figure 16:
Floquet Trace $\mathcal{F}_T$, for $\delta = 0.3$ and $\mu \approx 1.5$.

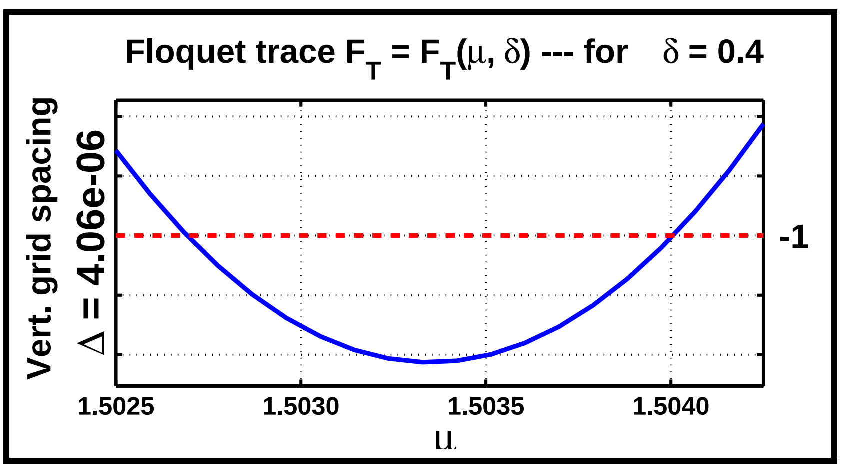

Figure 17:
Floquet Trace $\mathcal{F}_T$, for $\delta = 0.4$ and $\mu \approx 1.5$.

Figures 6 through 9 show plots of the Floquet Trace $\mathcal{F}_T$ in a neighborhood of $\mu = 0.5$, for the values $\delta = 0.1$, 0.2, 0.3, and 0.4. Thus **these figures show details of the lowest, and largest, instability window, for δ small and μ near $1/2$.** Note that the width of this window grows roughly linearly with δ (for small δ this can be shown using asymptotic expansion techniques). Of

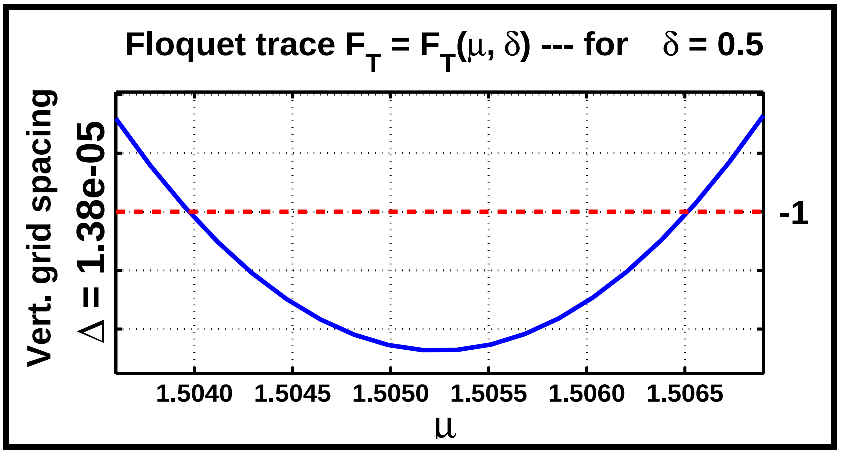

Figure 18:
Floquet Trace $\mathcal{F}_T$, for $\delta = 0.5$ and $\mu \approx 1.5$.

course, by $\delta = 0.5$ this is no longer true, and the window is so large as to have completely absorbed the stable "μ small" range.

Figures 10 through 14 show plots of the **Floquet Trace $\mathcal{F}_T$ in a neighborhood of $\mu = 1.0$,** for the values $\delta = 0.1$, 0.2, 0.3, 0.4, and 0.5. This window is much smaller than the $\mu \approx 0.5$ window, and it grows much more slowly. In fact, note that the width of this window grows roughly quadratically with δ (for small δ this can be shown using asymptotic expansion techniques).

Figures 15 through 18 show plots of the **Floquet Trace $\mathcal{F}_T$ in a neighborhood of $\mu = 1.5$,** for the values $\delta = 0.2$, 0.3, 0.4, and 0.5. This window is still smaller than the prior ones — so small, in fact, that I was un-able to resolve it for $\delta = 0.1$. The width of this window grows roughly cubically with δ (for small δ this can be shown using asymptotic expansion techniques).

Note: The figures were done using MatLab. To calculate the Floquet Trace $\mathcal{F}_T$, the ode solver ode113 was used to solve for the functions ϕ_1 and ϕ_2. To speed up the process, the calculation was "vectorized": for each value of δ, the solutions for all the calculated values of μ were calculated simultaneously.

THE END.