Finish remarks on min-cost flow.

- Strongly polynomial algorithms exist.
  - Tardos 1985
  - minimum mean-cost cycle
  - reducing ϵ-optimality
  - "fixing" arcs of very high reduced cost
  - best running running time roughly $O(m^2)$
  - best scaling time (double scaling) $O(mn \log\log U \log C)$.

1 Linear Programming

Problem description:

- motivate by min-cost flow
- bit of history
- everything is LP
- NP and coNP. P breakthrough.
- general form:
  - **variables**
  - **constraints:** linear equalities and inequalities
  - x **feasible** if satisfies all constraints
  - LP feasible if some feasible x
  - x **optimal** if optimizes objective over feasible x
  - LP is **unbounded** if have feasible x of arbitrary good objective value
  - **lemma:** every lp is infeasible, has opt, or is unbounded
  - (by compactness of R^n and fact that polytopes are closed sets).

Problem formulation:

- canonical form: $\min c^T x, Ax \geq b$
- matrix representation, componentwise $\leq$
- rows a_i of A are **constraints**
- c is **objective**
- any LP has transformation to canonical:
  - max/min objectives same

- move vars to left, consts to right
  - negate to flip $\leq$ for $\geq$
  - replace $=$ by two $\leq$ and $\geq$ constraints
- standard form: $\min c^T x, Ax = b, x \geq 0$
  - slack variables
  - splitting positive and negative parts $x \to x^+ - x^-$
- $Ax \geq b$ often nicer for theory; $Ax = b$ good for implementations.

How solve? First review systems of linear equalities.

- $Ax = b$. when have solution?
- baby case: A is squre matrix with unique solution.
- solve using, eg, Gaussian elimination.
- discuss polynomiality, integer arithmetic later
- equivalent statements:
  - A invertible
  - A^T invertible
  - $\det(A) \neq 0$
  - A has linearly independent rows
  - A has linearly independent columns
  - $Ax = b$ has unique solution for every b
  - $Ax = b$ has unique solution for some b.
- What if A isn't square?
- note that "$Ax = b$" means columns of A span b.
- use Gramm-Schmidt to find a basis of columns, check if b independent of them
- if not, some linear comb of A spans b
- in general, set of points $\{Ax \mid x \in \Re^n\}$ is a **subspace**
- so is $\{y \in \Re^m \mid Ay = b\}$
- anyone remember what we can say about dimensions?
- standard form LP asks for linear combo oto, but requires that all coefficients of combo be nonnegative!

Geometry

- canonical form: $Ax \geq b$ is an intersection of (finitely many) halfspaces, a **polyhedron**
- standard form: $Ax = b$ is an intersection of hyperplanes (thus a subspace), then $x \geq 0$ intersects in some halfspace. Also a polyhedron, but not full dimensional.
- polyhedron is **bounded** if fits inside some box.
- either formulation defines a **convex** set:
  - if $x, y \in P$, so is $\lambda x + (1 - \lambda)y$ for $\lambda \in 0, 1$.
  - that is, line from x to y stays in P.
- halfspaces define convex sets. Converse also true!
- let C be any convex set, $z \notin C$.
- then there is some a, b such that $ax \geq b$ for $x \in C$, but $az < b$.
  - **Proof** in 2D (for basic idea)
  - let y be closest point to z in C
  - let a be line from z to y.
  - note $ay < az$ (take $b = ay$)
  - claim $ax \leq b = ay$ for $x \in C$
  - suppose not. some $x \in C$ has $ax > b$
  - line from x to y is in C (convex)
  - but line from x to y goes into circle around z
  - that is, gets closer to z
- also true in higher dimensions (don't bother proving)
  - consider $u(\lambda) = \lambda x + (1 - \lambda)y$
  - $f(\lambda) = \|z - u\|^2$ distance from z to u
  - $f(0)$ supposed to be minimum
  - But

$$\begin{aligned} f'(0) &= \nabla f(u(0)) \cdot \frac{du}{d\lambda} \\ &= 2(z - y) \cdot (x - y) \\ &= -2a \cdot (x - y) \\ &= 2ay - 2ax \\ &\leq 0 \end{aligned}$$

  - so, can reduce f by increasing λ. Contra choice of y.
- deduce: every convex set is the intersection of the halfspaces containing it.

1.1 Vertices

Polyhedron has infinitely many points. Where are the optima? At "corners"

- a *extreme point* is a point that is not a convex combo of 2 other points in poly.
- a *vertex* is a point that is uniquely optimal in P for some cost function c
- vertex is extreme point:
  - vertex x unique opt for c
  - suppose $x = \lambda y + (1 - \lambda)z$
  - then

$$\begin{aligned} cx &= \lambda cy + (1-\lambda)cz \\ &\leq \max(cy, cz) \\ &< cx \end{aligned}$$

    by uniqe opt of x, a contradiction.
- soon, show extreme point is vertex

Claim: any LP in standard form with finite opt has opt at extreme point.

- suppose x opt, not extreme.
- find y so $x + y, x - y \in P$
- then $A(x + y) = A(x - y)$ so $Ay = 0$
- if $c^T y \neq 0$, can improve objective over x, contra
- Choose y so $y_j < 0$ some j (can since $y \neq 0$)
- note $c^T(x + \lambda y) = cx =$opt
- increase λ till new $x_i = 0$ becomes tight
- note get nonzero λ, since $\lambda = 1$ works
- so some new x_i becomes tight
- also by choice of y, if $x_i = 0$ then $y_i = 0$.
- so no zero x_i become nonzero.
- so one more zero.
- only can happen n times before stop at extreme point

Corollary:

- Actually showed, if x feasible, exists vertex with no worse objective.
- Note that in canconical form, might not have opt at vertex (optimize x_1 over (x_1, x_2) such that $0 \leq x_1 \leq 1$).
- but can deduce that any nonempty standard form poly has a vertex
- take arbitrary c, optimize over poly, get vertex.