Discuss project:

- four goals: read, implement, test, and exposite.
- choosing algorithm:
  - a nontrivial algorithm or data structure: Fib heap, bucket heap, VEB heap, splay tree, suffix tree, max-flow push/relabel, unusual shortest path, etc
  - source: advanced textbooks, or FOCS/STOC/SODA
  - maybe inherent interest, maybe useful in an application you are working on.
- read:
  - some article from a proceedings or journal, not yet digested in testbook.
  - goal: demonstrate you understood it (by explaining well).
- implement/test:
  - should not be too major to implement (don't write whole interior point algorithm!)
  - design/justify interesting inputs, or
  - use inputs from a real motivating application (still need to "design", but easier)
  - need a "control" experiment—dumb or prior implementation
  - testing should suggest changes. explain motivation and effect.
  - testing should suggest new inputs. explain motivation and effect.
  - perfectly all right for new version to work worse. But have to explain why.
- exposition:
  - clear explanation of algorith.
  - must demonstrate you understand it well
  - needn't include proofs in detail, but should give idea of "why works"
- meeting the goals: can either implement a new algorithm, or implement an old one but write about a new one. need to know can read journal article.

1 Review

Last time, characterization of optima/vertices/extreme points/basic feasible solutions. Saw simplex method, exponential algorithm for problem.

- If no negative reduced cost, optimum!
- If some negative reduced cost, "improve"
- wait: maybe can't move in imporving direction
- problem: **degeneracy**.
- degeneracy makes useless pivots
- Might be optimum but can't tell
- there are tricks to eliminate in theory: perturbation is one

We also left out one detail: how find starting basic feasible solution?

2 Duality

Quest for nonexponential algorithm: start at an easier place: how decide if a solution is optimal?

- decision version of LP: is there a solution with opt$> k$?
- this is in NP, since can exhibit a solution (assuming not too many bits...)
- is it in coNP? Ie, can we prove there is no solution with opt$> k$? (this would give an optimality test)
- even easier question: can we decide if an LP has any feasible solution?

2.1 Farkas' Lemma

Start with feasibility of system of linear equalities.

- $Ax = b$ has a *witness* for true: give x.
- How about a proof that there is no solution?
- claim: no solution iff for some y, $yA = 0$ but $yb \neq 0$.
- proof: if $Ax = b$, then $yA = 0$ means $yb = yAx = 0$.
- if no $Ax = b$, means columns of A don't span b
- set of points $\{Ax\}$ is subspace not containing b
- find part of b perpendicular to subspace, call y

- then $yb \neq 0$, but $yA = 0$,

How about feasiblility of system of inequalities?

- Farkas Lemma: Exactly one is true
  - $Ax = b, x \geq 0$ feasible
  - for some y, $yA \geq 0$ but $yb < 0$
- proof one way: $Ax = b$ and $yA \geq 0$ means $yb = yAx \geq 0$ since $x \geq 0$ and $yA \geq 0$.
- for other, suppose no solution.
- consider set $C = \{Ax \mid x \geq 0\}$
- this is a *convex set* (prove) not containing b
- in fact, it is a *cone*. draw picture, wave hands.
- remember, this means a hyperplane separates b from C.
- that is, for some y, $yb < yx \ \forall x \in C$
- in particular, $yb < 0$
- also, since $\lambda e_i \geq 0$, have $A(\lambda e_i) \in C$. So $yb < yA(\lambda e_i) = \lambda y A_i \ \forall \lambda > 0$
- so $yA_i \geq \lim_{\lambda \to \infty} yb/\lambda = 0$
- thus $yA \geq 0$

Similar for canonical form. Exactly one of:

- $yA \leq c$ feasible
- for some $y \geq 0$, $Ax = 0$ but $cx < 0$.
- prove by converting to standard form:
  - If $y^+A - y^-A + sI = c, y^+, y^-, s \geq 0$ infeasible means
  - exists x, $Ax \geq 0, -Ax \geq 0, Ix \geq 0$, but $cx < 0$

So, feasiblity is in $\mathcal{NP} \cap co-\mathcal{NP}$, since can exhibit feasible X or antifeasible y.

2.2 Duality

What about optimality?

- Intro *duality*, strongest result of LP
- give proof of optimality
- gives max-flow mincut, prices for mincost flow, game theory, lots other stuff.

Motivation: find a **lower** bound on $z = \min\{cx \mid Ax = b, x \geq 0\}$.

- try multiplying $a_i x = b_i$ by some y_i. Get $yAx = yb$
- if require $yA \leq c$, then $yb = yAx \leq cx$ is lower bound since $x_j \geq 0$
- so to get best lower bound, want to solve $w = \max\{yb \mid yA \leq c\}$.
- this is a new linear program, *dual* of original.
- just saw that dual is less than primal (weak duality)
- we will use Farkas to show equality (strong duality)

Note: dual of dual is primal:

$$\begin{aligned}
\max\{yb : yA \leq c\} &= \max\{by \mid A^T y \leq c\} \\
&= -\min\{-by \mid A^T y + Is = c, s \geq 0\} \\
&= -\min\{-by^+ + by^- \mid A^T y + (-A^T)y^- + Is = c, y^+, y^-, s \geq 0\} \\
&= -\max\{cz \mid zA^T \leq -b, z(-A^T) \leq -b, Iz \leq 0\} \\
&= \min\{cx \mid Ax = b, x \geq 0\} \quad (x = -z)
\end{aligned}$$

Weak duality: if P (min, opt z) and D (max, opt w) feasible, $z \geq w$

- $w = yb$ and $z = cx$ for some primal/dual feasible y, x
- x primal feasible ($Ax = b, x \geq 0$)
- y dual feasible ($yA \leq c$)
- then $yb = yAx \leq cx$

Note corollary:

- (restatement:) if P, D both feasible, then both bounded.
- if P feasible and unbounded, D not feasible
- if P feasible, D either infeasible or bounded
- in fact, only 4 possibilities. both feasible, both infeasible, or one infeasilbe and one unbounded.
- **notation:** P unbounded means D infeasible; write solution $-\infty$. D unbounded means P infeasilbe, write solution ∞.