1 Review

Farkas Lemma: Exactly one is true

- $Ax = b, x \geq 0$ feasible
- for some y, $yA \geq 0$ but $yb < 0$

Dual linear programs:

- Primal (P) $\min cx,\ \ Ax = b,\ \ x \geq 0$. Opt z.
- Pick any y such that $yA \leq c$
- then for any $x \geq 0$, $cx \geq yAx = yb$
- so yb is lower bound.
- Dual (D) $\max yb,\ \ yA \leq c$. Opt w.
- **weak duality:** $w \leq z$.
- also, saw dual of dual us primal.

2 Strong Duality

Strong duality: if P or D is feasible then $z = w$

- assume P feasible and $z > w$, show contra.
  - includes D infeasible via $w = -\infty$)
- recall $w = \max\{yb \mid yA \leq c\}$
- thus, no solution for $yA \leq c, yb \geq z$
- that is, $y(A\ (-b)) \leq (c\ (-z))$ infeasible
- so by Farkas, some $\binom{x}{q}$ with $x, q \geq 0$ has $Ax - bq = 0$ but $cx - zq < 0$.
- that is, $Ax = bq$ but $cx < zq$
- what if $q > 0$
  - then $A(x/q) = b$ (note $x/q \geq 0$) but $c(x/q) < z$
  - so x/q shows z not primal optimum.
- what if $q = 0$?
  - Then $Ax = 0$ but $cx < 0$.
  - Take any opt $Ax^* = b$, $cx = z$.
  - Then $x^* + x$ better! contra.

Neat corollary: Feasibility or optimality: which harder?

- given optimizer, can check feasiblity by optimizing arbitrary func.
- Given feasibility algorithm, can optimize by mixing primal and dual.

Interesting note: knowing dual solution may be useless for finding optimum (more formally: if your alg runs in time T to find primal solution given dual, can adapt to alg that runs in time $O(T)$ to solve primal without dual).

2.1 Rules for duals

General dual formulation:

- primal is

$$\begin{aligned} z &= \min c_1x_1 + c_2x_2 + c_3x_3 \\ A_{11}x_1 + A_{12}x_2 + A_{13}x_3 &= b_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 &\geq b_2 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 &\leq b_3 \\ x_1 &\geq 0 \\ x_2 &\leq 0 \\ x_3 & \quad UIS \end{aligned}$$

  (UIS emphasizes unrestricted in sign)

- means dual is

$$\begin{aligned} w &= \max y_1b_1 + y_2b_2 + y_3b_3 \\ y_1A_{11} + y_2A_{21} + y_3A_{31} &\leq c_1 \\ y_1A_{12} + y_2A_{22} + y_3A_{32} &\geq c_2 \\ y_1A_{13} + y_2A_{23} + y_3A_{33} &= c_3 \\ y_1 & \quad UIS \\ y_2 &\geq 0 \\ y_3 &\leq 0 \end{aligned}$$

- In general, variable corresponds to constraint (and vice versa):

PRIMAL	minimize	maximize	DUAL
constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	variables
variables	≥ 0 ≥ 0 free	$\leq c_j$ $\leq c_j$ $= c_j$	constraints

Derivation:

- remember lower bounding plan: use $yb = yAx \leq cx$ relation.
- If constraint is in "natural" direction, dual variable is positive.
- We saw A_{11} and x_1 case. $x_1 \geq 0$ ensured $yAx_1 \leq c_1 x_1$ for **any** y
- If some $x_2 \leq 0$ constraint, we want $yA_{12} \geq c_2$ to maintain rule that $y_1 A_{12} x_2 \leq c_2 x_2$
- If x_3 unconstrained, we are only safe if $yA_{13} = c_3$.
- if instead have $A_{21}x_1 \geq b_2$, any old y won't do for lower bound via $c_1 x_1 \geq y_2 A_{21} x_1 \geq y_2 b_2$. Only works if $y_2 \geq 0$.
- and so on (good exercise).
- This gives weak duality derivation. Easiest way to derive strong duality is to transform to standard form, take dual and map back to original problem dual (also good exercise).

Note: tighter the primal, looser the dual

- (equality constraint leads to unrestricted var)
- adding constraints create a new variable: more flexibility

2.2 Shortest Paths

A dual example:

- shortest path is a dual (max) problem:

$$\begin{aligned} w &= \max d_t - d_s \\ d_j - d_i &\leq c_{ij} \end{aligned}$$

- constraints matrix A has ij rows, i columns, ± 1 entries (draw)
- what is primal? unconstrained vars, give equality constraints, dual upper bounds mean vars must be positive.

$$\begin{aligned} z &= \min \sum y_{ij} c_{ij} \\ y_{ij} &\geq 0 \end{aligned}$$

thus

$$\sum_j y_{ji} - y_{ij} = 1(i = s), -1(i = t), 0\text{ow}$$

It's the minimum cost to send one unit of flow from s to t!

2.3 Simplex

We've actually seen duality before.

- recall simplex method.
- defined *reduced costs* of nonbasic vars N by

  $$\tilde{c}_N = c_N - c_B A_B^{-1} A_N$$

  and argued that when all $\tilde{c}_N \geq 0$, had optimum.
- Define $y = c_B A_B^{-1}$ (so of course $c_B = yA_B$)
- nonegative reduced costs means $c_N \geq yA_N$
- put together, see $yA \leq c$ so y is dual feasible
- but, $yb = c_B A_B^{-1} b = c_B x_B = cx$ (since $x_N = 0$)
- so y is dual optimum.
- more generally, y measures duality gap for current solution!
- another way to prove duality theorem: prove there is a terminating (non cycling) simplex algorithm.

3 Complementary Slackness

Another intuition:

- $\min\{yb \mid yA \geq c\}$ (note **flipped sign**)
- suppose b points straight up.
- so goal is to follow gravity.
- put a ball in the polytope, let it fall
- stops at opt y (no local minima)
- stops because in physical equilibrium
- equilibrium exterted by forces normal to "floors"
- that is, aligned with the A_i (columns)
- thus $b = \sum A_i x_i$ for some **nonnegative** force coeffs x_i.
- in other words, x feasible for $\max\{cx \mid Ax = b, x \geq 0\}$
- also, only walls touching ball can exert any force on it
- thus, $x_i = 0$ if $yA_i > c_i$

- that is, $(c_i - yA_i)x_i = 0$
- thus, $cx = \sum(yA_i)x_i = yb$
- so x is dual optimal.

Leads to another idea: *complementary slackness*:

- given feasible solutions x and y, $cx - by \geq 0$ is *duality gap.*
- optimal iff gap 0 (good way to measure "how far off"
- Go back to original primal and dual forms
- rewrite dual: $yA + s = c$ for some $s \geq 0$ (that is, $s = c_j - yA_j$
- The following are equivalent for feasible x, y:
  - x and y are optimal
  - $sx = 0$
  - $x_j s_j = 0$ for all j
  - $s_j > 0$ implies $x_j = 0$
- proof:
  - $cx = by$ iff $(yA + s)x = (Ax)y$, so $sx = 0$
  - if $sx = 0$, then since $s, x \geq 0$ have $s_j x_j = 0$ (converse easy)
  - so of course $s_j > 0$ forces $x_j = 0$ (converse easy)
- basic idea: opt cannot have a variable x_j and corresponding dual constraint s_j slack at same time: one must be tight.
- Another way to state: in arbitrary form LPs, feasible points optimal if:

$$\begin{aligned} y_i(A_x - b_i) &= 0 \forall i \\ (c_j - yA_j)x_j &= 0 \forall j \end{aligned}$$

- proof: note in definition of primal/dual, feasiblity means $y_i(a_i x - b_i) \geq 0$. Also $(c_j - yA_j)x_j \geq 0$. Also,

$$\begin{aligned} \sum y_i(a_i x - b_i) + (c_j - yA_j)x_j &= yAx - yb + cx - yAx \\ &= cx - yb \\ &= 0 \end{aligned}$$

  at opt. But since all terms are nonnegative, all must be 0