Systematic Procedure to Meet Specific Input/Output Constraints in the $\ell_1$-optimal Control Problem Design

by

Marcos Escobar Fernández de la Vega

Submitted to the Department of Electrical Engineering and Computer Science in partial fulfillment of the requirements for the degree of Master of Science in Electrical Engineering at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Abstract

This thesis is concerned with $\ell_1$ optimal control problem design in the presence of input/output constraints. A systematic procedure to include input/output constraints in the design process will be developed. The utility of these results will be shown in the solution of two problems: the minimization of the $\ell_1$ norm of a system subject to frequency and time domain constraints, and the exponentially weighted $\ell_1$ norm problem. Finally, a case study of the Earth Observing System (EOS) Satellite model will be studied. A comparison with previous designs for the EOS Satellite will be presented.

Thesis Supervisor: Munther A. Dahleh
Title: Associate Professor of Electrical Engineering
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## Contents

1 Introduction 9
   1.1 Motivation and Organization ........................................ 9
   1.2 Organization .................................................. 10
       1.2.1 Contributions ........................................... 11

2 Background 12
   2.1 Systems ..................................................... 12
       2.1.1 Stability and Maximum Amplification .......................... 13
       2.1.2 r-Transforms ............................................ 14
       2.1.3 Weighted Norms .......................................... 15
   2.2 Signal Spaces and Signal Norms ................................. 15
       2.2.1 The l∞ Signal Norm ...................................... 15
       2.2.2 l1 Norm Minimization for Controller Design ............... 16
   2.3 Performance Constraints ....................................... 22
       2.3.1 Worst-Case Performance ................................... 24
       2.3.2 Performance: Linear Constraints and Approximate Linear Constraints ........................................... 25

3 Improved Design Algorithm 26
   3.1 Linear Constraints ............................................ 26
       3.1.1 l1-Norm Constraints ...................................... 26
   3.2 Performance with fixed inputs - Time Domain Constraints ........ 27
   3.3 H∞-Norm Constraints - Frequency Domain Constraints ............ 28
3.4 Exponentially Weighted Norm Problem ................................................. 31
  3.4.1 Software Implementation ................................................................. 32
3.5 Mixed Performance Objectives ................................................................. 33
3.6 Design Examples .......................................................................................... 34
  3.6.1 Academic Example .................................................................................. 34
  3.6.2 Realistic Examples .................................................................................. 40

4 Case Study for the EOS Satellite .................................................................. 45
  4.1 The EOS Satellite ...................................................................................... 45
    4.1.1 Satellite Model ...................................................................................... 46
    4.1.2 Major Requirements .............................................................................. 49
  4.2 Problem Setup ............................................................................................. 50
    4.2.1 Parametric Uncertainty in the Plant ...................................................... 52
  4.3 Computing an $\ell_1$ sub-optimal controller .............................................. 57
  4.4 Analysis of Results ...................................................................................... 59
    4.4.1 Comparison with Previous Design Methodologies .................................. 65
  4.5 Summary for the EOS $\ell_1$ Design ........................................................... 73

5 Conclusions and Future Work ....................................................................... 75
  5.1 Conclusions ................................................................................................. 75
  5.2 Future Work ................................................................................................ 76

A Matlab Codes ................................................................................................ 78
  A.1 To Include Time Domain Constraints ....................................................... 78
  A.2 To Include Frequency Domain Constraints ............................................... 82
## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-1</td>
<td>Standard Problem</td>
<td>17</td>
</tr>
<tr>
<td>2-2</td>
<td>General Setup</td>
<td>23</td>
</tr>
<tr>
<td>3-1</td>
<td>Exponentially Weighted Problem</td>
<td>33</td>
</tr>
<tr>
<td>3-2</td>
<td>Problem 1</td>
<td>35</td>
</tr>
<tr>
<td>3-3</td>
<td>Response to a Step Input (solid) and to an Impulse (dashed) Without Constraints</td>
<td>36</td>
</tr>
<tr>
<td>3-4</td>
<td>Time Responses: (a) With Constraints to a Step Input (solid) and to an Impulse (dashed); (b) With Constraints (until $t = 10s$) to an Impulse</td>
<td>37</td>
</tr>
<tr>
<td>3-5</td>
<td>Frequency Responses: (a) With Constraints (solid) and Without Constraints (dashed); (b) Zoom of (a)</td>
<td>38</td>
</tr>
<tr>
<td>3-6</td>
<td>Frequency Response With Constraints (solid) and Without Constraints (dashed)</td>
<td>39</td>
</tr>
<tr>
<td>3-7</td>
<td>Frequency Responses: (a) With Constraints (solid) and Without Constraints (dashed); (b) Zoom of (a). $\gamma_{H_\infty} = 2.6$ (8.299 db.)</td>
<td>42</td>
</tr>
<tr>
<td>3-8</td>
<td>Time Response of the Control Signal with Constraints (solid), without (dashed) and the Vector $U_{max}$</td>
<td>44</td>
</tr>
<tr>
<td>4-1</td>
<td>EOS Satellite Configuration</td>
<td>47</td>
</tr>
<tr>
<td>4-2</td>
<td>Structural Frequency Responses</td>
<td>49</td>
</tr>
<tr>
<td>4-3</td>
<td>Plant Description</td>
<td>49</td>
</tr>
<tr>
<td>4-4</td>
<td>Standard Problem</td>
<td>51</td>
</tr>
<tr>
<td>4-5</td>
<td>Discretized Plant-Filter $T$</td>
<td>51</td>
</tr>
<tr>
<td>4-6</td>
<td>Flexible Mode Dynamics with Uncertainty</td>
<td>57</td>
</tr>
</tbody>
</table>
4-7 System Configuration ........................................ 58
4-8 Closed Loop System ........................................ 60
4-9 Roll Angle Time Response .................................. 62
4-10 Disturbance to Roll Angle Frequency Response ....... 63
4-11 Pre-Filtered Control Torque Time Response ............. 63
4-12 Disturbance to Pre-Filtered Control Torque Frequency Response .......................... 64
4-13 System Description for $\mathcal{H}_\infty$ and $\mu$-synthesis methods .................. 65
4-14 Performance/Robustness Tradeoff ....................... 70
4-15 Comparison for the Roll Angle Time Responses ........ 71
4-16 Comparison for the Frequency Response from Disturbance to Roll Angle ........... 71
4-17 Comparison for the Control Torque Time Response ........ 72
4-18 Comparison for the Frequency Response from Disturbance to Control Torque .......... 72
List of Tables

3.1 Results Problem 1. * = reordered outputs; † = did not improve re-ordering the outputs .................................. 39
3.2 Results Problem 2. † = did not improve reordering the outputs .................................. 42
3.3 Results Problem 3. .................................. 43
4.1 EOS Satellite Characteristics .................................. 48
4.2 Modal Admittances and Natural Frequencies .................................. 50
4.3 EOS $\ell_1$ Design Results .................................. 61
4.4 Comparison Table for Time Specifications .................................. 66
4.5 Comparison Table for Frequency Specifications .................................. 67
4.6 Comparison Table for the Level of Robustness .................................. 68
Chapter 1

Introduction

1.1 Motivation and Organization

In recent years, several papers and publications related to the $\ell_1$ optimal control design problem and in $\ell_1$ optimization in general have appeared, in particular \[4, 5\]. Several other papers have also addressed the constrained $\ell_1$ problem, i.e. \[11\]. Ideas of how to solve square (i.e., one-block) and non-square (i.e., multiblock) problems were presented in \[5\], including a method to compute approximate suboptimal solutions iteratively. In \[11\], a general treatment of the multiblock case was presented, and existence was discussed.

A more rigorous treatment of the multiblock problem was presented by Diaz-Bobillo \[6\], where one of the goals was to understand the structure of the optimal solution, and to provide an efficient and systematic method to compute (sub-)optimal controllers. Diaz-Bobillo developed the Delay Augmentation Algorithm, which is based on converting the rank interpolation conditions to zero interpolation conditions.

The latest case study was presented in \[9\] for an ill-conditioned plant, where general ideas concerning the design of robust controllers using norm-based methods were discussed. The most important contributions in this case study are:

1. Robustness (design and analysis) is studied with respect to structured uncertainty.
2. Dynamic first order performance weighting on the sensitivity is related to time domain characteristics such as rise time and overshoot.

Many different aspects of the state of the art in $\ell_1$ optimal control have motivated this research. Initially the motivation of my research was to create a complete software package able to include different kinds of constraints when designing a controller using $\ell_1$ norm methodologies.

At the same time, another interesting point to analyze was the effect of the exponentially weighted $\ell_1$ norm when designing controllers for plants with poles on the unit circle.

The main question is "How can the $\ell_1$ software available be improved to effectively and efficiently solve the constrained and/or weighted norm $\ell_1$-optimal control problem?"

1.2 Organization

Chapter 2 covers the basic background in signals, systems, stability and performance needed to understand the material covered in the subsequent chapters.

"Delay Augmentation" can be implemented to solve the constrained $\ell_1$ problem and the weighted norm problem. However, extra analysis is required to determine whether the controller order can be reduced without degrading the performance. It may be necessary to sacrifice some of the constraints in order to reduce the order of the controller. There will be several tradeoffs between number of constraints included in the problem and the degree of difficulty in the computation of the solution.

The main interest of this research is to obtain a methodology to handle different constraints on the inputs and outputs when designing controllers with $\ell_1$ norm methodologies. It is realistic to have one or more of the following types of constraints:

1. Time domain constraints, such as rise time, overshoot, final steady state value (see Chapter 3).

2. Frequency domain constraints, such as maximum value in the frequency re-
sponse, bandwidth (see Chapter 3).

3. Exponential weights applied to the system (see Chapter 3).

A complete design problem will be presented in Chapter 4.

1.2.1 Contributions

The contributions of this research are:

1. An efficient software package that allows for both time and frequency domain specifications. This new software is included in the general $\ell_1$ software developed by Diaz-Bobillo in [6].

2. A subroutine to apply exponential weights to the system is also included in the general $\ell_1$ software.

3. A case study of the Earth Observing System (EOS) satellite model is presented.
Chapter 2

Background

This chapter is intended to provide the basic notation and concepts needed to understand the work developed in chapters 3 and 4. Section 2.2 develops the signal norms and signal spaces required to use the $\ell_1$ methodologies. Section 2.1 describes the systems as operators between signal spaces. Section 2.3 covers the performance constraints commonly used in the $\ell_1$-optimal control problem design. The material included in this chapter was taken from [3], [6], [9] and [13].

2.1 Systems

Abstractly, systems are mathematical operators which map signal spaces to signal spaces. In symbols, the system’s output is $y = Tu$, where $u$ is the input signal. Note that $y$ and $u$ are representations of signals over all $t \in [0, \infty)$, not at a given instant.

Definitions for linearity, causality and time invariance are as follow:

$T$ is a linear operator from $X$ to $Y$ if it satisfies

$$T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2), \quad \text{for all } \alpha, \beta \in \mathbb{R} \quad (2.1)$$

In words, a system is linear if it exhibits both linearity and homogeneity (i.e. superposition).
Denote by $P_k$, $k \in \mathbb{Z}_+$, the standard truncation operator on $\ell^m$, i.e.,

$$P_k(x(0), x(1), ...) = (x(0), x(1), ..., x(k), 0, 0, ...) \quad (2.2)$$

**Definition 3.** An operator $T$ is causal (proper) if

$$P_tT = P_tTP_t \ for \ all \ t, \quad (2.3)$$

and is strictly causal (strictly proper) if

$$P_tT = P_tTP_{t-1} \ for \ all \ t. \quad (2.4)$$

In words, an operator $T$ is causal if the current output does not depend on future inputs. It is strictly causal, if the current output depends on past inputs, not including the current input.

Denote by $S$ the standard unit shift operator, i.e.

$$S(x(0), x(1), ...) = (0, x(0), x(1), ...) \quad (2.5)$$

**Definition 4.** An operator $T$ is time-invariant if it commutes with the unit shift operator, i.e.

$$ST = TS \quad (2.6)$$

In words, a system is time invariant if its action is independent of the starting time.

### 2.1.1 Stability and Maximum Amplification

Let $X$ and $Y$ be normed linear spaces and $T : X \rightarrow Y$. From [3] it follows that $T$ is a bounded operator from $X$ to $Y$ (i.e. continuous) if and only if its induced norm is finite, i.e.

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} < \infty \quad (2.7)$$
The induced norm of $T$ indicates the amount of amplification the operator exerts on the space $X$.

**Definition 5.** A linear system $T$ is stable with respect to some input/output space $X$ if it is bounded as a linear operator on $X$. In general the input space has a different dimension from the output space.

The induced norm for the $\ell_1$ space can be computed exactly:

$$|A|_1 = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|$$  \hspace{1cm} (2.8)

It is interesting to note that the notion of stability is tied to the particular space under consideration. One particular operator can be stable with respect to one space but not the other, i.e. $\ell_\infty$-stability for the SISO case is guaranteed if and only if the pulse response is in $\ell_1$. The $\ell_1$ norm of the pulse response sequence in the SISO case can be computed in the following way

$$\|R\| = \max_{1 \leq i \leq m} \sum_{j=1}^{\infty} |r_{ij}| = \max_{1 \leq i \leq m} \sum_{j=1}^{n} \|r_{ij}\|_1$$  \hspace{1cm} (2.9)

where $R$ is a time-invariant operator on $\ell_\infty^n$. Notice that for the MIMO case, the induced norm is the composition of the finite-dimensional matrix norm with the $\ell_1$-norm.

### 2.1.2 $\lambda$-Transforms

For every element $R \in \ell_1^{m \times n}$, the $\lambda$-Transform $\hat{R}$ is defined as:

$$\hat{R}(\lambda) = \sum_{i=0}^{\infty} R(i)\lambda^i.$$  \hspace{1cm} (2.10)

Then, $\hat{R}(\lambda)$ is analytic on the open unit disc, and continuous on the boundary. The collection of all such elements equipped with the norm defined in Equation 2.9 is traditionally denoted by $A$ (or $A^{m \times n}$). From this definition, it is clear that the $A^{m \times n}$ and $\ell_1^{m \times n}$ are different representations of the same space.
2.1.3 Weighted Norms

Other measures can be obtained by introducing weighting functions, both in time and frequency domains. Let $C(t)$ be a given matrix valued time function. The weighted $\ell_1$ norm is defined as:

$$\|R\|_C = \max_{1 \leq i \leq m} \sum_{j=1}^{n} \sum_{t=0}^{\infty} |c_{ij}(t)r_{ij}(t)|.$$  \hspace{1cm} (2.11)

If $c_{ij}(t) = a^t$, then the measure is known as the exponentially weighted $\ell_1$ norm. If $c_{ij}(t)$ is any time-varying function, then the measure is known as the time-varying weighted $\ell_1$ norm.

2.2 Signal Spaces and Signal Norms

The measure of signal size that this thesis is concerned with is the maximum amplitude. The $\ell_1$ control methodologies require that the signals are in the space of bounded amplitude signals.

2.2.1 The $\ell_\infty$ Signal Norm

Consider measuring a discrete-time signal by its maximum absolute value, or its $\ell_\infty$ norm defined as

$$\|f\|_\infty = \sup_k |f(k)|$$ \hspace{1cm} (2.12)

Thus signals of bounded $\ell_\infty$ norm are signals of bounded magnitude. The use of the $\ell_\infty$ norm as a measure of signal size results in the $A$ induced norm on stable systems, i.e. if $T$ is an $\ell_\infty$-stable system, then

$$\|T\|_A = \sup_{x \neq 0} \frac{\|Tx\|_\infty}{\|x\|_\infty} < \infty.$$ \hspace{1cm} (2.13)
The $\mathbf{A}$ norm of $T$ is given by the $\ell_1$ norm of the pulse response, $t$, associated with $T$. i.e. $\|T\|_{\mathbf{A}} = \|t\|_1$ where

$$\|t\|_1 = \sum_{k=0}^{\infty} |t(k)|$$  \hfill (2.14)

If $T$ is MIMO of dimension $m \times n$ then

$$\|T\|_{\mathbf{A}} = \max_{1 \leq i \leq m} \left( \sum_{j=1}^{n} \|t_{ij}(k)\|_1 \right)$$  \hfill (2.15)

The primary motivation for using the $\ell_\infty$ signal norm is the time domain interpretation of this norm for specifying performance specifications. For example, if the types of disturbances expected by the system are persistent (i.e. not of bounded energy) but do have bounded magnitude, (for example sinusoids, steps) then it makes sense to consider performance in terms of the $\ell_\infty$ signal norm of the output. Moreover, time domain specifications can be specified as constraints on the $\ell_\infty$ norm of the output.

### 2.2.2 $\ell_1$ Norm Minimization for Controller Design

We now collect several results from [6], which we use throughout this thesis. For our purposes it suffices to present the algorithm.

First we establish the notation used throughout the thesis. The problem being studied can be represented in the general form given in Figure 4-4 where $z$, $w$, $u$ and $y$ represent regulated outputs, exogenous disturbances, controls and measured outputs respectively. Problems will be classified as to the dimension of these signals. A problem is **one-block** if the number of exogenous disturbances is less than or equal to the number of measurements, and the number of regulated outputs is less than or equal to the number of controls. If a problem is not one block then it is **multiblock**.

The $\ell_1$ problem can be stated as

$$\nu^* = \inf_{\text{All } K-\text{Stabilizing}} \|\Phi\|_1$$  \hfill (2.16)

where $\Phi$ is the closed loop map from $w$ to $z$ in Figure 4-4.

By invoking the standard parametrization of all stabilizing controllers it is possible
to rewrite Equation 2.16 as

\[ \nu^o = \inf_{Q \in \ell_1^{n_u \times n_y}} \|H - UQV\|_1 \]  \hfill (2.17)

This can be rewritten as a minimum distance problem by defining the subspace \( S \) as:

\[ S = \{ R \in \ell_1^{n_u \times n_y} | R = UQV \text{ for some } Q \in \ell_1^{n_u \times n_y} \} \]  \hfill (2.18)

Now Equation 2.17 can be restated as

\[ \nu^o = \inf_{R \in S} \|H - R\|_1 \]  \hfill (2.19)

Thus the minimization problem is transformed into a minimum distance problem in \( \ell_1 \).

The subspace \( S \) can be uniquely determined with a set of linear constraints on the pulse response of \( R = H - \Phi \). Since \( H \) is known, the linear constraints can be written in terms of the pulse response of \( \Phi \). These linear constraints can be broken into two sets of constraints, each serving a different purpose. The first set of constraints are the interpolation conditions for zeros of \( U \) and \( V \) that ensure the internal stability of the closed loop maps; these results from the unstable poles and non-minimum phase zeros of \( T \) in Figure 4-4. The second set of constraints are rank interpolation
conditions which ensure the consistency of the problem if it is multiblock, and are not present in one-block problems. These conditions are required since the set of equations \( \Phi = H - UQV \) is over determined in \( Q \) for multiblock problems. It has been known that the \( \ell_1 \) problem can be posed as an infinite dimensional linear program [4]. Using duality theory, it was shown in [6] that if the problem is one block, then the solution reduces to a finite dimensional linear program. This equivalence is true even if there are interpolations on the unit circle; however, in this case the existence of the optimal solution is not guaranteed. “Delay Augmentation” is a methodology aimed at solving multiblock problems efficiently.

**The Delay Augmentation Algorithm**

The main idea of the delay augmentation algorithm, taken from [6] is as follows:

1. augment \( U \) and \( V \) with pure delays (i.e. right shifts) such that the augmented problem is one-block

2. solve the resulting one-block problem

3. recover the original system and compute the controller

This algorithm provides upper and lower bounds which converge, under certain conditions, to the optimal value of solution as the number of delays is increased.

A brief presentation of the mechanics of the delay augmentation algorithm will now be presented. First break the parametrization of \( \Phi \) into its blocks

\[
\begin{bmatrix}
\Phi_{11} & \Phi_{12} \\
\Phi_{21} & \Phi_{22}
\end{bmatrix} = \begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{bmatrix} - \begin{bmatrix}
U_1 \\
U_2
\end{bmatrix} Q \begin{bmatrix}
V_1 & V_2
\end{bmatrix}
\]

(2.20)

where \( \Phi_{11} \in \mathbb{C}^{n_u \times n_y} \). The \( U \) and \( V \) matrices are augmented with \( N \)th order shifts transforming the problem to a standard one-block. Consequently, parameter \( Q \) is augmented with extra degrees of freedom. In the following equation \( Q_{11,N} = Q \) in
Equation 2.20.

\[
\begin{bmatrix}
\Phi_{11,N} & \Phi_{12,N} \\
\Phi_{21,N} & \Phi_{22,N}
\end{bmatrix} =
\begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{bmatrix} -
\begin{bmatrix}
U_1 & 0 \\
U_2 & S_N
\end{bmatrix}
\begin{bmatrix}
Q_{11,N} & Q_{12,N} \\
Q_{21,N} & Q_{22,N}
\end{bmatrix}
\begin{bmatrix}
V_1 & V_2 \\
0 & S_N
\end{bmatrix}
\]

(2.21)

or in more compact notation,

\[
\Phi_N = H - U_N Q_N V_N = H - R_N
\]

(2.22)

This can be expanded as

\[
\Phi_N = H - U Q_{11} V - S_N \tilde{R}_N = \Phi - S_N \tilde{R}_N
\]

(2.23)

\[
\tilde{R}_N =
\begin{bmatrix}
0 & U_1 Q_{12} \\
Q_{21} V_1 & Q_{21} V_2 + U_2 Q_{12} + S_N Q_{22}
\end{bmatrix}
\]

(2.24)

These equations depend on the fact that these are all time invariant operators. The delay augmentation problem of order \( N \) is defined as

\[
\nu_N = \inf_{Q_N \in \ell_1 \times n_w} \| \Phi_N \|_1
\]

(2.25)

From Equation 2.25, \( \nu_N \) provides a lower bound for \( \nu^o \) due to the added degrees of freedom in \( Q_N \). It is obvious that the solution to Equation 2.25 is infeasible to the original problem for which \( Q_{12,N}, Q_{21,N}, \) and \( Q_{22,N} \) are not present. Thus the solution of the delay augmentation problem of order \( N \) produces a superoptimal solution to the original \( \ell_1 \)-optimization problem.

\[
\nu \leq \inf_{Q_{11} \in \ell_1^{n_u \times n_y}} \| \Phi_N \|_1 = \inf_{Q_{11} \in \ell_1^{n_u \times n_y}} \| \Phi \|_1 = \nu^o
\]

(2.26)

\[
Q_{12} = Q_{21} = Q_{22} = 0
\]

An upper bound for \( \nu^o \) can also be derived from the solution of Equation 2.26. By setting \( Q_{12,N}^o, Q_{21,N}^o, \) and \( Q_{22,N}^o \) all to zero and considering only \( Q_{11,N}^o \) it is possible to
obtain a feasible solution to the original problem. This feasible solution is obviously suboptimal,

$$\nu^o = \inf_{Q \in \ell_1^{\nu \times n_y}} \|H - UQV\|_1 \leq \|H - UQ_{11, N}^o V\|_1 = \bar{\nu}_N$$

(2.27)

Thus we have found lower and upper bounds to the optimal solution of the $\ell_1$ problem. Note that in the practical implementation of this algorithm, a check on the accuracy of solutions can be made from the previous information. Recall Equations 2.23 and 2.24. These equations show that $\Phi_{11}$ of the suboptimal and superoptimal solutions are equivalent. In fact, the value of $Q_{11}$ is actually obtained by using this information: $\Phi_{11, N} = \Phi_{11} = H - U_1 Q_{11, N} V_1$ where everything but $Q_{11, N} = Q$ is known.

The convergence of the lower and upper bounds, $\nu_N$ and $\bar{\nu}_N$, to the optimal solution $\nu^o$ is discussed and proved in [6]. A brief discussion is presented here. Convergence of both bounds depends on the assumption that $\tilde{U}_1(\lambda)$ and $\tilde{V}_1(\lambda)$ have no left and right zeros on the unit circle respectively. Note that from a practical point of view many systems contain integrators. The presence of these integrators will result in a violation of this assumption; however, making the following modification will alleviate this problem. If the pole at $\lambda = 1$ is a result of a weighting function outside the loop, then perturb the pole so that it is slightly stable. If the pole at $\lambda = 1$ is a result of a system inside the loop, then perturb the pole slightly unstable. Now, with the assumption of no interpolations on the unit circle, it can be shown that $\nu_N$ forms a non-decreasing sequence that converges to the optimal $\nu^o$. Under additional assumptions, the upper bound $\bar{\nu}_N$ also converges to $\nu^o$. The full proof of this can be found in [6]. The main additional assumption needed for the convergence of the upper bound is that the $\ell_1$ norm of the first $n_u$ rows of the optimal solution $\Phi^o = H - UQ^o V$ must be active, i.e., they must achieve the optimal norm $\nu^o$. Note that these are the rows which contain elements of $\Phi_{11}$. If this is not the case, then the convergence will not occur and the ordering of the rows must be changed.
Solution Structure

As mentioned in the previous paragraph, the ordering of the inputs and outputs of the closed loop $\Phi_N$ is very important in the optimization process. The ordering determines which subblock of $\Phi_N$ will become $\Phi_{11,N}$ and thus which block will be used to compute the controller. It was discussed in the previous section that the ordering of the outputs can also determine whether the upper and lower bounds converge. In cases where all rows are active, the ordering will not determine if the bounds will converge, but rather how fast. It will also determine the order of the resulting controller. Choosing the proper order of inputs and outputs is based on the issue of dominance which results from the support structure of the closed loop system. The support structure is defined as the length of each of the individual entries of the closed loop pulse response.

**Definition 1.** In a multiblock problem, a one-block partition is totally dominant (TD) if the optimal free parameter $Q^*$ obtained from its solution also solves the original multiblock optimization problem. [6]

If a totally dominant one-block partition exists, then the rows which do not contain elements of this partition are not active in the original problem, and therefore the $\ell_1$ norm of these rows will be smaller than the $\ell_1$ norm of the rows which are active. As was discussed previously, if $\Phi_{11,N}$ contains elements of a row which is not active, then the upper bound will not converge to the optimal solution.

The issue of partial dominance is more subtle and less well understood. It affects the speed of convergence and the order of the resulting controller. Again from [6]

**Definition 2.** In a multiblock problem, a one-block partition is partially dominant (PD) if all $\ell_1$ optimal solutions are polynomial in the entries corresponding to such partition.

As was explained in [6], the intuition behind this is that the multiblock problem can be viewed as one-block problem with extra constraints from the rank interpolations. If the rank interpolations do not have a "strong" effect on the nature of the solution, then the optimal solution for that partition will still have the polynomial
structure characteristic of one-block problems. Since the controller is calculated from $\Phi_{11}$, if its pulse response is polynomial, the controller can be expected to be of low order. This is the motivation for finding partially dominant one-block partitions and moving them to the position of $\Phi_{11}$. If it is seen in the preliminary delay augmentation iterations that the support of certain elements of $\Phi$ is small, or finite, then one should reorder the inputs and outputs so as to move these elements into $\Phi_{11}$. It is possible, though not usually the case, that elements of $\Phi$ will have small support in initial iterations; however after moving them to $\Phi_{11}$ they do not retain small support and the convergence is actually slowed by the move.

The support structure, which determines whether or not a totally or partially dominant partition exists, can be used to analyze the physical relationship between the entries in a given problem.

2.3 Performance Constraints

It is difficult to capture useful performance requirements as mathematical constraints for some optimization problem. In this section, a general setup is introduced in which some performance requirements can be captured. Consequently, performance specifications that lead to linear constraints on the closed loop function will be described. This is done in a general setting allowing multiple objectives for different input/output pairs, i.e. the $\ell_1$ problem.

Figure 2.3 shows a general setup for posing performance specifications. The variables as defined in the Figure are:

- $u =$ Control Inputs
- $y =$ Measured Outputs
- $w =$ Exogenous Inputs (Fixed commands, unknown commands, disturbances, noise, etc.)
**z** = Regulated Outputs (Tracking errors, control inputs, measured outputs, states, etc.)

The operator *G* is a $2 \times 2$ block matrix mapping the inputs *w* and *u* to the outputs *z* and *y*:

\[
\begin{bmatrix}
    z \\
    y
\end{bmatrix} =
\begin{bmatrix}
    G_{11} & G_{12} \\
    G_{21} & G_{22}
\end{bmatrix}
\begin{bmatrix}
    w \\
    u
\end{bmatrix}
\]

The actual process or plant is the submatrix $G_{22}$. Both the exogenous inputs and the regulated outputs are auxiliary signals that need not be part of the closed loop system. The feedback controller is denoted by $K$. The dimensions of the signal spaces will be denoted by $n_u, n_y, n_w$ and $n_z$, where the association is explicitly given by the subindex.

The map of interest is the map between *w* and *z*, denoted by $\Phi$:

\[
\Phi = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}
\]  

This particular mapping represents the performance objectives. Whenever it is necessary, we will denote this map by $T_{zw}$ to explicitly state the inputs and outputs of the map. For a given map $\Phi$, we will discuss two kinds of constraints:

- **feasibility constraints**, i.e. whether $\Phi$ can be written as in Equation 2.28 for some stabilizing controller, and
• performance constraints representing the performance objectives.

The discussion on the first can be looked in [6]. A map $\Phi$ satisfying the feasibility constraints will be referred to as feasible closed-loop map. The discussion on some of the performance constraints will be presented in the following subsections, and also in Chapter 3.

2.3.1 Worst-Case Performance

If the exogenous signal is not known exactly but is known to lie in a set, then a reasonable measure for performance is one which looks at the worst possible output. In particular, assume that the set of exogenous inputs is given by

$$\{ w \in \ell_p \mid \| w \|_p \leq 1 \}.$$ 

A good measure of performance is given by

$$\sup_w \| z \|_p$$

which is the norm of the worst possible output as the exogenous signal ranges over the allowable set. The controller design problem is given by

$$\inf_{K \text{ stabilizing}} (\sup_w \| \Phi w \|_p) = \inf_{K \text{ stabilizing}} \| \Phi \|_{\ell_p-\text{ind}}.$$ 

This performance objective is known as a minimax objective. The controller is designed to guard against all exogenous signals in the allowable set. Hence, any minimization problem involving an induced norm of the closed loop operator will be considered a worst-case design method.

Notice that this formulation does not require any probabilistic assumption on the exogenous signals. Classes of signals modelled in terms of a norm are known as unknown but bounded signals.
2.3.2 Performance: Linear Constraints and Approximate Linear Constraints

Some design specifications can be represented as linear constraints on the closed loop map, \( \Phi \). Let \( \mathcal{P} \) be a positive cone in \( \ell_p^{r \times s} \) defined as

\[
\mathcal{P} = \{ H \in \ell_p^{r \times s} \mid h_{ij}(k) \geq 0 \text{ for all } i, j, k \}.
\]

Let \( \mathcal{A} \) be a linear operator from \( \ell_1^{n \times s} \) to \( \ell_p^{r \times s} \) for some \( p \), and \( b \in \ell_p^{r \times s} \) be a fixed element. Then \( \Phi \) satisfies the set of linear constraints given by \( \mathcal{A} \) and \( b \) if

\[
b - \mathcal{A} \Phi \in \mathcal{P}.
\]

In a more familiar notation, the above condition takes the form

\[
\mathcal{A} \Phi \leq b \tag{2.29}
\]

where the inequality is with respect to the cone \( \mathcal{P} \), i.e. pointwise. It turns out that many performance specifications can be posed in terms of linear constraints as in Equation 2.29.

The \( \ell_1 \)-norm constraints and the performance with fixed inputs can be represented as linear constraints on the closed loop map \( \Phi \), as will be discussed in Chapter 3.

Not all design specifications yield to linear constraints on the closed loop map. Some examples where the constraints are convex but nonlinear can be approximated with linear constraints, such as the \( \mathcal{H}_\infty \)-norm constraints. The nonlinear constraints mentioned before will be addressed also in Chapter 3.
Chapter 3

Improved Design Algorithm

3.1 Linear Constraints

We will see that time and frequency constraints can be expressed as linear constraints on the closed loop map. One thus incorporates time and frequency domain constraints in the $\ell_1$ problem by augmenting these constraints to the interpolation/rank constraints. The augmented $\ell_1$ optimal control problem is then solved using the Delay Augmentation Algorithm.

3.1.1 $\ell_1$-Norm Constraints

The nominal $\ell_1$ performance problem is defined as:

$$\nu^\circ := \inf_{\text{stabilizing } K} (\sup_w \| \Phi w \|_{\infty}) = \inf_{\text{stabilizing } K} \| \Phi \|_1. \quad (3.1)$$

It is known [3] that the feasible set of closed loop maps, $\Phi$, is characterized by a set of linear equations. The nonlinearity of the norm can be avoided using a standard change of variables from linear programming. Let $\Phi = \Phi^+ - \Phi^-$ where $\Phi^+$ and $\Phi^-$ are sequences of $n_z \times n_w$ matrices with non-negative entries ($n_w$ is the number of exogenous inputs and $n_z$ is the number of regulated outputs). We then replace the
\[ \ell_1 \text{ norm of } \Phi \text{ by} \]

\[
\max_i \sum_{j=1}^{n_w} \sum_{t=0}^{\infty} (\phi_{ij}^+(t) + \phi_{ij}^-(t))
\]

which is linear in \((\Phi^+, \Phi^-)\). This expression is equal to the norm only if, for every \((i, j, t)\), either \(\phi_{ij}^+(t)\) or \(\phi_{ij}^-(t)\) is zero. The problem 3.1 can be restated as follows:

\[
\nu_0 = \inf_{\nu, \Phi^+, \Phi^-} \nu \quad \text{subject to} \quad \sum_{j=1}^{n_w} \sum_{t=0}^{\infty} (\phi_{ij}^+(t) + \phi_{ij}^-(t)) \leq \nu \quad \forall \ i = 1, \ldots, n_z
\]

This change of variables lead to a compact representation of the \(\ell_1\) norm constraints on the closed loop map, by defining an operator \(A_{\ell_1} : \ell_1^{n_z \times n_w} \rightarrow \mathbb{R}^{n_z}\) such that

\[
(A_{\ell_1} \Phi)_i = \sum_{j=1}^{n_w} \sum_{t=0}^{\infty} \phi_{ij}(t) \quad \text{for } i = 1, \ldots, n_z,
\]

and a vector with all elements equal to one, \(1 \in \mathbb{R}^{n_w}\). It follows that

\[
\sum_{j=1}^{n_w} \sum_{t=0}^{\infty} (\phi_{ij}^+(t) + \phi_{ij}^-(t)) \leq \nu \quad \text{for } i = 1, \ldots, n_z \iff A_{\ell_1}(\Phi^+ + \Phi^-) \leq 1\nu. \quad (3.3)
\]

The operator \(A_{\ell_1}\) is called norm operator since it replaces the \(\ell_1\) norm constraints.

The discussion in the following subsections allow us to write the time and frequency domain constraints in the same form as norm operators.

### 3.2 Performance with fixed inputs - Time Domain Constraints

In many control problems, the time specifications could be expressed in terms of maximum allowable overshoot or undershoot. Alternatively, the specification could take the form of any other prespecified track to follow. The controller is designed so
that the time response satisfies

\[ g_l(t) \leq z(t) \leq g_u(t) \]

where \( g_l(t) \) and \( g_u(t) \) are the lower and upper bounds respectively and \( z(t) \) represents an output variable of the closed loop system. \( z \) is feasible if \( z = \Phi w_f \) for some feasible \( \Phi \), where \( w_f \) is the specific fixed input. Examples of traditional importance in control design are: Maximum overshoot, settling time and maximum deviation. These kinds of specifications result in linear constraints on the closed loop map. All of the constraints can be combined to form a linear operator of the same form as \( A_{t_1} \), called \( A_{\text{time}} \) such that

\[ A_{\text{time}} \Phi \leq b_{\text{time}} \]  

(3.4)

for some fixed \( b_{\text{time}} \).

The Matlab code of the program to handle Time Domain Constraints is included in Section A.1.

### 3.3 \( \mathcal{H}_\infty \)-Norm Constraints - Frequency Domain Constraints

The \( \mathcal{H}_\infty \) norm is considered as an upper bound on the amplitude gain over persistent sinusoidal signals (i.e. bounded power signals) and not just a minimization of the energy of the regulated outputs, as defined in the following performance problem:

\[
\inf_{K \text{ stabilizing}} \left( \sup_w \| \Phi w \|_2 \right) = \inf_{K \text{ stabilizing}} \| \Phi \|_\infty.
\]

This interpretation of the \( \mathcal{H}_\infty \) norm leads itself to analysis via traditional loopshaping.

In the MIMO case, frequency constraints are defined as follows: given a closed loop map \( \Phi \), the maximum singular value of \( \hat{\Phi}(e^{i\omega_0}) \) is given by

\[
\sigma_{\text{max}}[\hat{\Phi}(e^{i\omega_0})] = \max_{u,v} \{ \Re[u^* \hat{\Phi}(e^{i\omega_0})v] \mid |u|_2 = |v|_2 = 1 \}. \tag{3.5}
\]
Say that the design specification is such that the $\mathcal{H}_\infty$ norm of $\Phi$ needs to be bounded from above by $\gamma > 0$, i.e.,

$$\sigma_{\text{max}}[\hat{\Phi}(e^{j\omega_0})] \leq \gamma \quad \text{for all } \omega_0 \in [0, 2\pi). \quad (3.6)$$

To approximate these constraints, first $N$ samples from the unit circle are obtained. Then, for each sample, Equation 3.5 is approximated by a polytope. This kind of approximation can yield a large set of linear inequalities. Usually a small number of constraints is sufficient to alter a given design.

For the frequency domain constraints, an approximation of SISO $\mathcal{H}_\infty$-norm constraints can be used, as defined also in [3]. The minimization of the $\mathcal{H}_\infty$-norm will give us the maximum upper bound on the amplitude gain in the frequency response of a system. This could be useful for the loopshaping ideas when designing a controller. The approximation in the SISO case is then as follows: Let $H$ be a scalar complex number, and let $H = H_R + iH_I$. If we impose a bound on the magnitude of $H$, we have

$$|H| \leq \gamma \iff H_R \cos \theta + H_I \sin \theta \leq \gamma \quad \text{for all } \theta \in [0, 2\pi).$$

Let $\hat{\Phi}(\lambda) = \sum_{k=0}^{\infty} \phi(k) \lambda^k$ be a SISO transfer function with impulse response $\{\phi(k)\}$. Define $\Phi_R(\omega_n) = \Re[\hat{\Phi}(e^{i\omega_n})]$ and $\Phi_I(\omega_n) = \Im[\hat{\Phi}(e^{i\omega_n})]$, where $\omega_n$ are samples of the unit circle. A set of linear constraints that approximates the $\mathcal{H}_\infty$-norm constraints is:

$$\Phi_R(\omega_n) \cos \theta_m + \Phi_I(\omega_n) \sin \theta_m \leq \gamma \quad \text{where} \quad \begin{cases} \omega_n \in [0, 2\pi), n = 1, \ldots, N \\ \theta_m \in [0, 2\pi), m = 1, \ldots, M \end{cases} \quad (3.7)$$

Note that the evaluation of the $\lambda$-transform at some frequency is a linear operation on $\Phi$:

$$\Phi_R(\omega_n) = \sum_{k=0}^{\infty} \phi(k) \cos(k\omega_n), \quad \Phi_I(\omega_n) = \sum_{k=0}^{\infty} \phi(k) \sin(k\omega_n).$$
And combined with Equation 3.7 gives

\[ \sum_{k=0}^{\infty} \phi(k) \cos(k\omega_n - \theta_m) \leq \gamma \quad \text{where} \quad \begin{cases} \omega_n \in [0, 2\pi), n = 1, \ldots, N \\ \theta_m \in [0, 2\pi), m = 1, \ldots, M \end{cases} \quad (3.8) \]

where \( \omega_n \) are the \( N \) samples on the unit circle at which \( \phi(\omega) \) is to be constrained and \( \theta_m \) is the angle of each of the \( M \) sides of the polytope inscribed within the unit circle at each sample.

A different approach to approximate the MIMO \( \mathcal{H}_\infty \)-norm constraints on the closed loop map \( \Phi \), could be as a linear combination of the inputs and outputs to constrain.

We can also represent the MIMO \( \mathcal{H}_\infty \)-norm constraints by selecting the SISO transfer functions \( \Phi_{i,j} \) (input \( j \) to output \( i \)) where the constraints are to be applied. In Equation 3.8, \( \phi(k) \) can be written as:

\[ u^*\Phi(k)v \quad (3.9) \]

where \( u \) and \( v \) are arbitrary vectors of unit length. If the vectors \( u \) and \( v \) have more than one entry different from zero, we will be constraining a linear combination of outputs and/or inputs respectively. If we combine Equations 3.8 and 3.9, the new set of linear constraints will be:

\[ \sum_{k=0}^{\infty} [u^*\Phi(k)v] \cos(k\omega_n - \theta_m) \leq \gamma \quad \text{where} \quad \begin{cases} w_n \in [0, 2\pi), n = 1, \ldots, N \\ \theta_m \in [0, 2\pi), m = 1, \ldots, M \\ \text{all } u, v \text{ with } \|u\|_2 = \|v\|_2 = 1 \end{cases} \quad (3.10) \]

The linear constraints in Equations 3.8 and 3.10 can be arranged (each one by separate) in a linear operator (infinite matrix), \( \mathcal{A}_{\mathcal{H}_\infty} : \ell_1^{n_x \times n_y} \rightarrow \mathcal{R}^{NM} \) such that

\[ \|\hat{\Phi}\|_\infty \leq \gamma \implies \mathcal{A}_{\mathcal{H}_\infty} \hat{\Phi} \leq \gamma 1. \quad (3.11) \]

If the samples \( w_n \) and \( \theta_m \) are dense, then the right hand side of the Equation 3.11
approximates the left hand side.

The Matlab code of the program to handle Frequency Domain Constraints is included in Section A.2.

3.4 Exponentially Weighted Norm Problem

The original software available to design controllers with the $\ell_1$ norm methodologies cannot handle plants with poles on the unit circle. The software will only consider the unstable poles (inside the unit circle) in order to obtain the interpolation conditions that will determine the linear constraints for the linear program to solve. Different ways of perturbing the plant were used in order to include the poles on the unit circle (i.e. slightly perturbing the poles by subtracting an $\epsilon \ll 1$). The exponentially weighted norm problem can be used to automatically include the poles on the unit circle.

The weighted norm problem is defined as

$$\|F\|_{(r)} = \sum_{t=0}^{\infty} r^t |f(t)|$$

for $r > 1$. Notice that not all functions in $\ell_1$ has a bounded $\| \cdot \|_{(r)}$ norm. Let the space $\ell_1(r)$ denote the subspace of $\ell_1$ that has elements with bounded $\| \cdot \|_{(r)}$ norm, and equipped with the $\| \cdot \|_{(r)}$ norm. The elements in this space must decay faster than $r^{-t}$ (hence, the associated $\lambda$-transform is analytic in the disc of radius $r$).

Then the minimization problem becomes:

$$\nu_r = \inf_{K \text{ stabilizing}} \|\Phi\|_{(r)} \quad (3.12)$$

Also by invoking the standard parametrization of all stabilizing controllers it is possible to rewrite Equation 3.12 as

$$\nu_r = \inf_{Q \in \ell_1(r)} \|H - UQ\|_{(r)} \quad (3.13)$$

31
where $\hat{U}$ may have zeros on the unit circle, but not on the circle of radius $r$. Denote all the zeros in the disc of radius $r$ by $a_1, ..., a_N$, and define the feasible space $S$ as

$$S = \{ R \in \ell_1(r) \mid R = UQ, \ Q \in \ell_1(r) \}.$$  

The dual problem of the primal defined in Equation 3.13 can be stated as a finite dimensional linear program:

$$\nu_r = \sup_{a_1, ..., a_N} \sum_{i=1}^{N} \alpha_i H(a_i)$$

subject to

$$\left| \sum_{i=1}^{N} \alpha_i \left( \frac{a_i}{r} \right)^k \right| \leq 1, \ \forall k = 0, 1, ..., K \text{ and } \alpha_i \in \mathbb{R}.$$  

It can also be shown that

$$\lim_{r \to 1} \nu_r = \nu_1$$

where $\nu_1$ denotes the solution of the standard $\ell_1$ problem.

### 3.4.1 Software Implementation

In the standard $\ell_1$ problem we solve the linear program with the norm operator $A_{\ell_1}$. Similarly, we can solve the exponentially weighted norm problem by modifying the norm operator: $A_{\ell_1} \rightarrow A_{\ell_1(r)}$, and the weighted problem will be equivalent to the following linear program

$$\nu_r^\circ = \inf_{K \text{ stabilizing}} \| \Phi \|_1(r)$$

subject to

$$A_{\ell_1(r)} \Phi \leq \nu_r 1$$

$$\Phi \text{ feasible}$$

The effect of modifying the $\ell_1$ norm can be observed in Figure 3-1, where the poles on the unit circle are now considered unstable in the $r$-sense since they are inside the
circle or radius $r$. Interpolation points on the unit circle now are strictly inside the disc of radius $r$.

![Diagram](image)

Figure 3-1: Exponentially Weighted Problem

The current version of the $\ell_1$ software has the option of selecting exponential weights. An example to illustrate the use of this methodology of handling the interpolations on the unit circle is covered in Chapter 4.

### 3.5 Mixed Performance Objectives

To guarantee that a closed loop map satisfies multiple constraints, we augment all the linear operators in one operator constraint. Notice that different linear constraints can be defined for different closed loop maps $\Phi = T_{z_i w_j}$, i.e., on the map between the $i$th input and the $j$th output. The augmented set of conditions may not have a feasible solution, indicating that there does not exist a controller that can meet all the stated specifications. A typical augmented operator will have the form:

\[
\begin{pmatrix}
A_{\ell_1} & A_{\ell_1} \\
A_{H_{\infty}} & -A_{H_{\infty}} \\
A_{\text{time}} & -A_{\text{time}}
\end{pmatrix}
\begin{pmatrix}
\Phi^+ \\
\Phi^-
\end{pmatrix}
\leq
\begin{pmatrix}
\nu 1 \\
\gamma 1 \\
b_{\text{time}}
\end{pmatrix}
\]  

(3.15)

Including exponential weights, the augmented operator in Equation 3.15 will be
modified to

\[
\begin{pmatrix}
\mathcal{A}_{\ell_1}(r) & \mathcal{A}_{\ell_1}(r) \\
\mathcal{A}_H & -\mathcal{A}_H \\
\mathcal{A}_{\text{time}} & -\mathcal{A}_{\text{time}}
\end{pmatrix}
\begin{pmatrix}
\Phi^+ \\
\Phi^-
\end{pmatrix}
\leq
\begin{pmatrix}
\nu_r 1 \\
\gamma 1 \\
\beta_{\text{time}}
\end{pmatrix}
\]  

(3.16)

If the plant is known to lie in a set, then part of the objectives is to guarantee robust stability. This is of course given by some norm constraints.

### 3.6 Design Examples

Several problems will be discussed in this subsection. One of them will be just an academic example and some others will be more realistic and complex. The important idea here is to see the effect of the constrained \( \ell_1 \) norm problem versus the original, and to make sure that the time and/or frequency responses follow the desired specifications.

It will be interesting to analyze the changes in the order of the compensator, the structure of the solution and the norm obtained in each design. Of course it will be important to see if the new designs are feasible or not.

#### 3.6.1 Academic Example

**Problem 1.** Consider the following problem of Figure 3-2 where:

\[
P(z) = \left( \frac{1}{z^2 - \frac{1}{2}z + \frac{5}{2}} \right)
\]

and

\[
W(z) = 1.5 \times \left( \frac{z}{z - \frac{7}{10}} \right)
\]

Notice that the plant is unstable. I will analyze how the closed loop system behaves after constraining either the time response or frequency response, or both. The input signal will be either a step or an impulse.

This problem has 1 input, 2 outputs and 3 states. The Figure 3-2 only shows the
weighted output, and the second output is the control signal \( u \).

The time domain constraints will be applied to the weighted output and the frequency domain constraints will be applied to both signals as follows:

**Time Domain Constraints applied to the Weighted Output**

Figure 3-3 shows the time response of the system to a step input (solid line) and to an impulse (dashed line). Note that if we set a constraint such that the maximum allowable overshoot and undershoot is \( |z| \leq 5 \) for the case of the step input, we will need to design a new compensator \( K \) to meet this requirement. The new problem would be

\[
\nu^o = \inf_{K \text{ stabilizing}} \left\| \begin{bmatrix} U \\ WC \end{bmatrix} \right\|_1 \\
\text{subject to}
\]

\[
WC \text{ feasible}
\]

\[
|z(t)_{\text{step}}| \leq 5 \text{ for all } t
\]

where \( WC \) is the weighted output. In this particular problem, we constrained the time response for all time, but we may be interested in constraining only a certain interval of the time response (i.e. from second 5 until 10 and/or then from 15 until
Figure 3-3: Response to a Step Input (solid) and to an Impulse (dashed) Without Constraints

50). Also from Figure 3-3 we can see that the impulse response can be constrained in the interval from 1 until 10 to be below 1.5, and the problem would be

\[ \nu^0 = \inf_{K \text{ stabilizing}} \left\| \begin{array}{c} U \\ WC \end{array} \right\|_1 \]

subject to

\[ WC \text{ feasible} \]

\[ |z(t)_{imp}| \leq 1.5 \text{ for } t \in (1,10). \]

Figure 3-4 (a) shows both constrained responses. In both cases the response was within the desired boundaries during the required time. Notice in Figure 3-4 (b) that the response blows away after the time \( t = 10 \) sec., where no constraints were added.

The numerical results are shown in Table 3.1 at the end of Problem 1.
Figure 3-4: Time Responses: (a) With Constraints to a Step Input (solid) and to an Impulse (dashed); (b) With Constraints (until $t = 10s$) to an Impulse

$H_\infty$ Constraints Applied to Both Outputs

In this case, we will constrain both signals separately. Figures 3-5 and 3-6 shows both frequency responses without constraints (dashed lines).

Let us say that we want to constrain the control signal to be less than 15 db. for the frequency range from 0.3 rad/sec. to 2.5 rad/sec.. We will just place 8 points in that interval in order not to increase the difficulty of the computation. Figures 3-5 (a) and (b) show both the constrained (solid) and unconstrained (dashed) responses along with the points that were used to constrain the response ("+"). Observe how the response may go above the limit of 15 db. but not in the points we marked. Figure 3-5 (b) is a zoom from the range of frequencies where we added the constraints. If we were interested in reducing the response where it goes beyond 15 db., we would need to include more points in the design, and of course it would be harder to compute.

Now consider the case where we want to constrain the low frequency region in the
Figure 3-5: Frequency Responses: (a) With Constraints (solid) and Without Constraints (dashed); (b) Zoom of (a)

weighted output, i.e. to be less than 16 db. in the range from 0.0001 rad/sec. until 0.01 rad/sec. Again we include some points (5 in this case) equally spaced in that range of frequencies and compute the compensator. Notice in Figure 3-6 the effect of the constraints in the frequency response.

The numerical results are also shown in Table 3.1.

**Numerical Results of Problem 1**

The following data were used in all the simulations: a) Number of Delays = 10; b) Maximum length of the closed loop map (Φ) = 31. Also all designs turned out to be stable and feasible. If we impose stronger constraints to the problem, we may obtain an infeasible solution.

Table 3.1 shows the numerical results for all simulations, with and without constraints. The number of rows of the constraint matrix is given by adding both inequalities and equalities. The upper and lower bounds are given in the columns of $\mu^o$ and $\mu_o$ respectively. Notice how the norms augmented due to the increased number
of constraints (column 2). In almost all cases the gap between upper and lower norms was small except in the case when we impose a hard constraint to an impulse response. Recall the discussion about the Figure 3-4, where the constrained response increased in magnitude after $t = 10$ seconds. As expected, the order of the compensator and the structure of the solution increased for the constrained problems.

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Ineq</th>
<th>Eq</th>
<th>$\mu_0$</th>
<th>$\mu^c$</th>
<th>Ord($K$)</th>
<th>$\text{len}(\Phi_N^c)^T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without</td>
<td>2</td>
<td>13</td>
<td>11.6602</td>
<td>11.6602</td>
<td>3</td>
<td>(5 6)*</td>
</tr>
<tr>
<td>Time (to a Step)</td>
<td>62</td>
<td>16</td>
<td>29.8615</td>
<td>29.8637</td>
<td>12</td>
<td>(10 9)$\dagger$</td>
</tr>
<tr>
<td>Time (to an Imp.)</td>
<td>22</td>
<td>13</td>
<td>302.3303</td>
<td>507.2672</td>
<td>10</td>
<td>(13 12)$\dagger$*</td>
</tr>
<tr>
<td>Freq. (Control)</td>
<td>114</td>
<td>13</td>
<td>13.1027</td>
<td>13 1397</td>
<td>7</td>
<td>(9 30)*</td>
</tr>
<tr>
<td>Freq. ($WC$)</td>
<td>66</td>
<td>13</td>
<td>12.55</td>
<td>12.55</td>
<td>5</td>
<td>(7 8)*</td>
</tr>
</tbody>
</table>

Table 3.1: Results Problem 1. * = reordered outputs; $\dagger$ = did not improve reordering the outputs
3.6.2 Realistic Examples

The following two problems were addressed in [6] and [3] and we will pursue the same design objectives but using the new functions shown in Appendix A to handle frequency and time domain constraints. For the full description of the problems refer to [3].

(Note: Remember that the idea of the problems in this Chapter is to show how we can include different specifications in the \( \ell_1 \) problem design. Chapter 4 will cover a full example were we will not just include all kind of specifications but we will try to obtain the best possible design using \( \ell_1 \) methodologies.)

**Problem 2. Pitch Axis Control of the X29 Aircraft.**

The X29 aircraft possess an interesting control problem due to its revolutionary forward-swept wing design. With such a configuration, the center of gravity lies behind the aerodynamic center of pressure, rendering the aircraft statically unstable. Consequently, a control system has to actively stabilize the aircraft during flight. These type of wings have some desirable aerodynamic characteristics such as better maneuverability and reduced drag when compared with the more classical wing design.

The simplified model of the aircraft is approximately represented by the following continuous time SISO plant:

\[
P(s) = \frac{(s + 3)}{(s + 10)(s - 6)} \times \frac{20}{(s + 20)} \times \frac{(s - 26)}{(s + 26)}
\]

where \( s \) is the Laplace variable. The first factor corresponds to a simplified model of the pitch dynamics of the airplane (rigid body) flying at a low altitude and with an air speed of approximately 0.9 Mach. The second factor corresponds to a model of the equivalent actuators. The third factor lumps the equivalent low frequency phase
lag introduced by the dynamics that are neglected in deriving the reduced model. For the discrete model, the sampling period was $T_s = \frac{1}{30}$ seconds. The design objective that we are interested in is: $\ell_1$ performance with frequency domain constraints.

### $\ell_1$ Performance Objectives with Frequency Domain Constraints

Consider the following minimization problem:

$$
\nu^* = \inf_{K \text{ stabilising}} \left\| \begin{array}{c}
W_1 KS \\
W_2 S
\end{array} \right\|_1
$$

subject to

$$
\|\hat{W}_2 \hat{S}\|_\infty \leq \gamma_{\mathcal{H}_\infty}
$$

where $S := (I - PK)^{-1}$ denotes the discrete-time sensitivity function. For the discrete equivalent of Equation 3.19 we used a zero order hold at the plant input and a synchronized sampling of the plant output. The two weights are generally chosen to reflect the tradeoffs between low frequency disturbance rejection and the control effort. In this case the weights are $\hat{W}_1 = 0.01$ and $\hat{W}_2 = \frac{(s+1)}{(s+0.001)}$. Notice that we are including $\mathcal{H}_\infty$-norm constraints on the transfer function $\hat{W}_2 \hat{S}$. This situation may arise if the specifications include tracking performance for sinusoidal type inputs.

From Figure 3-7, we see that the maximum peak in the unconstrained response for $\hat{W}_2 \hat{S}$ is 11.1725 db (3.619) at a high frequency (dashed). Let us assume that the design specifications are such that the frequency response should never exceed $\gamma_{\mathcal{H}_\infty} = 8.299$ db (2.6). We included 6 points in the range of frequencies from 0.3 until 1.5 rad/sec. The same overshoot effect occurred between the chosen points as in Problem 1 (see Figure 3-7). If we increase the number of points in frequency we may attain the desired $\mathcal{H}_\infty$ constraints but the number of this points need not be too dense. Table 3.2 shows the results of this problem for 9 delays. The added constraints demand more degrees of freedom from the controller which increases its order from $6^{th}$ to $10^{th}$. Even reordering the outputs, the structure of the solution did not improve.
Figure 3-7: Frequency Responses: (a) With Constraints (solid) and Without Constraints (dashed); (b) Zoom of (a). $\gamma_{\infty} = 2.6$ (8.299 db.)

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Ineq</th>
<th>Eq</th>
<th>$\mu_\infty$</th>
<th>$\bar{\mu}$</th>
<th>Ord($\mathcal{K}$)</th>
<th>len($\Phi_N^T$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without</td>
<td>2</td>
<td>12</td>
<td>4.0513</td>
<td>4.3142</td>
<td>6</td>
<td>(6 9)</td>
</tr>
<tr>
<td>Freq. ($W_2S$)</td>
<td>82</td>
<td>12</td>
<td>4.4033</td>
<td>4.8148</td>
<td>13</td>
<td>(12 9)†</td>
</tr>
</tbody>
</table>

Table 3.2: Results Problem 2. † = did not improve reordering the outputs

**Problem 3. Flexible Beam.**

A flexible beam has one end pinned to the shaft of a big torque DC motor and the other end free. A position measurement of the free end of the beam is available. The objective is to design a controller to regulate the position of the tip of the beam.

For the design we will use a reduced fourth order continuous model:

$$\hat{P}(s) = \frac{-6.475s^2 + 4.0302s + 175.77}{s(\bar{5}s^3 + 3.5682s^2 + 139.5091s + 0.0929)}.$$  

It represents the rigid body motion and the first flexible mode of the beam. The plant
is unstable and non-minimum phase. The rigid body motion is slightly damped by the back electromagnetic force in the DC motor, and the non-minimum phase character of the plant is a direct consequence of the uncollocated sensor. The problem in consideration is as follows:

\textbf{\( \ell_1 \) Performance Objective with Maximum Deviation Constraints}

The specifications of the problems are: obtain the best tracking performance in the \( \ell_1 \) sense, subject to the magnitude of the control signal not exceeding a given value \( U_{\text{max}} \), when the input disturbance is a unit step. The optimization problem is then

\[
\nu^* = \inf_{K \text{ stabilizing}} \left\| \begin{array}{c} W_1 S \\ W_2 KS \end{array} \right\|_1
\]

subject to

\[
\| KS w_{\text{step}} \|_{\infty} \leq U_{\text{max}}
\]

where \( W_1 \) and \( W_2 \) are chosen to reflect the tradeoffs between tracking performance and control effort, and the expected spectral characterization of the exogenous disturbance. Again \( w_{\text{step}} \) refers to a unit step. For simplicity, let \( W_1 = W_2 = 1 \).

The unconstrained step response has a maximum value of 0.5 (dashed line in Figure 3-8) and suppose we want to bound the control signal with the vector \( U_{\text{max}} \) (+ marks in Figure 3-8). Notice how the control signal (solid line in Figure 3-8) follows exactly the bound given by \( U_{\text{max}} \), and no overshoot is present. Table 3.3 shows the solution of the problem for 30 delays.

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Ineq</th>
<th>Eq</th>
<th>( \mu_0 )</th>
<th>( \mu^* )</th>
<th>Ord(( K ))</th>
<th>len(( \Phi_N^o )^T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without 2 31 2.0069 2.0087 13 (10 23)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Time (( u(t) )) 62 31 2.0107 2.0150 19 (26 26)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.3: Results Problem 3.
Figure 3-8: Time Response of the Control Signal with Constraints (solid), without (dashed) and the Vector $U_{\text{max}}$. 
Chapter 4

Case Study for the EOS Satellite

In this chapter we will apply the systematic procedure developed in Chapter 3 to design a compensator for the EOS Satellite. Two interesting factors motivated the choice of this example: first, it contains mixed performance constraints, the plant description has poles on the unit circle and modal frequencies are uncertain; and second, several controllers were designed using different methodologies and comparisons will be made with the previous results.

4.1 The EOS Satellite

The objective of this case study is to design a controller as a contingency against uncertainty at the solar array frequencies and disturbance torques being larger than currently projected.

The solar array (SA) flexible modal frequencies can change for several reasons. The major source of uncertainty is due to the inability for the developers to reproduce the zero-g environment and actually measure the frequencies on the ground. The frequencies are predicted from finite-element models. When the array rotates, the roll modes are transferred to yaw with lower frequencies and vice-versa. The solar array rotates to track the sun, which can cause about 10% uncertainty in the modal frequencies. Also, a new analysis indicates that modal frequencies can be reduced if there is a bias momentum on the satellite. Jet firings for orbital maneuvers will
change of the mass and inertial of the spacecraft. Loosened blanket tension can also cause a reduction in frequency. The control system is designed before the solar array and the rest of the satellite design is final, so an important source of uncertainty is if the solar array vendor does not meet the specifications, which may cause the frequencies to be lower.

At the orbital rate, the disturbances are gravity gradient and aerodynamic drag torques. There are many sources of transient disturbances: high gain antenna, high speed tape recorders, and from the motion of the on-board scientific instruments. The instruments, along with other mechanisms such as the solar array and the high-gain antenna, produce force and torque disturbances to the spacecraft which induces undesirable attitude motion. Thermal snap can occur twice on each orbit as the satellite goes from into or out of the sunlight. For a more detailed and technical description of the EOS Satellite see [1], [2] and [14].

The satellite which the controller will control is shown in Figure 4-1 and is described in Table 4.1. Section 4.1.1 describes the model of the satellite of this study and Section 4.1.2 provides with the major requirements to be accomplished. We are interested in designing a digital controller for the roll axis of the spacecraft.

4.1.1 Satellite Model

The structural dynamics as defined by NASTRAN model EOS 7 are detailed in Figure 4-2. The two structural frequency responses are for: i) continuous plant model with eight important modes, and ii) discrete plant model with eight important modes. The modal admittance variations and natural frequencies of the modes are detailed in Table 4.2.

The simplified satellite model consists of a rigid body, an integrator and 8 flexible modes. The continuous plant description is as in Figure 4-3. All flexible modes (FM) are second order transfer functions of the form

\[ FM_k(s) = \frac{\phi_k^2 s}{s^2 + 2\xi_k \omega_k s + \omega_k^2} \]  \hspace{1cm} (4.1)
Figure 4-1: EOS Satellite Configuration
<table>
<thead>
<tr>
<th>Parameter/Component</th>
<th>Value/Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.- Roll Axis Inertia</td>
<td>13,325 slug – ft² Roll axis only considered. Roll is the worst case axis and has the least inertia. Cross-coupling of the axes is small.</td>
</tr>
<tr>
<td>2.- Flexible Modes</td>
<td>Fundamental SA mode is 0.22 Hz. Modal admittances ((\phi)) are detailed in Table 4.2. SA tension is 35 lb.</td>
</tr>
<tr>
<td>3.- Location of Sensors</td>
<td>Assumed collocated. Celestial sensors, gyros, and reaction wheels assumed rigidly connected. Allows use of (\phi^2) for admittances (simplifies analysis)</td>
</tr>
<tr>
<td>and Actuators</td>
<td></td>
</tr>
<tr>
<td>4.- Modal Damping</td>
<td>C/Cc = 0.001 for all modes.</td>
</tr>
<tr>
<td>5.- SA Thermal Snap</td>
<td>0.1 in-lb for 10 sec. could cause about 23 asec jitter based on an early SA thermal/dynamic model.</td>
</tr>
<tr>
<td>Dist. Profile</td>
<td></td>
</tr>
<tr>
<td>6.- Reaction Wheel</td>
<td>Max. Torque: 2.64 in-lb.</td>
</tr>
<tr>
<td>Characteristics</td>
<td></td>
</tr>
<tr>
<td>7.- Sampling Time</td>
<td>0.512 sec.</td>
</tr>
</tbody>
</table>

Table 4.1: EOS Satellite Characteristics
where \( \omega_k = 2\pi f_k \)

\[
\begin{array}{c}
\sum_{k=1}^{8} FM_k \\
\frac{1}{Js} \\
\end{array}
\]

\[
\rightarrow W \rightarrow \frac{1}{s} \rightarrow Z
\]

4.1.2 Major Requirements

The major requirements in this design are as follow:

1. Improve peak to peak roll response to an impulse of height 0.1 in-lb for 10 sec. by a factor of 5 over the current baseline response (baseline has 22.8 arcsec. peak to peak response).

2. Reject sinusoidal torques at 0.001 rad/s orbital rate at least as well as baseline (2.8 arcsec/in-lb). This is a characteristic of current PDR controller and serves
### Table 4.2: Modal Admittances and Natural Frequencies

<table>
<thead>
<tr>
<th>( \phi_k )</th>
<th>( \phi_k' )</th>
<th>( f_k ) (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.502504013583195e-03</td>
<td>6.26252633800e-06</td>
<td>0.221863042671016</td>
</tr>
<tr>
<td>1.996977720456590e-03</td>
<td>3.98792001600e-06</td>
<td>0.346979643170415</td>
</tr>
<tr>
<td>2.48319461172946e-03</td>
<td>6.16625473200e-06</td>
<td>0.452696931688918</td>
</tr>
<tr>
<td>1.45719398265512e-03</td>
<td>2.12341430300e-06</td>
<td>0.452696931688918</td>
</tr>
<tr>
<td>4.73894266682561e-04</td>
<td>2.24575776000e-07</td>
<td>1.441548307352709</td>
</tr>
<tr>
<td>6.657968624137545e-04</td>
<td>4.43285462000e-07</td>
<td>2.936731813527805</td>
</tr>
<tr>
<td>1.744488486921023e-03</td>
<td>3.04324008100e-06</td>
<td>7.127517929322115</td>
</tr>
<tr>
<td>7.18538369845124e-03</td>
<td>5.16297389140e-05</td>
<td>20.451315475352060</td>
</tr>
<tr>
<td>1.621575281024597e-02</td>
<td>2.62950639203e-04</td>
<td>25.126453072489207</td>
</tr>
</tbody>
</table>

as a specific requirement for the contingency controller.

3. Avoid reaction wheel saturation (max. torque 2.64 in-lbs). The control torque is produced using four reaction wheels.

4. Satisfy gain and phase margins of ± 3db and 30 degrees respectively.

5. Maintain stability with up to 15% variation of modal frequencies.

Currently, the thermal snap is not considered a problem and the 0.1 in-lb, 10 second torque should be considered a typical disturbance, not necessarily due to thermal snap. The uncertainty in the plant is reflected in the parameter \( \omega_k \) at each flexible mode. Section 4.2.1 shows how this uncertainty can be embedded in a structured uncertainty set.

Note that for this case study, the peak to peak value will denote the value of the difference of the maximum value minus the minimum value.

### 4.2 Problem Setup

The setup corresponds to a standard disturbance rejection problem formulated as linear fractional transformation from the disturbance input (exogenous input \( w \)) to
the regulated outputs \( z \), with the controller, \( K \), in the lower loop (see Figure 4-4). The problem is represented via an LTI finite dimensional operator, \( T \), that maps the roll disturbance \( w \) (dimension 1), and the control torque \( u \) (dimension 1), to the regulated output vector \( z \) (dimension 3: roll angle, roll rate and control torque), and the measurement vector \( y \) (dimension 2: roll angle and roll rate).

![Figure 4-4: Standard Problem](image)

In order to prevent exciting high frequency unmodelled dynamics, we need to include a low pass filter at the plant input, as shown in Figure 4-5, that has a transfer function of the form

\[
F(s) = \frac{6\pi}{s + 2\pi 3} \quad (4.2)
\]

The filter in Equation 4.2 also has the function of an anti-aliasing filter and to provide the desired roll off of the transfer function from \( w \) to \( u' \).

![Figure 4-5: Discretized Plant-Filter T](image)

The discrete version of the plant of Table 4.2 with the filter of Equation 4.2 was obtained by using a zero-order hold on the inputs and sample time of 0.512 seconds. Let the discretized plant-filter of Figure 4-5 be \( T \) in Figure 4-4, and now we want to
find a stabilizing controller such that the $\ell_1$ norm of the transfer function from the input disturbance, $w$, to the pre-filtered control torque, $u$, the roll angle, $z_1$, and roll rate, $z_2$. That is,

$$\inf_{K \text{ stabilizing}}\begin{bmatrix} T_{z_1w} \\ T_{z_2w} \\ T_{uw} \end{bmatrix}_1$$

(4.3)

### 4.2.1 Parametric Uncertainty in the Plant

The controller designed for the plant described in Section 4.2 should be robust against parameter variations. The uncertain parameters in the plant are present in the flexible modes, and each flexible mode will experience variations in the value of $\omega$ independently from each mode. In this way, we can represent the uncertainty of the plant as $\Delta$, with the following structure:

$$\hat{\Delta} = \{\Delta = \text{diag}(\Delta_1, ..., \Delta_k)\}$$

(4.4)

where $\Delta_k$ represent the uncertainty at the $k^{th}$ flexible mode. In the sequel, the general description of the uncertainty at each mode will be presented.

**Structured Uncertainty Description**

The transfer function for the $k^{th}$ flexible mode, given in Equation 4.1 can be written as

$$FM_k(\omega_k, s) = \frac{\phi_k^2 s}{s^2 + 2\xi \omega_k s + \omega_k^2}$$

(4.5)

where $s$ is the Laplace variable and $\omega_k$ is the uncertain parameter. Now I will introduce some new notation. For simplicity the subindex $k$ will be omitted and rather than letting $\omega$ be the uncertain variable that is bounded from above and below ($\pm 15\%$ of nominal value $\hat{\omega}$), $\delta \in \mathbb{R}$ will now denote an uncertain variable in the nominal model that can take on any value between $+1$ and $-1$, that is

$$|\delta| < 1.$$
There is no loss of generality in switching the uncertain variables in the model, as it is simply a matter of notation to consider the $\delta$ as the uncertain variable rather than the actual physical variable. Specifically, each uncertain variable will be represented as

$$\omega = \dot{\omega} + \delta q \quad \text{with} \quad |\delta| < 1$$

(4.6)

where $q$ is a scalar variable that quantifies the amount of error in $\omega$. By letting $\delta$ take on its maximum or minimum possible value, it is simple to evaluate $q$ from Equation 4.6 and the upper and lower bounds of the uncertain parameter $\omega$

$$q = \max\{\bar{\omega} - \dot{\omega}, \dot{\omega} - \omega\}$$

(4.7)

In this way, the level of error in the uncertain variable is now directly reflected to $q$.

In using this kind of description of the uncertainty, we want to be able to define a fictitious input, $v$, and output, $p$, for the state space model of the flexible mode

$$\dot{x}(t) = A(\omega)x(t) + Bw(t)$$

$$y(t) = Cx(t)$$

(4.8)

where

$$A(\omega) = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{bmatrix} ; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} ; \quad C = \begin{bmatrix} 0 & \phi^2 \end{bmatrix}$$

(4.9)

so that

$$v = \delta p$$

(4.10)

Note that in this particular case, $\Delta = \delta$. Before defining the fictitious input and output, it is necessary to switch to the $\delta$ notation for the uncertain variable. This can be done by substituting the $\delta$ uncertainty description of Equation 4.6 for $\omega$ into the matrix $A(\omega)$ in Equation 4.9. Doing so decomposes $A(\omega)$ into the nominal $A$ matrix,
denoted as $\hat{A}$, and a perturbation matrix, $\Delta A$, that depends on the uncertain $\delta$

$$A(\omega) = \begin{bmatrix} 0 & 1 \\ -(\omega + \delta q)^2 & -2\xi(\omega + \delta q) \end{bmatrix} = \hat{A} + \Delta A$$ (4.11)

where

$$\hat{A} = \begin{bmatrix} 0 & 1 \\ -\dot{\omega}^2 & -2\xi\dot{\omega} \end{bmatrix}$$ (4.12)

and

$$\Delta A = \begin{bmatrix} 0 & 0 \\ -(2\dot{\omega}\delta + \delta^2) & -2\xi\delta \end{bmatrix}$$ (4.13)

Notice from this description that the uncertain $\delta$ parameter do not appear in a linear fashion in $\Delta A$. Hence there is no way to define fictitious inputs and outputs to arrive at a linear relation of Equation 4.10. There are few techniques that can be used to arrive at a linear combination of the uncertain parameters in a system. The approach that will be applied here, which is used in many other parameter uncertainty methods (see [10], [7], [8]), is more conventional. Basically, the non-linear structure of the actual uncertain variable that may appear in $\Delta A$ is avoided by considering the individual elements in the state space matrices to be the parametrically uncertain variables. As long as the bounds on the uncertain elements in the state space matrices cover the original uncertainty description of the model, this method, while more conservative than the realistically defined error model, is safe to use. By further assuming that there is no dependence among these uncertain variables, a quite simple procedure for casting such parametric errors into the system description is available (see [12]). For the case of the flexible mode, the new uncertain parameter will be as

$$\alpha = \begin{pmatrix} \omega_n^2 \\ 2\xi\omega_n \end{pmatrix}$$ (4.14)

to get a linear relation amongst the uncertainty. Each uncertain variable will be represented as

$$\alpha_i = \hat{\alpha}_i + \delta_i q_i \quad \text{with} \quad |\delta_i| < 1$$ (4.15)
and the level of error will now be

\[ q_i = \max\{\hat{\alpha}_i - \hat{\alpha}_i, \hat{\alpha}_i - \alpha_i\} \quad (4.16) \]

In this case, we will have to define two fictitious inputs and outputs

\[ v_i = \delta_i p_i \quad i = 1, 2. \quad (4.17) \]

If we substitute the uncertainty description of Equation 4.15 into \( A(\alpha) \) in Equation 4.9 (note that \( A(\omega) \) becomes \( A(\alpha) \) since the new uncertainty variable is \( \alpha \)), we obtain

\[ A(\alpha) = \begin{bmatrix} 0 & 1 \\ -(\dot{\omega}^2 + \delta_1 q_1) & -(2\xi \dot{\omega} + \delta_2 q_2) \end{bmatrix} = \hat{A} + \Delta_A \quad (4.18) \]

where \( \hat{A} \) is the same as in 4.12 and

\[ \Delta_A = \begin{bmatrix} 0 & 0 \\ -\delta_1 q_1 & -\delta_2 q_2 \end{bmatrix} = \delta_1 \begin{bmatrix} 0 & 0 \\ -q_1 & 0 \end{bmatrix} + \delta_2 \begin{bmatrix} 0 & 0 \\ 0 & -q_2 \end{bmatrix} \quad (4.19) \]

Notice that the structure of \( \delta_1 \) and \( \delta_2 \) in \( \Delta_A \) (Equation 4.19) allows \( \Delta_A \) to be decomposed into the sum of two rank one matrices, denoted by \( A_i \), weighted by \( \delta_i \)

\[ \Delta_A = \sum_{i=1}^{2} \delta_i A_i \quad (4.20) \]

where \( A_i \) is the outer product of two vectors,

\[ \Delta_A = \sum_{i=1}^{2} \delta_i a_i b_i^T. \quad (4.21) \]

Also notice that from Equation 4.19 that for \( A(\alpha) \in \mathbb{R}^{2 \times 2} \) with an uncertain element in the \((m, n^{th})\) location of \( A(\alpha) \), a possible choice for \( a_i \) and \( b_i \) are two-length vectors of zeros except for a \( q_i \) in the \( m^{th} \) row of \( a_i \) and a \(-1\) in the \( n^{th} \) row of \( b_i \). Now, \( A(\alpha) \)
in Equation 4.18 can be also written as

$$A(\alpha) = \hat{A} + \sum_{i=1}^{2} \delta_i a_i b_i^T. \quad (4.22)$$

We can define the fictitious outputs of the system to be

$$p_i(t) = b_i^T x(t) \quad i = 1, 2, \quad (4.23)$$

recalling that the fictitious inputs needed to arrive at $\Delta$, where the $\Delta$’s are 2×2 matrices of the form

$$\Delta = diag(\delta_1, \delta_2) \quad \text{with} \quad \delta_i \in \mathbb{R}, |\delta_i| < 1 \quad (4.24)$$

are

$$v_i(t) = \delta_i p_i(t) \quad i = 1, 2, \quad (4.25)$$

and using this information in Equation 4.22 produces the state dynamics in terms of the inputs and outputs of the system

$$\begin{pmatrix} \dot{x} \\ p \\ y \end{pmatrix} = \begin{bmatrix} \hat{A} & B_{\Delta} & B \\ C_{\Delta} & D_{11} & D_{12} \\ C & D_{21} & 0 \end{bmatrix} \begin{pmatrix} x \\ v \\ u \end{pmatrix} \quad (4.26)$$

where

$$\begin{align*}
\dot{x}(t) &= \hat{A}x(t) + \sum_{i=1}^{2} a_i v_i(t) + Bw(t) \\
y(t) &= Cx(t) \quad (4.27)
\end{align*}$$

Further stacking the $a_i$ and $b_i$ vectors for the 2 uncertain elements in the flexible mode model into matrices

$$v(t) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{and} \quad p(t) = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \quad (4.28)$$
produces the fictitious inputs and outputs of Equation 4.26. Augmenting Equation 4.28 to the inputs and outputs of Equation 4.27 then provides the values of matrices for the system description of the flexible mode model with parametric uncertainty in its $A$ matrix, Equation 4.26,

$$B_\Delta = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$$

$$C_\Delta = \begin{bmatrix} b_1^T \\ b_2^T \end{bmatrix}$$

\[D_{11} = D_{12} = D_{21} = 0\] (4.29)

This is the desired model that could be used in the $\ell_1$ framework, to deal with the parametric uncertainty in the flexible modes. Figure 4-6 shows the interconnection between the flexible mode dynamics and the uncertainty block $\Delta$.

![Figure 4-6: Flexible Mode Dynamics with Uncertainty](image)

### 4.3 Computing an $\ell_1$ sub-optimal controller

With the information in previous sections, we are ready to complete the problem set-up including the plant dynamics derived from Table 4.2

$$\begin{pmatrix} \dot{x} \\ z \\ y \end{pmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{pmatrix} x \\ w \\ u \end{pmatrix}$$ (4.30)
and the uncertainty at all flexible modes (see Section 4.2.1). This means we will include up to 8 uncertainty blocks $\Delta_k$, where each block will increase by 2 the number of fictitious inputs and outputs. The augmented system will be

$$
\begin{pmatrix}
\dot{x} \\
\dot{z} \\
\dot{p} \\
y
\end{pmatrix} =
\begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_\Delta & 0 & D_{22} \\
C_2 & 0 & 0
\end{bmatrix}
\begin{pmatrix}
x \\
w \\
v \\
u
\end{pmatrix}
$$

(4.31)

where

$$B_\Delta = \begin{bmatrix} a_1 & \ldots & a_i \end{bmatrix} \quad \text{and} \quad C_\Delta = \begin{bmatrix} b_1^T \\
\vdots \\
b_i^T \end{bmatrix} \quad i = 1, 2, \ldots, 16 \quad (4.32)$$

The representation of the system of Equation 4.31 is shown in Figure 4-7.

![System Configuration Diagram](Image)

Figure 4-7: System Configuration

With this problem set-up we are ready to apply the delay augmentation algorithm as described in Chapter 2 and in [6]. Due to the limitations of the computers available to design the controller, a reduced model was used. We are not able to include all flexible modes nor all uncertainty blocks for the flexible modes included. The number of flexible modes included in the design was reduced to 2, the lower modes in Table
4.2 and only one uncertainty block was used. Including the uncertainty block for the second flexible mode gave better results than in the first one. It was also necessary to reduce the original problem with 3 outputs to 2 outputs, where the roll angle and control torque were the outputs with specific requirements to satisfy.

The new problem to solve is

\[ \nu_r = \inf_{K \text{ stabilizing}} \begin{bmatrix} T_{z_1 w} & T_{z_1 v_1} & T_{z_1 v_2} \\ T_{u w} & T_{u v_1} & T_{u v_2} \\ T_{p_1 w} & T_{p_1 v_1} & T_{p_1 v_2} \\ T_{p_2 w} & T_{p_2 v_1} & T_{p_2 v_2} \end{bmatrix} \|_r \] \tag{4.33}

Note that since the plant has poles on the unit circle, we needed to use the exponentially weighted norm to solve this problem. The problem defined in Equation 4.33 can be translated to the following linear program

\[ \nu^o = \inf \nu_r \]

subject to

\[ A_{t_1}(r) \Phi \leq \nu_r 1 \]

\[ \| T_{u w} w_{imp} \|_\infty \leq 2.64 \]

\[ \bar{\sigma}(T_{z_1 w}(j\omega)) \leq 2.8 \text{ at } \omega = 0.001 \text{ rad/s} \]

The problem of Equation 4.34 was solved using the delay augmentation algorithm, with a maximum length of the impulse response of 25 and a total number of delays of 20. The length of the impulse response was an extra limiting factor, since increasing the length would also increase the number of constraints in the linear program in Equation 4.34.

4.4 Analysis of Results

The problem discussed in the previous section was solved and a controller was obtained. Then we proceeded to close the loop with the original plant, including
the eight flexible modes, the rigid body, the integrator and the anti-aliasing filter (in discrete time), with the discrete time controller (see Figure 4-8). In this section we will discuss the results obtained after closing the loop.

![Diagram of closed loop system](image)

Figure 4-8: Closed Loop System

Table 4.3 shows the numerical results for the \( \ell_1 \) design. For the time domain specifications, the roll angle peak to peak value was considerably small compared with the maximum required of 4.6 arcsec., and the control torque did not reach the saturation limit of 2.64 in-lbs.. It was expected to obtain good performance in time domain since \( \ell_1 \) is a time domain based optimization that minimizes maximum error over time. Not all the frequency domain specifications were completely satisfied. The sinusoidal torques at 0.001 rad/s. are rejected better than the maximum specified of 2.8 arcsec/in-lbs., and the gain margin satisfies the requirement of \( \pm 3 \) dB. In the case of the phase margin, the design is below the minimum of 30 degrees. Different approaches were tried to improve the phase and gain margins, i.e. including weights and different filters, but the results were not improved. Note that there is no method to incorporate gain and phase margin in the design procedure. The robustness for the first three flexible modes were the only ones considered since the concern was to satisfy the minimum frequencies of the solar array. All three modes met the robustness specification of \( \pm 15\% \). The \( \ell_1 \) design provided a high order compensator.

The response of the roll angle to the impulse disturbance is shown in Figure 4-9 and the frequency response from the disturbance to the roll angle is presented in Figure 4-10. In the low frequency region we have a flat response, below the value
<table>
<thead>
<tr>
<th>Specification</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.- Time Domain:</td>
<td></td>
</tr>
<tr>
<td>Roll Angle peak-to-peak</td>
<td>0.5247 arcsec.</td>
</tr>
<tr>
<td>Control Torque peak-to-peak</td>
<td>0.2117 in-lbs.</td>
</tr>
<tr>
<td>2.- Frequency Domain:</td>
<td></td>
</tr>
<tr>
<td>$\sigma(T_zw(j\omega)), \omega = 0.001rad/s$</td>
<td>2.6095 arcsec/in-lbs.</td>
</tr>
<tr>
<td>Gain Margin</td>
<td>+3.8775 dB, -3.0141 dB</td>
</tr>
<tr>
<td>Phase Margin</td>
<td>12.416 degrees</td>
</tr>
<tr>
<td>3.- $\ell_1$ Norms</td>
<td></td>
</tr>
<tr>
<td>Roll Angle</td>
<td>7.3029</td>
</tr>
<tr>
<td>Control Torque</td>
<td>11.0458</td>
</tr>
<tr>
<td>Closed Loop</td>
<td>11.0458</td>
</tr>
<tr>
<td>4.- Robustness:</td>
<td></td>
</tr>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt; Flexible Mode</td>
<td>-90%, &gt; 100%</td>
</tr>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt; Flexible Mode</td>
<td>-93%, +84%</td>
</tr>
<tr>
<td>3&lt;sup&gt;rd&lt;/sup&gt; Flexible Mode</td>
<td>-95%, +22%</td>
</tr>
<tr>
<td>5.- Order of Compensator</td>
<td>38</td>
</tr>
</tbody>
</table>

Table 4.3: EOS $\ell_1$ Design Results
of 2.8 arcsec/in-lbs. and at high frequency we can see that the response rolls off. The time response of the pre-filtered control torque (Figure 4-11) shows the presence of high frequencies, that was expected from the corresponding frequency response of Figure 4-12. The response of Figure 4-12 has high bandwidth and does not roll off at high frequencies. The pre-filtered control torque is the signal that enters the low pass filter discussed in Section 4.2, and this signal is presented instead of the actual control torque since the second was not available for measurement, due to the joint discretization of the filter and the continuous plant. The control torque should roll off after passing through the filter. Also the time response of the control torque is expected to be smooth since the high frequencies are filtered.

![Roll Angle Response](image)

**Figure 4-9: Roll Angle Time Response**
Figure 4-10: Disturbance to Roll Angle Frequency Response

Figure 4-11: Pre-Filtered Control Torque Time Response
Figure 4-12: Disturbance to Pre-Filtered Control Torque Frequency Response
4.4.1 Comparison with Previous Design Methodologies

Previous designs had been presented for the EOS Satellite. In this subsection, a comparison between the different design methodologies will be addressed. The previous designs were carried out with classical control, $\mathcal{H}_\infty$ and $\mu$-synthesis methods. The goal for the $\mathcal{H}_\infty$ method is

$$\inf_{K \text{ stabilizing}} \| \Phi \|_\infty$$

and uses the Small Gain Theorem for robustness: given $\Delta, M$ both stable, the closed loop system $(I - \Delta M)^{-1}$ is stable if $\| \Delta M \|_\infty < 1$. The $\mu$-synthesis method is

$$\min_{K \text{ stabilizing}} \sup_{\omega} \mu_{\Delta}(M(j\omega)).$$

where

$$\mu_{\Delta}(M(j\omega)) := \frac{1}{\min\{\sigma_{\text{max}}(\Delta) \mid \Delta \in \Delta, \det(I - \Delta M) = 0\}}$$

and the system is stable iff $\mu_{\Delta}(M(j\omega)) < 1$. Both methods require a system description as in Figure 4-13.

![Figure 4-13: System Description for $\mathcal{H}_\infty$ and $\mu$-synthesis methods](image)

Table 4.4 shows a comparison for the time domain specifications, as well as the order of the compensators. Only two previous designs, Draper $\mathcal{H}_\infty$ and Draper $\mu \neq 1$ satisfied the requirement for the roll angle response. The performance for the $\ell_1$ Design shows a substantial improvement over all previous designs with an improvement.
<table>
<thead>
<tr>
<th>Compensator</th>
<th>Compensator Dimension</th>
<th>Roll Response to Unit Impulse (asec pk-to-pk)</th>
<th>Roll Response to Unit Impulse (improvement over baseline)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Design Requirements of Study</td>
<td>–</td>
<td>4.6</td>
<td>5.0</td>
</tr>
<tr>
<td>Baseline Design (Classical)</td>
<td>10</td>
<td>22.8</td>
<td>1.0</td>
</tr>
<tr>
<td>Enhanced Baseline</td>
<td>12</td>
<td>23.8</td>
<td>0.96</td>
</tr>
<tr>
<td>Draper Filtered FW-LQG</td>
<td>10</td>
<td>15.4</td>
<td>1.5</td>
</tr>
<tr>
<td>Draper $\mathcal{H}_\infty$</td>
<td>10</td>
<td>2.7</td>
<td>8.4</td>
</tr>
<tr>
<td>Draper $\mu$#1 (nominal design)</td>
<td>10</td>
<td>4.3</td>
<td>5.3</td>
</tr>
<tr>
<td>Draper $\mu$#2 (extra robustness)</td>
<td>11</td>
<td>5.9</td>
<td>3.9</td>
</tr>
<tr>
<td>$\ell_1$ Design</td>
<td>38</td>
<td>0.52</td>
<td>43.2</td>
</tr>
</tbody>
</table>

Table 4.4: Comparison Table for Time Specifications
<table>
<thead>
<tr>
<th>Compensator</th>
<th>Roll/Dist. 0.001 rad/s Orbital Frequency (asec/in-lb)</th>
<th>Gain Margin (dB)</th>
<th>Phase Margin (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Design Requirements of Study</td>
<td>2.8</td>
<td>-3, 3</td>
<td>30</td>
</tr>
<tr>
<td>Baseline Design (Classical)</td>
<td>2.8</td>
<td>-9.6, 13.9</td>
<td>37</td>
</tr>
<tr>
<td>Enhanced Baseline</td>
<td>2.8</td>
<td>-9.6, 10.7</td>
<td>36</td>
</tr>
<tr>
<td>Draper Filtered FW-LQG</td>
<td>0.51</td>
<td>-6.9, 3.5</td>
<td>38</td>
</tr>
<tr>
<td>Draper $\mathcal{H}_\infty$</td>
<td>1.6</td>
<td>-13.9, 3.8</td>
<td>19</td>
</tr>
<tr>
<td>Draper $\mu$#1 (nominal design)</td>
<td>2.4</td>
<td>-10.7, 4.2</td>
<td>26</td>
</tr>
<tr>
<td>Draper $\mu$#2 (extra robustness)</td>
<td>1.6</td>
<td>-5.5, 3.6</td>
<td>19</td>
</tr>
<tr>
<td>$\ell_1$ Design</td>
<td>2.6</td>
<td>-3.0, 3.9</td>
<td>12.4</td>
</tr>
</tbody>
</table>

Table 4.5: Comparison Table for Frequency Specifications
<table>
<thead>
<tr>
<th>Compensator</th>
<th>Robustness Mode 1 Frequency (percent)</th>
<th>Robustness Mode 2 Frequency (percent)</th>
<th>Robustness Mode 3 Frequency (percent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline Design (Classical)</td>
<td>-29, &gt;100</td>
<td>-53, &gt;100</td>
<td>-67, &gt;100</td>
</tr>
<tr>
<td>Enhanced Baseline</td>
<td>-49, &gt;100</td>
<td>-66, &gt;100</td>
<td>-76, &gt;100</td>
</tr>
<tr>
<td>Draper Filtered FW-LQG</td>
<td>-8, &gt;100</td>
<td>-40, &gt;100</td>
<td>-56, &gt;100</td>
</tr>
<tr>
<td>Draper $\mathcal{H}_\infty$</td>
<td>-8, 4</td>
<td>-11, 10</td>
<td>-34, &gt;100</td>
</tr>
<tr>
<td>Draper $\mu$#1 (nominal design)</td>
<td>-16, &gt;100</td>
<td>-39, &gt;100</td>
<td>-59, &gt;100</td>
</tr>
<tr>
<td>Draper $\mu$#2 (extra robustness)</td>
<td>-23, &gt;100</td>
<td>-45, &gt;100</td>
<td>-64, &gt;100</td>
</tr>
<tr>
<td>$\ell_1$ Design</td>
<td>-90, &gt;100</td>
<td>-93, 84</td>
<td>-95, 22</td>
</tr>
</tbody>
</table>

Table 4.6: Comparison Table for the Level of Robustness
over the baseline controller by a factor of 43.2. The price of achieving very high performance in the $\ell_1$ Design is being paid in the order of the compensator, that compared with all previous designs is considerably higher. Table 4.5 details a comparison for the frequency domain specifications and Table 4.6 shows a comparison for the level of robustness for each design. For the maximum singular value at 0.001 rads/sec. from the disturbance to roll angle response, the $\ell_1$ Design has a value close to the maximum allowed and is higher compared with the designs using modern control methods. For the gain margin requirement, the $\ell_1$ Design is just satisfying the requirement of -3 dB. Amongst the designs that do not satisfy the phase margin specification, the $\ell_1$ Design is clearly the worst design. Only half of the designs were above the 30 degrees requirement, and most of the controllers using modern design methodologies were below 30 degrees. The design with better robustness against variations below the nominal frequencies (of the first three flexible modes) is the $\ell_1$ Design. For variations above the nominal frequencies, the $\ell_1$ Design has a good performance for the first two flexible modes, which is similar to most of the designs, and for the third flexible mode, it has the worst robustness, but this value is still acceptable over the requirement of 15%.

We can see in Figure 4-14 a plot that compares improvement in the roll angle response (measured in point errors) versus the lowest level of robustness in the first three flexible modes (measured in percent error). The design goal was to have a point error below 4.6 and to have at least 15% variation in all flexible modes. As Figure 4-14 shows, only two designs satisfied the design goal, and the $\ell_1$ Design satisfies better the performance/robustness goal.

Since the specifications were based on improvement over the baseline controller, Figures 4-15, 4-16, 4-17 and 4-18 show the time and frequency domain responses for the $\ell_1$ Design compared with the Baseline Design. In Figure 4-15 the improvement of the $\ell_1$ Design over the Baseline Design is not only on the maximum and minimum values of the response but also in the time of the transients to reach the final value. Figure 4-16 shows that the $\ell_1$ Design has better disturbance rejection over a broader range of frequencies than the Baseline Design. Both designs have similar roll off at
high frequencies. At this point it is important to mention that the comparison in
Figure 4-17 does not correspond to the same signal, since the one for the \( \ell_1 \) Design
is pre-filtered and the one for the Baseline Design is the signal that enters the actual
plant (similar comment applies to the comparison in Figure 4-18). As mentioned in
the beginning of Section 4.4, the high frequencies will be filtered before the signal
enters the actual plant (in the \( \ell_1 \) Design). As it was expected, the response of the
\( \ell_1 \) Design is faster than the Baseline Design because the higher bandwidth showed
in the frequency response from the disturbance to the control torque (Figure 4-18).
Asking for better performance required a higher bandwidth in the control action. The
frequency response of the Baseline Design has better roll off at high frequencies than
the \( \ell_1 \) Design, even after the filter, since the filter is only a first order filter.
Figure 4-15: Comparison for the Roll Angle Time Responses

Figure 4-16: Comparison for the Frequency Response from Disturbance to Roll Angle
Figure 4-17: Comparison for the Control Torque Time Response

Figure 4-18: Comparison for the Frequency Response from Disturbance to Control Torque
4.5 Summary for the EOS $\ell_1$ Design

At this point, after the results for the $\ell_1$ Design have been presented and a comparison between different design methodologies used for the EOS Satellite was studied, we want to present as a summary, the strengths and weaknesses of the $\ell_1$ Design.

Strengths

The overall performance for the $\ell_1$ design is satisfactory. In particular, we want to mention the improvement in the peak-to-peak value of the roll response that was 43.2 times better than the baseline controller. It is important that the overall performance was achieved with a controller designed with a reduced plant with only one uncertainty block, and the results were obtained with the full model of the satellite. With this methodology we have the ability to augment additional constraints that allowed us to satisfy other specifications, i.e., set saturation levels in the control torque and impose a limit on sinusoidal torques for the roll response. The case study showed that using the $\ell_1$ design methodology, the controller provides significant improvement in performance over other design methodologies (classical, $\mathcal{H}_\infty$ and $\mu$-synthesis). We were also able to meet the robustness requirements and to maintain performance with uncertain low-frequency structural modes, even though the uncertainty description used was very conservative.

The high order compensator is not necessary a disadvantage in the design. The 38th order digital controller can be programmed on the computer, if high pointing precision needed. It is an advantage that the design methodology tells us the limit of achievable performance.

Weaknesses

The most significant drawback for the $\ell_1$ Design is the fact that the phase margin requirement was not fulfilled. The design presented here was affected by a major limiting factor: the computational facilities. This factor did not allowed us to perform a design using the full model of the plant, and to include all the uncertainty blocks.
for the flexible modes, thus high frequency modes and unmodelled dynamics were not considered while designing the controller, therefore we were in the need to include the filter in the discrete plant.
Chapter 5

Conclusions and Future Work

5.1 Conclusions

A complete set of subroutines to augment the $\ell_1$ software were developed. The new $\ell_1$ software provided two new features to the original $\ell_1$ software. First, it allows us to systematically include different kinds of constraints, such as time domain constraints and approximate $H_\infty$ constraints; and second, it gives us the choice to include exponential weights in the design procedure. By including all subroutines, a complete procedure for computing (sub)optimal controllers is available. Several examples were used to test the software and some of them were discussed in Chapter 3.

A comprehensive study of controller design for the Earth Observing System Satellite was presented in this thesis using the results obtained with the augmented $\ell_1$ software. The $\ell_1$ design showed that using a simplified model of the plant with one structured uncertainty block is sufficient to obtain a controller for the full model of the plant. The time domain results satisfy all specifications, i.e. the improvement of the roll response was 43.2 times over the baseline design (required was 5 times better) and the maximum control torque was 0.2117 in-lbs well below the saturation levels (2.64 in-lbs). In the case of the frequency domain constraints, we were able to reject the sinusoidal torques at 0.001 rad/sec. with 2.6095 arcsec/in-lb., better than the requirement of 2.8 arcsec/in-lb. of the baseline design. The gain margin specification was met with 3.8775 dB and -3.0141 dB, just above the required $\pm 3$ dB, and the
phase margin was 12.416 degrees, not enough to meet the required 30 degrees. Due to the lack of a method to incorporate both gain and phase margins in the design procedure, we were not able to improve the actual results for the $\ell_1$ controller. In terms of robustness, the $\ell_1$ design showed to maintain stability for up to 90%, 84% and 22% variation (lowest percentage above or below the nominal value) in the modal frequencies for the first, second and third modes respectively, while the requirement was to maintain stability for at least $\pm 15\%$ variation from the nominal frequencies. Notice that the robustness was achieved using a very conservative description of the parametric uncertainty.

An important comparison between different design methodologies was covered in Section 4.4.1. The phase margin requirement was the hardest specification for most of the modern control designs, since only one out of five modern designs satisfied the 30 degrees requirement. Also from this comparison, we can conclude that the $\ell_1$ design methodology is more suitable to accomplish better performance in time domain, as it was expected from the nature of the $\ell_1$ optimal control problem.

5.2 Future Work

As a recommendation for future work, the improvement of the software to handle exact $\mathcal{H}_\infty$ constraints can be obtained by combining the existing $\ell_1$ software and the software to solve Linear Matrix Inequalities (LMI). In this way, it could be avoided the use of a finite set of constraints to approximate $\mathcal{H}_\infty$ constraints at each sample on the unit circle that we want to constrain.

The analysis of applying time varying weights, including exponential weights, into the $\ell_1$ methodologies should be addressed. The effect in the time and frequency domain responses using time varying weights should be studied.

For the EOS Satellite, different designs including frequency weights can be tried in order to achieve better phase margin than the one obtained in this case study. Also, different modal reduction techniques on the actual controller can be tried. With the use of better and faster computers, designs for a full plant model, with more
uncertainty blocks, can be performed. Also the 3 axis model of the satellite can be considered for design.
Appendix A

Matlab Codes

In this Appendix are included the Matlab programs necessary to handle time and frequency domain constraints in the $\ell_1$ optimal control problem design. Both codes will create an operator in the form in Sections 3.2 and 3.3 ready to be included in the linear program to solve the $\ell_1$ problem.

A.1 To Include Time Domain Constraints

```matlab
function [a,b] = timecon(ni,no,k,outcon,ubound,usup,lbound,lsup,db,wf)
% Function to add time domain constraints to l-1 software
% This function [a,b] = timecon(ni,no,k,outcon,bounds,db,wf)
% creates the A and B matrices to meet the time domain constraints
% for l-1 problems.
% The input data to the function is as follows:
% ni = number of inputs of the system
% no = number of outputs of the system
% k = order of solution (phi)
% outcon = vector whos elements indicate the input/output pair to
% apply the constraints. The first element is the number of
% the input and the second element correspond to the output.
% ubound = vector with the upper bound.
% usup = vector with the indices where the constraints are to
```
be applied, i.e. \[15 \ 20 \ 25 \ 30\] will apply the constraints of \(ubound\) from \(t=15\) till \(t=20\), and from \(t=25\) till \(t=30\).

The number of elements should be even.

\(lbound\) = vector with the lower bound.

\(lsup\) = vector with the indices where the constraints are to be applied, i.e. \[15 \ 20 \ 25 \ 30\] will apply the constraints of \(ubound\) from \(t=15\) till \(t=20\), and from \(t=25\) till \(t=30\).

The number of elements should be even.

\(db\) = case of bounds. For upper and lower bounds, \(db = 1\); for upper or lower bound, \(db = 2\) or \(3\) respectively.

\(wf\) = input function to the system, of dimensions \(1 \times (k+1)\)

The output data of the function is as follows:

\(a\) = matrix of dimensions \((1 \ or \ 2) \times (num \ of \ cons)\) \(\times \ [2 \times (k+1) \times ni \times no]\)

ready to be included in the general \(A\) operator for the LP.

\(b\) = vector of dimensions \((1 \ or \ 2) \times (num \ of \ cons)\) \(\times 1\) ready to be included in the general \(B\) vector of constraints.

WRITTEN BY MARCOS ESCOBAR 8/9/93... 5/11/94

if nargin < 10,
    disp('*** The arguments in timecon should be 10 ***')
    return
else
    if db==0 | db > 3,
        disp('**** Check the bounds and try again ****')
        return
    end
    if db==1,
        doble=2;
    end
    if db==2 | db==3,
        doble=1;
    end
    dusup = max(size(usup));
    disup = max(size(lsup));
Aa = zeros(doble*(k+1),no*ni*(k+1));
l = 0;
if outcon(1) > ni,
    % Constructing the A matrix
    disp(sprintf('** There are only %g inputs **',ni))
    return
end
if outcon(2) > 0,
    % number of output to constrain
    if outcon(2) > no,
        disp(sprintf('** Output %g does not exist **',outcom(2)))
        return
    else,
        kk = 0;
        for i = 1:k+1,
            if db == 1,
                Aa(i,outcon(2)*(k+1)-k:outcon(2)*(k+1)) = [wf(1,k+1-kk:k+1) zeros(1,k-kk)];
                Aa(i+k+1,outcon(2)*(k+1)-k:outcon(2)*(k+1)) = [-wf(1,k+1-kk:k+1) zeros(1,k-kk)];
                % Changing sign to invert the inequality direction.
            end
            if db == 2,
                Aa(i,outcon(2)*(k+1)-k:outcon(2)*(k+1)) = [wf(1,k+1-kk:k+1) zeros(1,k-kk)];
            end
            if db == 3,
                Aa(i,outcon(2)*(k+1)-k:outcon(2)*(k+1)) = [-wf(1,k+1-kk:k+1) zeros(1,k-kk)];
            end
            kk = kk + 1;
        end
    end
    % of i
end
% of if outcon > no
end
% of if outcon > 0

aac = [];
bbc = [];
init = 0;
if db == 1,
    for counter1 = 1:dusup/2,
        prov = usup(2*counter1) - usup(2*counter1-1) + 1;
        aac(init+1:prov+init,:) = Aa(usup(2*counter1-1):usup(2*counter1),:);
bbc(init+1:prov+init,1) = ubound(init+1:prov+init)';
init = init + prov;
end
% of counter1

for counter2 = 1:dlsup/2,
    int = 0;
    prov = lsup(2*counter2) - lsup(2*counter2-1) + 1;
    aac(init+1:prov+init,:) = -Aa(lsup(2*counter2-1):lsup(2*counter2),:);
    bbc(init+1:prov+init,1) = -lbound(init+1:prov+init)';
    init = init + prov;
    int = int + prov;
end
% of counter2

elseif db == 2,
    for counter1 = 1:dusup/2,
        prov = usup(2*counter1) - usup(2*counter1-1) + 1;
        aac(init+1:prov+init,:) = Aa(usup(2*counter1-1):usup(2*counter1),:);
        bbc(init+1:prov+init,1) = ubound(init+1:prov+init)';
        init = init + prov;
    end
% of counter1

else
    for counter2 = 1:dlsup/2,
        prov = lsup(2*counter2) - lsup(2*counter2-1) + 1;
        aac(init+1:prov+init,:) = -Aa(lsup(2*counter2-1):lsup(2*counter2),:);
        bbc(init+1:prov+init,1) = -lbound(init+1:prov+init)';
        init = init + prov;
    end
% of counter2
end
% of db

aa=[];
[nraac,ncaac]=size(aac);
for p=0:ncaac-1,
    aa(:,2*p+1)=aac(:,p+1); % Separate in columns to match the phi structure of the program
    aa(:,2*p+2)=-aac(:,p+1);
end

clear Aa Bb aac
[nraa,ncaa] = size(aa);
a = [zeros(nraa,1) aa];
b = bbc;

---

A.2 To Include Frequency Domain Constraints

function [a,b]=freqcon(ni,no,k,u,v,wn,gamma,points)
% Function to add frequency constraints to l-1 software
% % This function [a,b]=freqcon(ni,no,k,u,v,wn,gamma,points)
% creates the A and B matrices to meet the frequency domain constraints
% for l-1 problems.
% % The input data to the function is as follows:
% %  ni = number of inputs of the system
% %  no = number of outputs of the system
% %  k = order of solution (phi)
% %  u = matrix with the same number of columns as outputs of the system
% %      and number of rows as many input/output (IO) combinations to be
% %      constrained.
% %  v = matrix with the same number of rows as input to the system
% %      and number of columns as many input/output (IO) combinations to
% %      be constrained.
% %  wn = Matrix with columns (N) as the frequency points to impose
% %        constraints and rows as number of input/output (IO) combinations
% %        to be constrained. Row entries should be in increasing order.
% %        And all the entries should be less than pi (3.141592.....)
% %  gamma = Matrix which row entries are the bounds at each frequency (N)
% %          to be constrained, and number of rows as many input/output (IO)
% %          combinations to be constrained.
% %  points = Number of linear constraints (M) at each frequency
% %
% % The output data of the function is as follows:
% a = matrix of dimensions [(N*M*IO)*(k+1)] X [2*(k+1)*ni*no]
% ready to be included in the general A operator for the LP.
% b = vector of dimensions [(N*M*IO)*(k+1)] X 1 ready to be included
% in the general B vector of constraints.

% WRITTEN BY MARCOS ESCOBAR  7/18/93-7/29/93

if nargin ~= 8,
    disp('*** Parameters are wrong, we need 8 in freqcon ***')
    return
end
[nru,ncu] = size(u);
[nrv,ncv] = size(v);
[IO,N] = size(wn);
if ncu ~= no | nrv ~= ni | nru ~= ncv | nru ~= IO,
    disp(' **** Check dimensions of u, v and wn, and try again ****')
    return
end
M = points;
Aa = zeros(M*N*IO,(k+1)*ni*no);
Bb = zeros(M*N*IO,1);
theta = linspace(0,2*pi,M);
for io = 0:IO-1,
    for n = 0:N-1,
        if n > 0
            if wn(io+1,n+1) > wn(io+1,n), % Skip filling rows of constraints for % of n>0
                u(io+1,:) = u(io+1,:)/norm(u(io+1,:));
                v(:,io+1) = v(:,io+1)/norm(v(:,io+1));
            end
            for m = 1:M, % frequencies that has no bound.
                for j = 0:no-1,
                    for l = 0:mi-1,
                        for p = 1:k+1,
Aa(m+n*M+io*N*M,p+1*(k+1)+j*ni*(k+l1)) = ...

\[ u(io+1,j+1) \cdot v(l+1,io+1) \cdot \cos((p-1) \cdot wn(io+1,n+1) - \theta(m)) \];

end \% of p
end \% of l
end \% of j

if gamma(io+1,n+1) < 0,
    disp('*** The bounds can not be negative, try again ***')
    return
end \% of if

Bb(m+n*M+io*N*M,1) = gamma(io+1,n+1);

end \% of m
end \% of if
end \% of n
end \% of io

% eliminate the rows that has all zeros in A and B
[nra,nca] = size(Aa);
[nrb,ncb] = size(Bb);
if nra ~= nrb,
    disp('*** Number of linear constraints and bounds doesnt match ***')
    return
end

AA=[];
kk=1;
for eli = 1:nra,
    com1 = Aa(eli,:);
    com2 = Bb(eli,1);
    comp1 = [zeros(1,(k+1)*ni*no)];
    comp2 = [zeros(1,1)];
    if sum(abs(com1 = comp1)) & com2 = comp2,
        AA(kk,:) = Aa(eli,:); \% Eliminate useless
        BB(kk,1) = Bb(eli,1); \% rows of constraints
        kk = kk+1;
end
end

aa=[];
bb=BB;
[nrAA,ncAA]=size(AA);
for p=0:ncAA-1,
    aa(:,2*p+1)=AA(:,p+1);  % Separate in columns to match the
    aa(:,2*p+2)=-AA(:,p+1);  % phi structure of the program
end

clear Aa;
clear Bb;
clear AA;
clear BB;
a = [zeros(nrAA,1) aa];
b = bb;
Bibliography


