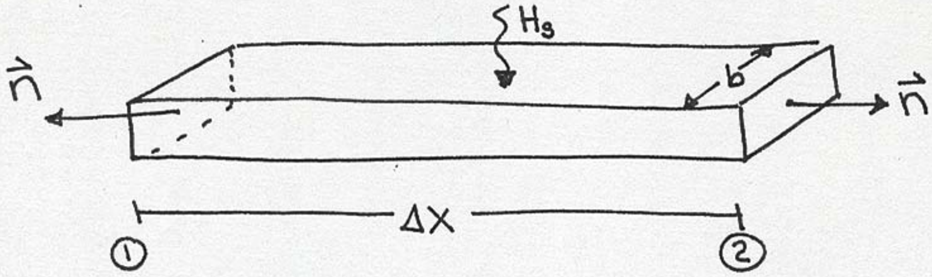
ANSWER 1.9

CONTROL-VOLUME APPROACH:

SELECT A SHORT LENGTH OF RIVER, ΔX, AND EVALUATE THE CONTROL VOLUME (INTEGRAL) FORM OF THE CONSERVATION EQUATION. FOR CONSERVATION OF HEAT ENERGY, REPLACE $C = \rho C_p T$ IN EQ. 4

$$\text{(A)} \quad \frac{\partial}{\partial t}\int_{CV} \rho C_p T \, d\forall = -\int_{CS} \rho C_p T \, \vec{V} \cdot \vec{n} \, dA + \int_{CS} D_n \frac{\partial C}{\partial n} \, dA + H_s \Delta X b$$



AS THE PROBLEM STATEMENT DOES NOT INDICATE ANY UNSTEADINESS, WE ASSUME STEADY FLOW, I.E. $\frac{\partial}{\partial t} = 0$.

EVALUATING THE FLUX TERMS IN (A)

$$(B) \quad 0 = -\rho C_p U b h (T_2 - T_1) + \rho C_p D b h \left(\frac{\partial T}{\partial X}\bigg|_2 - \frac{\partial T}{\partial X}\bigg|_1 \right) + H_s \Delta x b$$

USING A TAYLOR EXPANSION, ASSUMING T IS CONTINUOUS IN X

$$T_2 = T_1 + (\partial T/\partial X)\Delta X$$

$$\partial T/\partial X\big|_2 = \partial T/\partial X\big|_1 + \frac{\partial}{\partial X}(\partial T/\partial X)\Delta X$$

PLUG THESE EXPANSIONS IN (B), AND DIVIDE OUT THE COMMON TERMS, $\Delta X\, b$

$$(C) \quad 0 = \rho C_p h \left(-U \frac{\partial T}{\partial X} + D \frac{\partial^2 T}{\partial X^2} \right) + H_s$$

FROM WHICH, ONE COULD SOLVE FOR $\partial T/\partial X$.

IT IS USEFUL TO CONSIDER THE RELATIVE IMPORTANCE OF THE ADVECTIVE AND DIFFUSIVE FLUXES. HERE, SPECIFICALLY THE RELATIVE MAGNITUDES OF $U\partial T/\partial X$ AND $D\partial^2 T/\partial X^2$. THE SCALE OF EACH TERM CAN BE ESTIMATED FOR THIS SYSTEM. CONSIDER THE CONTROL VOLUME LENGTH, ΔX, AS AN APPROPRIATE LENGTH-SCALE, THEN

$$U \frac{\partial T}{\partial X} \sim U \frac{\Delta T}{\Delta X}$$

$$D \frac{\partial^2 T}{\partial X^2} \sim D \frac{\Delta T}{\Delta X^2},$$

WHERE ΔT IS THE TEMPERATURE CHANGE ACROSS ΔX.

THE RELATIVE MAGNITUDE OF THESE TERMS IS THEN,

$$\frac{\text{ADVECTIVE FLUX}}{\text{DIFFUSIVE FLUX}} = \frac{U\frac{\Delta T}{\Delta X}}{D\,\Delta T/\Delta X^2} = \boxed{\frac{U\Delta X}{D}}$$

THIS DIMENSIONLESS PARAMETER IS CALLED THE PECLET NUMBER. IT IS DISCUSSED IN DETAIL IN CHAPTER 5.

IF $\frac{U\Delta X}{D} \gg 1$, THEN ADVECTIVE FLUXES DOMINATE DIFFUSIVE FLUXES, AND WE CAN DROP THE TERM $D\frac{\partial^2 T}{\partial X^2} \ll U\frac{\partial T}{\partial X}$.

IF $\frac{U\Delta X}{D} \ll 1$, DIFFUSIVE FLUXES $\left(D\frac{\partial^2 T}{\partial X^2}\right)$ ARE MUCH LARGER THAN ADVECTIVE FLUXES $\left(U\frac{\partial T}{\partial X}\right)$, AND WE CAN DROP $U\frac{\partial T}{\partial X}$.

SINCE ΔX IS NOT SPECIFICALLY DEFINED, WE ASK, EG, FOR WHAT LENGTH-SCALE WILL ADVECTION DOMINATE TRANSPORT?

$$\frac{U\Delta X}{D} \gg 1 \quad \text{IFF} \quad \Delta X \gg \frac{D}{U} = \frac{0.1\,m^2 s^{-1}}{1\,m s^{-1}} = 0.1\,m$$

$\therefore$ OVER ANY LENGTH-SCALE, $\Delta X \gg 0.1\,m$, WE MAY NEGLECT THE IMPACT OF DIFFUSION IN (A) FOR THIS SYSTEM. THE PROBLEM ASKS FOR A DESCRIPTION OF $\partial T/\partial X$ ALONG A RIVER CHANNEL. IN SUCH A SYSTEM, THE LENGTH SCALES OF INTEREST ARE MUCH LARGER THAN 10 cm!, AND ARE MORE LIKE 100-m TO Km's. THEREFORE, FOR THIS SYSTEM,

WE CAN SAFELY DROP THE DIFFUSIVE TRANSPORT TERM. THEN, (C) REDUCES TO

$$(D) \quad 0 = -\rho C_p h U \frac{\partial T}{\partial x} + H_S$$

FROM WHICH,

$$(E) \quad \frac{\partial T}{\partial x} = \frac{H_S}{\rho C_p h U} = \frac{J s^{-1} m^{-2}}{(kg\, m^{-3})(J\, kg^{-1}\, {}^{\circ}K^{-1})(m)(m\, s^{-1})} = \frac{{}^{\circ}K}{m}$$

UNITS CHECK OUT.

USING THE STATED PARAMETERS

$$(F) \quad \frac{\partial T}{\partial x} = \frac{800\, W m^{-2}}{(1000 \frac{kg}{m^3})(4200 \frac{J}{kg K})(1 m)(1 \frac{m}{s})} = 2 \times 10^{-4} \frac{{}^{\circ}K}{m} = 0.2 \frac{{}^{\circ}C}{km}$$

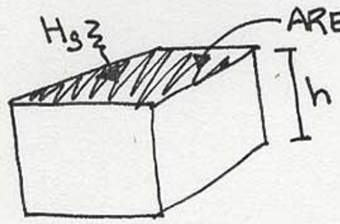
DIFFERENTIAL APPROACH:

THE CONSERVATION OF MASS EQUATION CAN BE APPLIED TO THE TRANSPORT OF HEAT ENERGY BY NOTING THE CONCENTRATION OF HEAT ENERGY, $C\,[J/m^3] = \rho C_p T$. THEN, THE DIFFERENTIAL FORM OF THE CONSERVATION EQUATION IS, FOR INCOMPRESSIBLE FLOW

$$(g) \quad \frac{\partial}{\partial t}(\rho C_p T) + u \frac{\partial}{\partial x}(\rho C_p T) + v \frac{\partial}{\partial y}(\rho C_p T) + w \frac{\partial}{\partial z}(\rho C_p T) =$$

$$\frac{\partial}{\partial x} D_x \frac{\partial}{\partial x}(\rho C_p T) + \frac{\partial}{\partial y} D_y \frac{\partial}{\partial y}(\rho C_p T) + \frac{\partial}{\partial z} D_z \frac{\partial}{\partial z}(\rho C_p T) \pm S$$

- IF WE NEGLECT $\rho = f(T)$, THEN $\rho \neq f(x,y,z,t)$.
- THE PROBLEM STATEMENT GIVES US $V = W = 0$, AND ISOTROPIC, HOMOGENEOUS $D = D_x = D_y = D_z \neq f(x,y,z)$
- IF WE ASSUME THE SYSTEM IS UNIFORM (WELL-MIXED) IN Y & Z, THEN $T \neq f(y,z)$
- THE SOURCE TERM IS GIVEN AS A <u>SURFACE</u> FLUX, $H_s = [J\,s^{-1}m^{-2}]$. SINCE THE EQUATION DEALS IN <u>VOLUME</u> CONCENTRATION, WE MUST DIVIDE BY DEPTH TO PUT THE SOURCE TERM IN CONSISTENT UNITS.



$$H_s = \frac{J/m^2}{s} = \frac{\text{ENERGY PER SURFACE AREA}}{s}$$

$$\frac{H_s}{h} = \frac{J/m^3}{s} = \frac{\text{ENERGY PER VOLUME}}{s}$$

APPLYING THE ABOVE POINTS, (g) REDUCES TO

$$\text{(h)} \quad \rho C_p \left[\frac{\partial T}{\partial t} + \underbrace{u\frac{\partial T}{\partial X}}_{1.}\right] = \underbrace{D\frac{\partial^2 T}{\partial X^2}}_{2.} + \frac{H_s}{h}$$

FOR TYPICAL LENGTH SCALES OF INTEREST ALONG A RIVER CHANNEL, $\Delta X \sim 100\,m$ TO km's, IT IS EASY TO SHOW THAT THE DIFFUSION TERM (2) IS SMALL COMPARED TO (1) THE ADVECTION TERM. THUS WE WILL DROP $D\,\partial^2T/\partial X^2 \ll u\,\partial T/\partial x$.

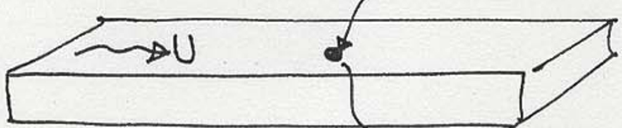
SEE SCALING ARGUMENTS GIVEN ABOVE

FINALLY, NOTE THAT THE BRACKETED TERM IN (h) IS THE <u>TOTAL</u> <u>DERIVATIVE</u>

$$\text{(i)} \qquad \frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \frac{H_s}{\rho C_p h} = \left[\frac{°C}{T}\right]$$

THIS EQUATION MAY BE READ IN THE LAGRANGIAN CONTEXT AS, FOLLOWING A PARTICULAR FLUID PARTICLE, WE WOULD OBSERVE ITS TEMPERATURE TO INCREASE AT THE RATE $\left[H_s/\rho C_p h\right]$ °C/S.

$$T(t) = T_0 + \left[\frac{H_s}{\rho C_p h}\right] t$$



IF THE FLOW/THERMAL CONDITONS ARE <u>STEADY</u>, $\partial T/\partial t = 0$, THEN (i) ALSO PROVIDES A SIMPLE DESCRIPTION OF SPATIAL GRADIENT.

$$\text{(j)} \qquad \frac{\partial T}{\partial x} = \frac{H_s}{u \rho C_p h} = \left[\frac{°C}{L}\right]$$