Answer 9.2
A perfectly absorbing boundary can be treated like a dissolving boundary with $C_{eq} = 0$. The boundary is a sink rather than a source, otherwise the process of exchange between the bed and the water column is the same. Here, dye is injected as a continuous point source and the evolving plume experiences a sink at the absorbing boundary. If the system has fast mixing, then we can assume that $\partial C/\partial z = \partial C/\partial y = 0$ and use the fast-mixing model for bed-exchange. Then the effects of the boundary sink are modeled as a distributed sink S. For steady-state conditions and Pe >> 1, the transport equation

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = \frac{\partial}{\partial x}K_x\frac{\partial C}{\partial x} + \frac{\partial}{\partial y}D_y\frac{\partial C}{\partial y} + \frac{\partial}{\partial z}D_z\frac{\partial C}{\partial z} \pm S,$$

will reduce to,

(1) $$u\frac{\partial C}{\partial x} = -S.$$

However, if the dye mixes slowly over the cross-section, we cannot assume $\partial C/\partial z = \partial C/\partial y = 0$ in the channel. Under these conditions we would use the solution for a 3-D, steady, continuous release (equation 8, chapter 6), with positive image sources to account for the no-flux side-boundaries, and a negative image source to account for the perfectly absorbing bed.

Before proceeding with (1), we must check all the assumptions. First, we will determine if the flow is turbulent, and if it is we will estimate turbulent diffusivities. The hydraulic radius is (5cm x 10cm)/ (10cm + (2 x 5 cm)) = 2.5 cm. The Reynolds number based on hydraulic radius is, $Re_H = (10 cms^{-1} x 2.5 cm)/(0.01\ cm^2 s^{-1}) = 2500$, which indicates the flow is likely to be turbulent. Next to each boundary there is a laminar sub-layer with thickness, $\delta_s = 5\ \nu/u_*$. The friction velocity is estimated as $u_* \approx 0.1U = 1\ cms^{-1}$. This gives $\delta_s = 0.05$ cm.

Now, we estimate the coefficients of turbulent diffusion using the empirical relations for a straight channel given in Table 1 of Chapter 7.

$$D_{t,x} = 0.45\ u_*h = 2.3\ cm^2s^{-1}$$

$$D_{t,y} = 0.15\ u_*h = 0.75\ cm^2s^{-1}$$

$$D_{t,z} = 0.067\ u_*\ h = 0.34\ cm^2s^{-1}$$

Confirm Fast-Mixing Bed Exchange Model

To determine if the system will follow a fast-mixing or slow-mixing model of bed-exchange, we compare the time scale required for the channel to mix vertically with the time scale for diffusive flux to cross the laminar sub-layer.

$$\frac{T_{\delta s}}{T_L} = \frac{\delta_s^2 / D}{h^2 / D_t} = \frac{(0.05\text{cm})^2 (0.34\text{cm}^2\text{s}^{-1})}{(5\text{cm})^2 (10^{-5}\text{cm}^2\text{s}^{-1})} = 3.4$$

To be very confident that the fast-mixing model is appropriate, we require that $T_{\delta s}$ is an order of magnitude greater than T_L. Here the time scales only differ by a factor of three. However, the system is closer to the fast-mixing model then the slow-mixing model, so we proceed with that assumption.

Confirm well-mixed conditions ($\partial C/\partial y = \partial C/\partial z = 0$) for plume evolution.

To use (1) to describe plume evolution, we must confirm that the plume rapidly mixes over the channel cross-section. We need to find the distance from the source at which the plume is uniform in y and z. These distances are,

$$X_{mix,y} = b^2 u / (4 D_{t,y}) = (10\text{cm x } 10\text{cm x } 10 \text{ cm}^2\text{s}^{-1}) / (4 \text{ x } 0.75 \text{ cm}^2\text{s}^{-1}) = 333 \text{ cm}$$

$$X_{mix,z} = h^2 u / (4 D_{t,z}) = (5\text{cm x } 5\text{cm x } 10 \text{ cm}^2\text{s}^{-1}) / (4 \text{ x } 0.34 \text{ cm}^2\text{s}^{-1}) = 183 \text{ cm.}$$

This indicates that for distances greater than 333 cm from the source, the plume will be uniform in y and z. We are interested in the position x = 2000 cm, so we can model the concentration as if it originated from a one-dimensional source at x = 0. That is, we can assume $\partial C/\partial y = \partial C/\partial z = 0$.

Confirm assumption of Pe >>1

If $Pe = UL_x/K_x >> 1$, we can neglect longitudinal dispersion relative to longitudinal advection. The relevant length-scale is the distance at which we want to predict the concentration, L = 2000 cm. The longitudinal dispersion is $K_X = 5.9u_*h = 30 \text{ cm}^2\text{s}^{-1}$. Then, $Pe = (10 \text{ cms}^{-1} \text{ x } 2000 \text{ cm})/(30 \text{ cm}^2\text{s}^{-1}) = 666 >> 1$. So, this assumption is confirmed.

We have confirmed the assumptions that led to (1). Now, we can replace the sink term, S, in (1) with the form given in equation (16) in Chapter 9. That is,

$$(2) \qquad S = -\left\lfloor \frac{D_m A}{V\delta s} \right\rfloor (C - Ceq) = -k(C - Ceq).$$

With V/A = h, we estimate the bed-exchange rate constant $k = 10^{-5} \text{ cm}^2\text{s}^{-1} / (5\text{cm x } 0.05\text{cm}) = 4 \text{ x } 10^{-5} \text{ s}^{-1}$. In this system $C_{eq} = 0$, such that (1) becomes

$$(3) \qquad u\frac{\partial C}{\partial x} = -kC$$

As shown in Chapter (6), leading to and including equation 13, the initial concentration at the source will be C(x = 0) = $\dot{m}/ubh$. With this initial condition, the solution to (3) is.

$$(4) \qquad C(x) = \frac{\dot{m}}{ubh}\exp\left(-\frac{kx}{u}\right)$$

Note, a generic form of equation (4) was also given for a 1-D, steady, continuous release with first-order reaction in equation (11) of chapter 9.

Using (4) we find the concentration at x = 2000 cm to be,

$$C(x = 2000\ \text{cm}) = \frac{1\text{gs}^{-1}}{(10\text{cms}^{-1})(10\text{cm})(5\text{cm})}\exp\left(-\frac{(4\text{x}10^{-5}\text{s}^{-1})(2000\text{cm})}{10\text{cms}^{-1}}\right) = 0.00198\text{gcm}^{3}$$

In fact, the boundary sink does not make a significant contribution between x = 0 and 2000 cm, as the initial concentration is 0.002 gcm^{-3}. Barely 1 percent of the dye has been lost to the bed.