LECTURE 13
Graph algorithms
• Graph representation
• Minimum spanning trees
Greedy algorithms
• Optimal substructure
• Greedy choice
• Prim’s greedy MST algorithm

Prof. Charles E. Leiserson
Graphs (review)

**Definition.** A *directed graph (digraph)* $G = (V, E)$ is an ordered pair consisting of
- a set $V$ of *vertices* (singular: *vertex*),
- a set $E \subseteq V \times V$ of *edges*.

In an *undirected graph* $G = (V, E)$, the edge set $E$ consists of *unordered* pairs of vertices.

In either case, we have $|E| = O(V^2)$. Moreover, if $G$ is connected, then $|E| \geq |V| - 1$, which implies that $\lg |E| = \Theta(\lg V)$.

(Review CLRS, Appendix B.)
Adjacency-matrix representation

The *adjacency matrix* of a graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, is the matrix $A[1 \ldots n, 1 \ldots n]$ given by

$$A[i, j] = \begin{cases} 1 & \text{if } (i, j) \in E, \\ 0 & \text{if } (i, j) \not\in E. \end{cases}$$
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$\Theta(V^2)$ storage $\Rightarrow$ *dense* representation.
Adjacency-list representation

An *adjacency list* of a vertex $v \in V$ is the list $Adj[v]$ of vertices adjacent to $v$.

$$
\begin{align*}
\text{Adj}[1] &= \{2, 3\} \\
\text{Adj}[2] &= \{3\} \\
\text{Adj}[3] &= \{} \\
\text{Adj}[4] &= \{3\}
\end{align*}
$$
Adjacency-list representation

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<table>
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<tr>
<td>1</td>
<td>2, 3</td>
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<td>2</td>
<td>3</td>
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<td></td>
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For undirected graphs, $|Adj[v]| = \text{degree}(v)$.
For digraphs, $|Adj[v]| = \text{out-degree}(v)$. 
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For undirected graphs, $|Adj[v]| = \text{degree}(v)$. For digraphs, $|Adj[v]| = \text{out-degree}(v)$.

\textbf{Handshaking Lemma:} \( \sum_{v \in V} = 2|E| \) for undirected graphs $\Rightarrow$ adjacency lists use $\Theta(V + E)$ storage — a \textit{sparse} representation (for either type of graph).
Minimum spanning trees

**Input:** A connected, undirected graph $G = (V, E)$ with weight function $w : E \rightarrow \mathbb{R}$.

- For simplicity, assume that all edge weights are distinct. (CLRS covers the general case.)
Minimum spanning trees

Input: A connected, undirected graph $G = (V, E)$ with weight function $w : E \rightarrow \mathbb{R}$.
- For simplicity, assume that all edge weights are distinct. (CLRS covers the general case.)

Output: A spanning tree $T$ — a tree that connects all vertices — of minimum weight:

$$w(T) = \sum_{(u, v) \in T} w(u, v).$$
Example of MST
Example of MST
Optimal substructure

MST $T$: (Other edges of $G$ are not shown.)
Optimal substructure

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Remove any edge $(u, v) \in T$. 
Optimal substructure

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MST $T$:  

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Remove any edge $(u, v) \in T$. Then, $T$ is partitioned into two subtrees $T_1$ and $T_2$. 
Optimal substructure

MST $T$:

(Other edges of $G$ are not shown.)

Remove any edge $(u, v) \in T$. Then, $T$ is partitioned into two subtrees $T_1$ and $T_2$.

**Theorem.** The subtree $T_1$ is an MST of $G_1 = (V_1, E_1)$, the subgraph of $G$ induced by the vertices of $T_1$:

$$V_1 = \text{vertices of } T_1,$$

$$E_1 = \{(x, y) \in E : x, y \in V_1\}.$$

Similarly for $T_2$. 

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*Introduction to Algorithms* 

October 27, 2004 

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Proof of optimal substructure

Proof. Cut and paste:

\[ w(T) = w(u, v) + w(T_1) + w(T_2). \]

If \( T_1' \) were a lower-weight spanning tree than \( T_1 \) for \( G_1 \), then \( T' = \{(u, v)\} \cup T_1' \cup T_2 \) would be a lower-weight spanning tree than \( T \) for \( G \). □
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Do we also have overlapping subproblems?

\textbullet Yes.
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Do we also have overlapping subproblems?
• Yes.

Great, then dynamic programming may work!
• Yes, but MST exhibits another powerful property which leads to an even more efficient algorithm.
Hallmark for “greedy” algorithms

**Greedy-choice property**
*A locally optimal choice is globally optimal.*
Hallmark for “greedy” algorithms

**Greedy-choice property**

A locally optimal choice is globally optimal.

**Theorem.** Let $T$ be the MST of $G = (V, E)$, and let $A \subseteq V$. Suppose that $(u, v) \in E$ is the least-weight edge connecting $A$ to $V - A$. Then, $(u, v) \in T$. 
Proof of theorem

Proof. Suppose \((u, v) \notin T\). Cut and paste.

\[ T: \]

\[ \in A \]

\[ \in V - A \]

\((u, v) = \text{least-weight edge connecting } A \text{ to } V - A\)
**Proof of theorem**

**Proof.** Suppose \((u, v) \notin T\). Cut and paste.

Consider the unique simple path from \(u\) to \(v\) in \(T\).
Proof of theorem

Proof. Suppose \((u, v) \notin T\). Cut and paste.

Consider the unique simple path from \(u\) to \(v\) in \(T\). Swap \((u, v)\) with the first edge on this path that connects a vertex in \(A\) to a vertex in \(V - A\).
Proof of theorem

Proof. Suppose \((u, v) \notin T\). Cut and paste.

Consider the unique simple path from \(u\) to \(v\) in \(T\).
Swap \((u, v)\) with the first edge on this path that connects a vertex in \(A\) to a vertex in \(V - A\).
A lighter-weight spanning tree than \(T\) results.
Prim’s algorithm

**IDEA:** Maintain $V - A$ as a priority queue $Q$. Key each vertex in $Q$ with the weight of the least-weight edge connecting it to a vertex in $A$.

$$Q \leftarrow V$$
$$\text{key}[v] \leftarrow \infty \text{ for all } v \in V$$
$$\text{key}[s] \leftarrow 0 \text{ for some arbitrary } s \in V$$

while $Q \neq \emptyset$

    do $u \leftarrow \text{EXTRACT-MIN}(Q)$

    for each $v \in \text{Adj}[u]$

        do if $v \in Q$ and $w(u, v) < \text{key}[v]$

        then $\text{key}[v] \leftarrow w(u, v)$  \[\text{DECREASE-KEY}\]

        $\pi[v] \leftarrow u$

At the end, $\{(v, \pi[v])\}$ forms the MST.
Example of Prim’s algorithm
Example of Prim’s algorithm

\[ \in A \quad \in V - A \]
Example of Prim’s algorithm

\[ \in A \]
\[ \in V - A \]
Example of Prim’s algorithm

∈ \( A \)

∈ \( V - A \)

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Example of Prim’s algorithm
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\[ \epsilon A \quad \epsilon V - A \]
Example of Prim’s algorithm

\[ A \subseteq V \]
Example of Prim’s algorithm

\[ \in A \]
\[ \in V - A \]
Example of Prim’s algorithm

∈ A
∈ V − A

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Introduction to Algorithms
Example of Prim’s algorithm

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Analysis of Prim

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Analysis of Prim

$\Theta(V)$ total

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$|V|$ times

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\( \Theta(V) \) total

\( |V| \) times

\( \text{degree}(u) \) times
Analysis of Prim

$\Theta(V)$ total

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Handshaking Lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY’s.
Analysis of Prim

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Handshaking Lemma \( \Rightarrow \Theta(E) \) implicit \( \text{DECREASE-KEY}'s \).

Time = \( \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}} \)
Analysis of Prim (continued)

Time = $\Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$
Analysis of Prim (continued)

Time = $\Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$

| Q | $T_{\text{EXTRACT-MIN}}$ | $T_{\text{DECREASE-KEY}}$ | Total |
Analysis of Prim (continued)

Time = $\Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$

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Analysis of Prim (continued)

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Analysis of Prim (continued)

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<tr>
<td>Fibonacci heap</td>
<td>$O(lg V)$</td>
<td>$O(1)$</td>
<td>$O(E + V \cdot lg V)$</td>
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MST algorithms

Kruskal’s algorithm (see CLRS):
- Uses the *disjoint-set data structure* (Lecture 10).
- Running time $= O(E \lg V)$. 
MST algorithms

Kruskal’s algorithm (see CLRS):
• Uses the disjoint-set data structure (Lecture 10).
• Running time $= O(E \lg V)$.

Best to date:
• Karger, Klein, and Tarjan [1993].
• Randomized algorithm.
• $O(V + E)$ expected time.