Lecture 3: Expectation, Kolmogorov’s Theorem about Consistent Families of Distributions

Lemma 1. \( X_i : \Omega \to \mathbb{R} \)

1. \( \mathbb{P}(X_1 \leq t_1, \ldots, X_n \leq t_n) = \mathbb{P}(X_1 \leq t_1) \times \cdots \times \mathbb{P}(X_n \leq t_n) \) \( \Leftrightarrow X_i \)'s are independent \( \text{indpt.} \)

2. If \( X_i \) has density \( f_i(x) \), then \( X_i \)'s are indpt. \( \Leftrightarrow \exists f(x_1, \ldots, x_n) = \prod f_i(x_i) \) joint density

Proof of property (2)

\[ \text{\( \Rightarrow \) } \mathbb{P}(\cap \{X_i \in A_i\}) = \mathbb{P}(\bar{X} \in A_1 \times \cdots \times A_n) = \prod_{i \leq n} \int_{A_i} f_i(x_i) dx_i \text{ by Fubini’s Theorem} \]

\[ = \prod_{i \leq n} \mathbb{P}(X_i \in A_i) \]

\[ \text{\( \Rightarrow \) } \mathbb{P}(\bar{X} \in A_1 \times \cdots \times A_n) = \prod \mathbb{P}(X_i \in A_i) \text{ } \int_{A_1 \times \cdots \times A_n} \prod f_i(x_i) dx_i \]

for \( B \in \mathcal{B} - \text{Borel sets in } \mathbb{R}^n \)

Borel sets start with algebra \( A = \{ \text{finite unions of disjoint } A_1 \times \cdots \times A_n \} \Rightarrow \mathcal{B} = \sigma(A) \)

\( \mathbb{P}(\bar{X} \in B) = \int_B \prod f_i(x_i) dx \), \( B \in A \)

Both \( \mathbb{P}(B), \int_B \prod f_i(x_i) dx \) - countably additive, finite \( \mathbb{P}(\mathbb{R}^n) = \int_B \prod f_i(x_i) dx = 1 \)

By Caratheodory theorem \( \Rightarrow P, \int_B \prod f_i(x_i) dx \) uniquely extends to \( B = \sigma(A) \)

Expectation

\( X : \Omega \to \mathbb{R} \) random variable on \( (\Omega, \mathcal{A}, \mathbb{P}) \) then expectation of \( X \) is \( \mathbb{E}X = \int_{\Omega} X d\mathbb{P} \) has all the properties of the integral.

Lemma 2. \( F = \text{distribution or law of } X \) on \( \mathbb{R} \)

1. \( \mathbb{E}g(x) = \int_{\mathbb{R}} g(x) dF(x) \)

2. \( X - \text{discrete, } \mathbb{P}(X \in \{S_i\}_{i \geq 1}) = 1, \mathbb{E}g(X) = \sum_{i \geq 1} g(x_i) \mathbb{P}(X = S_i) \)

3. \( X \) has density \( f(X), \mathbb{E}g(X) = \int g(x) f(x) dx \)

\[ \mathbb{E}g(X) = \int_{\Omega} g(X(\omega)) d\mathbb{P}(\omega) \circ X^{-1}(x) = \int_{\mathbb{R}} g(\omega) d\mathbb{P} = \int_{\mathbb{R}} g(x) dF(x) \text{ where } x = X(\omega) \text{ and } \mathbb{P}x = F \]

\( g(x) = \mathbb{I}(x \in B) \)

\[ \mathbb{E} \mathbb{I}(X \in B) = \mathbb{P}(X \in B) = F(B) = \int_{\mathbb{R}} \mathbb{I}(x \in B) dF(x) \Rightarrow g(x) = \sum_{i \geq n} w_i \mathbb{I}(x \in B_i), B_i \text{ disjoint} \]

Kolmogorov’s theorem about consistent distributions
(The existence of random processes theory)

\( (\Omega, \mathcal{A}, \mathbb{P}) \quad X : \Omega \to \mathbb{R}, (\mathbb{R}, \mathcal{B}, \mathbb{P}_X) \)

\( \bar{X} : \Omega \to \mathbb{R}^k, (\mathbb{R}^k, \mathcal{B}, \mathbb{P}_X), B - \text{Borel set in } \mathbb{R}^k \)

\( \{X_i\}_{i \leq 1}, \{\sup_{i \leq 1} X_i \leq 1\}, \{X_i\}_{i \in T} \in \mathbb{R}^T = \prod_{t \in T} \mathbb{R} = \{f : t \to \mathbb{R}\} \)
\[ N \subseteq T, \text{ } N \text{ finite}, \{X_t\}_{t \in N} \]

\[ B \in B_N - \text{ Borel sets on } \mathbb{R}^N, \mathbb{P}_N(B) \text{ distribution} \]

Consistency: \( N \subseteq M - \text{ finite subset} \)

\[ \mathbb{P}_N(B) = \mathbb{P}_M(B \times \mathbb{R}^{M-N}) \]

Algebra of sets on \( \mathbb{R}^T; A = \{B \times \mathbb{R}^{T-N} : B \in B_N\} \)

Is \( A \) an algebra? Yes.

\[ B_1 \times \mathbb{R}^{T-N_1}, B_2 \times \mathbb{R}^{T-N_2}, B_1 \subseteq \mathbb{R}^{N_1}, B_2 \subseteq \mathbb{R}^{N_2} \implies B_1 \times \mathbb{R}^{(N_1 \cup N_2)\setminus N_1} \subseteq \mathbb{R}^{N_1 \cup N_2} \]

\[ \mathbb{P}(B \times \mathbb{R}^{T-N}) = \mathbb{P}_N(B) - \text{ defined correctly} \]

Let \( A = \sigma(A) - \sigma\text{-algebra generated by } A \)

\( A \) - cylindrical algebra, \( A \) - cylindrical \( \sigma\) - algebra

\( B \times \mathbb{R}^{T-N} \) cylinder, \( B \) - base of the cylinder

\[
\begin{array}{c}
\mathbb{R}^{T-N} \\
\text{B} \\
\mathbb{R}^N
\end{array}
\]

Can we extend \( \mathbb{P} \) from \( A \) to \( A = \sigma(A) \)?

\( \mathbb{P} - \sigma\text{-additive on } A \)

\[ \uparrow \text{ continuity of measures} \]

\[ A \ni B_n \supseteq B_{n+1} \]

\[ A \ni B = \bigcap_{i \geq 1} B_i, \lim_{n \to \infty} \mathbb{P}(B_n) = \mathbb{P}(B) \]

\[ B'_n = B_n \setminus B, B' = \bigcap_{i \geq 1} B'_i = \emptyset \]

\[ \lim_{n \to \infty} \mathbb{P}(B'_n) = \mathbb{P}(\emptyset) = 0 \]

\[ A \ni B_n \supseteq B_{n+1}, \bigcap_{i \geq 1} B_i = \emptyset, \mathbb{P}(B + n) \to 0 \]

**Proof.** By negation.

\[ \forall n, \mathbb{P}(B_n) > \epsilon > 0 \implies \bigcap_{i \geq 1} \neq \emptyset \]

\( B_n = C_n \times \mathbb{R}^{T-N_n}, N_n - \text{ finite, } C_n \in B_{N_n} \)

\[ \exists K_n \text{ (compact in } \mathbb{R}^{N_n}) \subseteq C_n, \mathbb{P}_{N_n}(C_n \setminus K_n) \leq \frac{\epsilon}{2^n+1} \]

\[ \epsilon/2^k \]

As \( R \to \infty \), what’s outside the ball \( \to \emptyset \) by continuity of measure.

Can approximate any set by compact on the inside.