Fluid Models for Traffic and Pricing

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Abstract—Fluid dynamics models provide a powerful deterministic technique to approximate stochasticity in a variety of application areas. In this paper, we study two classes of fluid models, investigate their relationship as well as some of their applications. This analysis allows us to provide analytical models of travel times as they arise in dynamically evolving environments, such as transportation networks as well as supply chains. In particular, using the laws of hydrodynamic theory, we first propose and examine a general second order fluid model. We consider a first-order approximation of this model and show how it is helpful in analyzing the dynamic traffic equilibrium problem. Furthermore, we present an alternate class of fluid models that are traditionally used in the context of dynamic traffic assignment. By interpreting travel times as price/inventory-sojourn-time relationships, we are also able to connect this approach with a tractable fluid model in the context of dynamic pricing and inventory management. Finally, we investigate the relationship between these two classes of fluid models.

I. INTRODUCTION

In this paper, we study a variety of fluid dynamics models as they arise in diverse application domains such as transportation and dynamic pricing. This analysis allows us to investigate and provide insights to dynamic phenomena that arise in a variety of systems that share similar characteristics. Such systems include transportation networks as well as supply chain and inventory management systems. An underlying common characteristic in these systems is some form of travel time. These systems are dynamic in nature. In particular, due to their service time and the inherent disequilibrium between demand and supply, these systems give rise to dynamic delays. For example, in ground transportation, poor-quality roads and congested traffic conditions cause travelers to experience delays in traversing a network’s path. In inventory management systems, a high unit price and a high level of inventory of a good may cause a newly produced unit of that good to incur a delay before it is sold.

As a result, it is important for traffic planners to understand and manage the nature of travelers’ delays (costs) in urban and highway transportation systems, and for the supply chain industry to design optimal pricing and inventory management strategies that maximize profits, reduce inventory levels, and effectively manage the delays that goods incur before being sold.

Therefore, understanding the nature of the dynamic phenomena arising in these systems, exploring their common characteristics, and designing mechanisms to manage them effectively, have potential for tremendous economic, social and political impacts. In this paper, we will explore the relationship between these systems.

The literature in transportation takes two approaches when modeling travel times. The first and more traditional approach assumes a predetermined functional form that describes the relationship between travel times and flow rates. This is typically determined through a statistical analysis. Practitioners in the transportation community have been using several functional forms to describe travel times. These include the BPR function (Bureau of Public Roads, [9]) which is used to estimate travel times at priority intersections, and is a polynomial function. Menejguer et al. [4] proposed an exponential travel time function for all-way-stop intersections. Akcelik [1] also proposed a polynomial-type travel time function for links at signalized intersections. Nevertheless, considering a specific link travel time function in advance has some drawbacks. These travel time functions may not describe accurately peak period traffic dynamics especially since there are dramatic changes in traffic conditions in a short period of time. Travel times in a dynamic transportation network depend both on prevailing traffic conditions and future traffic conditions relative to the departure time. As a result, this approach can lead to controversial results (see Daganzo, [6]). A second approach for modeling travel times considers travel time functions as an output rather than an input in the model. One can achieve this by determining the functional forms for travel times through an analytical method. Perakis [16] and [10] provide a first order model through the hydrodynamic model of Lighthill and Whitham [13]. As a result they do not assume in advance specific link travel time functions. This latter approach is attractive due to the fact that it takes into account the traffic flow dynamics but also provides analytical solutions. This paper extends this approach by proposing a second order model that incorporates additional phenomena. Furthermore, we examine its relation to alternate fluid formulations in the literature.

A further motivation for using fluid models is that they have provide good policies in a variety of settings. Furthermore, these models have been shown to approximate well the underlying stochasticity of problems in a deterministic way (see for example, Bertsimas and Paschalidis [2]).

The contributions in this paper are the following: 1. We develop a general second order model for travel times in a dynamic transportation network. 2. Using this model, we derive closed-form solutions for travel times. These travel times correspond to the ones used in practice. We illustrate and connect these travel time functions through a numerical example. 3. We establish a connection between the general second order model and a Dynamic Pricing Model. 4. We consider the applications of the fluid models in this paper in the dynamic user-equilibrium problem as well in dynamic pricing and inventory management of non-
perishable products.

The paper by McGill and van Ryzin [15], and the references therein, provide a thorough review of revenue management and pricing models.

The structure of this paper is as follows. In Section II, we consider a general second order fluid dynamics model for determining travel times. We study this model and some of its simplifications that allow us to propose closed form solutions for travel times. Furthermore, to illustrate these results, we consider a numerical example. In Section III, we illustrate how our results can be useful in the context of dynamic user-equilibrium. Although our focus is on transportation networks, in Section IV, we also demonstrate how we can use our results in the context of dynamic pricing and inventory management. We establish a connection between the two classes of fluid models. Finally we discuss our conclusions.

II. A Second Order Fluid Model

In this section we present and study a second order fluid model for determining travel times in dynamic transportation networks and illustrate our results through a numerical example.

A. Notation

We represent the physical transportation network through a directed network $G = (N, I)$, where $N$ is the set of nodes and $I$ is the set of directed links. Index $w$ denotes an Origin-Destination (O-D) in the set $W$ of origin destination pairs. Index $P$ denotes the set of paths and index $P_w$ denotes the set of paths between O-D $w$.

Path variables:

- $|P|$: number of paths in the network;
- $x_p$: position on path $p$;
- $L_p$: length of path $p$;
- $T_p(L_p, t)$: path travel time on path $p$ starting at $t$;

Link variables:

- $|I|$: number of directed links;
- $x_i$: position on link $i$;
- $L_i$: length of link $i$;
- $f_i(x_i, t)$: flow rate on link $i$ at time $t$;
- $f(0, t)$: vector of departure link flow rates;
- $T_i(x_i, t)$: travel time on $i$ at time $t$;
- $T_i(L_i, t)$: traversal time on $i$ at time $t$;
- $u_i(x_i, t)$: speed on link $i$ at time $t$;
- $k_i(x_i, t)$: density on link $i$ at time $t$;
- $u_{i\text{max}}$: maximum speed on $i$;
- $k_{i\text{max}}$: maximum density on $i$;
- $k_{i\text{max}}$: storage capacity rate of $i$;

Link-path flow variables:

- $ip$: link-path pair;
- $i'p$: predecessor of $i$ on $p$;
- $\delta_{ip} = 1$ if $i$ belongs to $p$, and 0 otherwise;
- $L_{ip}$: length from origin of $p$ until start of $i$;
- $T_{ip}(L_{ip}, t)$: partial path travel time from the origin of $p$ until the beginning of $i$ at time $t$;

B. A Generic Second Order Model

In what follows we use the laws of hydrodynamic theory to determine the delay (travel time in this model) to traverse a link as a function of the rate of flow on the links of the transportation system. In the mid-1950’s Lighthill and Whitham [13] and independently Richards [18], proposed a model for traffic flow on a single stretch of road. Based on this approach, Perakis [16] proposed an analytical model for travel times and connected it to the dynamic network equilibrium problem. Using a macroscopic fluid dynamics approach, this model extends the hydrodynamic theory of traffic flow on a single stretch of road to general networks.

In the mid-1950’s Lighthill and Whitham [13] and independently Richards [18], proposed the assumption that the velocity at any point depends only on the density. In mathematical terms $u_i = \hat{u}_i(k)$. The function $\hat{u}_i(.)$ is empirically measured and is input to the model. Several models have been proposed in the literature for the velocity function $\hat{u}_i(.)$. Reichschel [17], Mahmassani and Herman [14], and others proposed the linear separable model $\hat{u}_i(k) = u_i^{\text{max}}(1 - \frac{k_i}{k_{i\text{max}}})$, where they assume that at free flow, speed is at its maximum (i.e. $\hat{u}_i(0) = u_i^{\text{max}}$), and at maximum density, the speed is zero (i.e. $\hat{u}_i(k_{i\text{max}}) = 0$).

We design a generic model (Model 1 below) to account for

(i) Drivers’ reaction to upstream congestion or decongestion. In particular, when a driver realizes the formation of a queue upstream, he/she starts slowing down. Similarly, drivers start accelerating when the queue starts dissipating.

(ii) Effects on a link of densities as well as variations in densities of neighboring links.

To account for the two phenomena, we replace the commonly used speed-density relationship $u_i = \hat{u}_i(k)$ with a more general function: $u_i = \overline{u}_i(k, \nabla k)$. The variables $k$ and $\nabla k$ contain the term $\frac{\partial k}{\partial x_j}$ that allows us to model the reaction of drivers to changes in the link density. They also contain the terms $k_j$ and $\frac{\partial k_j}{\partial x_j}$ for the set of links $j$ in the neighborhood of link $i$, that allow us to effectively model link interaction. We propose the following general form of the velocity of link $i$, at position $x_i$ and at time $t$:

\[
\overline{u}_i(k, \nabla k) = u_i^{\text{max}} - b_i(u_i^{\text{max}})^2 k_i(x_i, t) - \frac{\lambda_i(x_i)}{k_i(x_i, t)} \frac{\partial k_i(x_i, t)}{\partial x_j} + \sum_{j \in I} \alpha_{ij}(x_i) k_j(\mathbf{x}_j, t - \Delta_{ij}) + \sum_{j \in I} \beta_{ij}(x_i) \frac{\partial k_j(\mathbf{x}_j, t - \Delta_{ij})}{\partial x_j},
\]

where $\alpha_{ij}(x_i)$ and $\beta_{ij}(x_i)$ are density correlation functions between link $i$ and link $j$ and depend on the position $x_i$ on link $i$: $\mathbf{x}_j$ is a fixed position of a detector of density on link $j$ and $\Delta_{ij}$ is a propagation time between link $i$ and link $j$.

The term $- \frac{\lambda_i(x_i)}{k_i(x_i, t)} \frac{\partial k_i}{\partial x_i}$ is borrowed from heat transfer and accounts for the drivers’ awareness of upstream and
downstream conditions. The heat transfer term $\lambda_i(x_t)$ is a positive term expressed in squared miles per unit of time. The propagation term $\frac{\partial u_i(x_t)}{\partial x_t}$ expresses the variation in the speed induced by a variation in the density. For instance, when a queue is expanding on link $i$, the term $\frac{\lambda_i(x_t)}{k_i(x_t)} \frac{\partial u_i}{\partial x_t}$ is negative and hence the velocity function $u_i(x_t, t)$ decreases. We develop several approximations of the general model that make the analysis more tractable. Model 1 can be formulated as follows. For all $t \in [0, T]$.

**Model 1**

\[
\begin{align*}
\frac{\partial q_i(x_t, t)}{\partial t} + \frac{\partial q_i(x_t, t)}{\partial x_t} &= 0, &\text{for all } i, \\
(f_i(0, t + T_i) = h_i^0(T_i), &\text{for all } i, \\
q_i(x_t, t) &= k_i(x_t)u_i(x_t, t), &\text{for all } i, \\
u_i = \bar{u}_i(k_i), &\text{for all } i, \\
\frac{d u_i(x_t)}{dx_t} &= \frac{1}{u_i}, &\text{for all } i, \\
T_i(0, t) &= 0, &\text{for all } i, \\
T_p(L_p, t) &= \sum_{i \in P} T_i(L_i, t + T_{ip}(L_{ip}, t))\delta_{ip}, &\text{for all } p \in P.
\end{align*}
\]

When terms $\lambda_i(\cdot)$, $\alpha_i(\cdot)$ and $\beta_i(\cdot)$ become 0, the above speed-density relationship becomes $u_i = \bar{u}_i(k_i) = u_{i}^{max} - b_i(u_{i}^{max})^2 k_i(x_t, t)$. The latter corresponds to the first-order model proposed by Perakis [16].

Model 1 is very hard to analyze in its current form. For this reason, in the following subsection, we consider two simplified models of Model 1.

**C. Two Simplified Second-Order Separable Models of Travel Time Functions**

In this subsection, we focus on a separable version of the velocity function (1). Our goal is to solve Model 1 and propose specific travel time functions. To achieve this, as a first step, we eliminate some of the variables involved in the model. We eliminate the density variables by expressing them as functions of the flow rates. This leads us to simplifying Model 1. We first impose the following assumptions:

(A1) The speed on link $i$ is a function of the density $k$ and its gradient $\nabla k$ (that is, $\bar{u}_i(k, \nabla k)$) is a separable function that depends solely on the density $k_i$ of link $i$). Assumption (A1) suggests that $u_i = \bar{u}_i(k, \nabla k)$. In particular, we consider $\bar{u}_i(k, \nabla k) = u_{i}^{max} - b_i(u_{i}^{max})^2 k_i(x_t, t) - \frac{\lambda_i(x_t)}{k_i(x_t)} \frac{\partial u_i}{\partial x_t}$, where $b_i$ is a constant. An example of $b_i$ is $\frac{1}{u_{i}^{max}}$. This assumption will allow us to express the density as a function of the flow rate.

(A2) The density $k_i \leq \frac{1}{2b_i u_i^{max}}$. When $b_i = \frac{1}{u_i^{max}}$, then this assumption becomes $k_i \leq k_i^{max}$. Notice that this assumption is not too restrictive since it still allows the density on link $i$ to be high, especially when the maximum density on link $i$, $k_i^{max}$ is very high.

**Lemma 1:** Under Assumptions (A1), (A2), the link density as a function of the link flow rate and the queu propagation term can be expressed as:

\[
k_i = \frac{1}{2b_i u_i^{max}}(1 - 4b_i(f_i + \lambda_i(x_t) \frac{\partial k_i}{\partial x_t}))^{\frac{1}{2}}.
\]

**Proof:** Combining the speed-density and the flow-speed-density (i.e. $f_i = k_i u_i$) relationships, we derive $f_i = u_{i}^{max} k_i - b_i(u_{i}^{max})^2 k_i^2 - \lambda_i(x_t) \frac{\partial k_i}{\partial x_t}$ (*). By solving in terms of $k_i$, for low densities, or equivalently for stable flows, we obtain the result of the lemma. Q.E.D.

**Remark:** When assumption (A2) is violated (i.e. $k_i > \frac{1}{2b_i u_i^{max}}$) that is, when the system operates under the regime of very high densities (i.e. close to the maximum density), then the density $k_i$ is equal to the other root of the second-degree polynomial (*), that is, equation (9) is replaced by $\frac{1}{2b_i u_i^{max}}(1 + (1 - 4b_i(f_i + \lambda_i(x_t) \frac{\partial k_i}{\partial x_t}))^{\frac{1}{2}}$). The analysis of the model under densities close to $k_i^{max}$ is similar to the one under Assumption (A2), and is not included for the sake of brevity.

Next, we impose some additional assumptions that will allow us to simplify (9).

(A3) The term $\frac{1}{u_{i}^{max}} << 1$. (Indeed, consider for example the case where $u_{i}^{max} = 40$ miles / hr, then, $\frac{1}{u_{i}^{max}} = \frac{1}{40} << 1$)

(A4) The term $\lambda_i(x_t) \frac{\partial k_i}{\partial x_t} << 1$. (This assumption implies that when compared to other components of the velocity function (1) since as the effect of link densities, the reaction of drivers to upstream congestion or decongestion (e.g. the gradient of the density) has only a first-order effect).

(A5) The link flow rate $f_i(0, t + \tau_i)$ can be approximated through a continuously differentiable function $h_i^0(T_i)$ of $\tau_i$. In particular, using Assumptions (A1)-(A5) in (9) will allow us to consider two approximations of Model 1. These approximations enable us to describe the conservation law of cars (2) only in terms of the link flow rates. In particular, the first model we describe is a second order polynomial travel time model as it gives rise to a polynomial-type family of travel times.

**The Second Order PTT Model**

\[
\begin{align*}
(1 + 2b_i f_i) \frac{\partial f_i}{\partial T_i} + \frac{\partial f_i}{\partial x} &= \lambda_i(x_t) \frac{\partial^2 f_i}{\partial x^2}, &\text{for all } i, \\
(1 + 2b_i f_i) \frac{\partial f_i}{\partial x} &= \lambda_i(x_t) \frac{\partial^2 f_i}{\partial x^2}, &\text{for all } i, \\
f_i(0, t + T_i) = h_i^0(T_i), &\text{for all } i, \\
k_i &= \frac{f_i + \lambda_i(x_t) \frac{\partial k_i}{\partial x}}{u_{i}^{max}} + \frac{b_i(f_i + \lambda_i(x_t) \frac{\partial k_i}{\partial x})^2}{u_{i}^{max}} \\
u_i = \frac{f_i}{b_i}, &\text{for all } i, \\
\frac{d T_i(x_t, t)}{d x_t} &= \frac{1}{u_i}, &\text{for all } i, \\
T_i(0, t) &= 0, &\text{for all } i, \\
T_p(L_p, t) &= \sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t))\delta_{ip}, &\text{for all } p \in P.
\end{align*}
\]
for all $t \in [0,T]$.

The second model we describe (called a second order exponential travel time model) gives rise to an exponential family of travel times. It is perhaps a more accurate description of reality as we will illustrate through examples in the next subsection. Nevertheless, as these examples will illustrate, the two models coincide in some situations.

The Second Order ETT Model

$$\frac{\partial f_i}{\partial t} + u_i^{\max}(1 - 2b_i f_i) \frac{\partial f_i}{\partial x_i} = \lambda_i(x_i) \frac{\partial^2 f_i}{\partial x_i^2}, \text{ for all } i \in I,$$

$$f_i(0, t + T_i) = h_i^0(T_i), \text{ for all } i \in I,$$

$$k_i = \frac{f_i + \lambda_i(x_i) \frac{\partial f_i}{\partial x_i} + b_i(f_i + \lambda_i(x_i) \frac{\partial f_i}{\partial x_i})^2}{u_i^{\max}},$$

$$u_i = \frac{f_i}{k_i}, \text{ for all } i \in I,$$

$$\frac{dT_i(x_i, t)}{dx_i} = \frac{1}{u_i}, \text{ for all } i \in I,$$

$$T_i(0, t) = 0, \text{ for all } i \in I,$$

$$T_p(L_p, t) = \sum_{i \in I} T_i(L_i, t + T_p(L_{ip}, t)) \delta_{ip}, \text{ for all } p \in P,$$

for all $t \in [0, T]$.

In what follows we will solve these models and provide examples illustrating how the models connect.

D. First-Order Separable Models

In Kachani and Peralis [10], we study the travel time models we proposed in the previous subsection for the case of a separable first-order approximation of the velocity function (1). This further simplifies the Second Order PTT Model giving rise to the Separable PTT Model and the Second Order ETT Model giving rise to the Separable ETT Model. This allows us to propose specific families of travel time functions.

We assume that during time periods $[t_j, t_{j+1}]$, users make the approximation that the link flow rates $f_i(x_i, t)$ at the head of link $i$ ($x_i = 0$) for subsequent times $t_j + T_i$ are linear ($f_i(0, t_j + T_i) = A_i(t_j) + B_i(t_j)T_i$) or quadratic ($f_i(0, t_j + T_i) = A_i(t_j) + B_i(t_j)T_i + C_i(t_j)T_i^2$) in terms of the link travel time $T_i$. Based on these approximations of link departure flow rates, we propose several classes of travel time functions. These apply to both the PTT and the ETT models.

The simplest example of a link travel time function we derive corresponds to the case of separable velocity functions (i.e. link velocity function $v_i(k) = u_i^{\max}(1 - \frac{k}{k_i^{\max}})$) with a constant approximation of link departure flow rates (i.e. $f_0(0, t_j + T_i) = f_0(0, t_j)$). Naturally, we would expect that the travel time function decreases as the maximum speed increases or as the maximum density increases. Indeed, the travel time function we derive $T_i(x_i, t) = \frac{A_i(0)}{u_i}[(1 + \frac{A_i(t)}{u_i^{\max}})x_i]$ satisfies this property. Another simple example is the case of separable velocity functions with a piecewise linear approximation of link departure flow rates, where $|B_i(t)| < \frac{u_i^{\max}}{|2f_i|}$. The travel time function we derive is $T_i(x_i, t) = \frac{1}{u_i^{\max}}[(1 + \frac{A_i(t)}{u_i^{\max}})x_i - \frac{A_i(t)B_i(t)}{2u_i^{\max}}x_i^2]$.

In general, the Linear PTT Model leads to the following polynomial family of travel time functions

$$T_i(x_i, t) = \frac{x_i}{u_i^{\max}} + A_i(t)B_i(t)(1 + \frac{2b_iB_i(t)x_i}{u_i^{\max}})^2 - 1),$$

and the Linear ETT Model leads to the exponential family of travel time functions

$$T_i(x_i, t) = \frac{\theta_1 x_i}{\theta_2(u_i^{\max})^2} - \frac{\theta_1 x_i}{\theta_2(u_i^{\max})^2},$$

where, $\theta_i, i \in \{1, 2, 3\}$ are appropriate parameters (see [10]).

It is very important to note that equations (10) and (11) coincide when $|B_i(t)| < \frac{u_i^{\max}}{|2f_i|}$ holds. That is, they possess the same second-order Taylor expansion

$$T_i(x_i, t) = \frac{1}{u_i^{\max}}[(1 + A_i(t)b_i)x_i - \frac{A_i(t)B_i(t)b_i^2}{2u_i^{\max}}x_i^2].$$

We refer to the travel time function above as the limit of the linear PTT and ETT.

This relationship shows that the assumptions made for both the Linear PTT Model and the Linear ETT Model are indeed reasonable.

Furthermore, the Quadratic PTT Model gives rise to a more complicated expression of link travel time functions. The third degree Taylor expansion leads to

$$T_i(x_i, t) = \frac{1}{u_i^{\max}}[(1 + A_i(t)b_i)x_i - \frac{A_i(t)B_i(t)b_i^2}{2u_i^{\max}}x_i^2 + \frac{11A_i(t)B_i(t)b_i^3}{6u_i^{\max}}x_i^3 - \frac{4A_i(t)B_i(t)c_i(t)B_i(t)^2b_i^4}{3u_i^{\max}}x_i^4].$$

We observe that if the quadratic term is neglected (i.e. $C_i = 0$), then a second-order approximation of equation (13) leads to equation (12) and, as one would expect, we fall in the case of the Linear PTT Model. Hence, it appears that the assumptions made for the Quadratic PTT Model are also reasonable.

Below, we illustrate these families of travel time functions through a numerical example.

Example

We first consider a quadratic profile of a link departure flow rate function during a one hour period. This profile is depicted in Figure 1. It corresponds in practice to a rush hour period that starts at time zero and ends an hour later.
Its peak is attained after 30 minutes. The departure flow rate function is given by

\[ f_i(t) = 1600 - 6400(t - 0.5)^2, \]

where \( t \) is expressed in hours, and \( f_i(t) \) is expressed in vehicles per hour.

We derive a piecewise quadratic approximation of the link departure flow rate as follows

\[ A_i(t) = 1600 - 6400(t - 0.5)^2, \quad B_i(t) = -12800(t - 0.5), \]

\[ C_i(t) = -6400, \quad f_i(0,t + T_i) = A_i(t) + B_i(t).T_i + C_i(T_i).T_i^2. \]

We consider two scenarios. The first scenario corresponds to \( u_i^{\text{max}} = 25 \text{ miles/hr}, k_i^{\text{max}} = 175 \text{ vehicles/mile} \) and \( L_i = 8 \text{ miles} \). In this case, \( B_i(t) \) is of the same order of magnitude as \( u_i^{\text{max}} \). As a result, we expect the PTT and the ETT models to lead to different travel times.

We discretize the time interval into intervals of 20 seconds. For each time interval, we compute \( A_i(t), B_i(t) \), and \( C_i(t) \). We also compute the link traversal times using the expressions of the four travel time functions derived in this section. Figure 2 provides a plot of the Linear PTT, the Linear ETT, the limit of the linear PTT and ETT, and the Quadratic PTT travel time functions during the one hour period. Notice that a finer discretization would lead to the same plots. This example leads us to the following observations:

- The travel time functions of the Linear PTT and the limit of the linear PTT and ETT models almost coincide.
- Since the Quadratic PTT Model takes into account the quadratic term \( C_i(t) \) in the departure flow rate function, the travel time of this model is more accurate than the travel time of the Linear PTT Model.
- The Linear PTT, the limit of the linear PTT and ETT, and the Quadratic PTT models lead to a symmetric quadratic shape that is similar to the profile of the link departure flow rate function. However, the Linear ETT model displays an asymmetric behavior.
- For moderate departure flow rates, the Linear ETT Model yields lower travel times than the other three models. However, for high departure flow rates, the Linear ETT Model yields higher travel times than the other three models. The asymmetric treatment of congestion depicted in Figure 2 by the Linear ETT Model seems to correspond to what is experienced in transportation networks. As a result, the Linear ETT Model seems to provide the most realistic travel times.

The second scenario corresponds to \( u_i^{\text{max}} = 40 \text{ miles/hr}, k_i^{\text{max}} = 200 \text{ vehicles/mile} \) and \( L_i = 4 \text{ miles} \). Furthermore, we consider the same discretization intervals of 20 seconds. In this case, \( B_i(t) \) is very small compared to \( u_i^{\text{max}} \). As a result, we expect the PTT and the ETT models to yield the same travel time function, that is, the travel time function of the limit of the Linear PTT and ETT. Figure 3 illustrates this observation.

III. Application to the Dynamic User-Equilibrium Problem

In this section, we illustrate how the second order fluid model we examined above for determining travel times connects with the dynamic user-equilibrium problem. We first need to consider the following assumption:

A6 Every user in the network has full information over the departure time period \([0,T]\).

Then, after re-ordering the path indices, the dynamic user-equilibrium problem seeks to solve:

**DUE Model**

\[ T_{1w}^t(L_{1w}) = \ldots = T_{m_w}^t(L_{m_w}) \leq T_{m_w+1}^t(L_{m_w+1}) \leq \ldots \leq T_{n_w}^t(L_{n_w}), \quad (14) \]

for all \( w \in W \) and \( t \in [0,T] \),

\[ F_{1w}(t), \ldots, F_{m_w}(t) > 0, F_{m_w+1}(t) = \ldots = F_{n_w}(t) = 0, \quad (15) \]

for all \( w \in W \) satisfying

\[ \sum_{p \in F_w} F_p(t) = d_w(t), \quad \text{for all } w \in W, \quad (16) \]

\[ T_{p}^t(L_p) = \sum_{i \in I} T_{ip}^t(L_{ip}) \delta_{ip}, \quad \text{for all } p \in P, \]

\[ f_i(x_i,t) = \sum_{p \in P} \delta_{ip} F_p(x_i,t), \quad \text{for all } i \in I, \]

and the Second Order DTT Model

for all \( t \in [0,T] \).

We abbreviated the notation in this formulation as follows: The travel time at path \( p_0 \) when starting the trip on the path at time \( t \) becomes \( T_{p_0}^t(L_{p_0}) = T_{i_0}(L_{1w}, t) \) and the flow rate at the beginning of path \( p \) at departure time \( t \) becomes \( F_p(t) = F_p(0,t) \).

The dynamic user-equilibrium problem (DUE Model) seeks to determine the path flows \( F_p(0,t), p \in P \), the link flows \( f_i(0,t), i \in I \) and densities \( k_i(0,t), i \in I \) that satisfy the feasibility conditions of traffic flow together with the constraints describing the Second Order DTT Model. Notice that this model does not assume that the travel time functions on the links are known in advance but rather includes the Second Order Model as a submodel for determining them.

IV. An Application to Dynamic Pricing and Inventory Control

In what follows, we briefly review the continuous-time model for the dynamic pricing problem using a delay approach as introduced in Kachani and Perakis [12]. We consider a make-to-stock production environment of a single manufacturer producing several products with dynamic shared production capacity. This model aims to address how to simultaneously determine optimal dynamic production, inventory and pricing policies.
Similarly to the previous section, we build a model by taking a fluid dynamics approach that expresses link dynamics, flow conservation, flow propagation and boundary constraints. In fact, this formulation extends the Dynamic Network Loading (DNL) model which is widely used in the context of the dynamic traffic equilibrium problem (see for example, Friesz et al. [3], and Wu et al. [4]).

The key modeling aspect in this model is that we consider the delay of a product in inventory before being sold and how it is affected by the price and inventory of the product as well as competitors’ prices. Traditional approaches in the literature consider how the demand is affected by prices. The approach we take is motivated from: (1) The widespread recording, by barcode readers, of entrance times and exit times of products in inventory systems, which makes delay data easily available. (2) The delay data being internal and easily extractable from data warehouses, as opposed to demand data, which is external, and therefore not controlled by the manufacturer. (3) In an environment where price does not vary a lot with time, the estimation of the relationship between price and demand, which is used as an input to the pricing models in the literature, can be quite inaccurate. However, because of the moderate to high variability of inventories with time, the estimation of the relationship between inventory level and sojourn time can be more accurate.

A. Notation

In this section, we study a multi-product inventory system that we represent conceptually by a directed network with two nodes O and D, and n links joining these two nodes. Node O represents the arrival of a product to the warehouse and node D represents the delivery of this product to the customer. Each link joining O and D corresponds to a distinct product that the company is selling and is indexed by i, i ∈ {1, ..., n}. We assume that the company under study is a Stackelberg leader, and as a result is a price setter. Therefore, competitors’ prices are functions of the price of the company under study. These functions can be estimated in practice using regression on the competitors’ prices and the prices of the company under study, as illustrated in [12]. Below, we describe the inputs and the outputs of the Dynamic Pricing Model.

Inputs of the Dynamic Pricing Model

**Link variables:**
- \( CFR(t) = \) shared production capacity rate at time t; \( p_i(t) = \) price of product i at time t; \( D_i(t) = T_i(I_i, p_i(t)) \), vector of prices of competitors on i; \( c_i(t) \) : product sojourn time; \( h_i(t) \) : inventory cost of i at time t.

**Time variables:**
- \( t \) : index for continuous time; 
- \([0, T]\) : production period.

Outputs of the Dynamic Pricing Model

**Link variables:**
- \( U_i(t) \) : cumulative production flow of i during \([0, t]\); 
- \( u_i(t) \) : production flow rate of i at t; 
- \( V_i(t) \) : cumulative sales flow of i during \([0, t]\); 
- \( v_i(t) \) : sales flow rate of i at t; 
- \( I_i(t) \) : inventory (number of units) of i at t; 
- \( p_i(I_i(t)) \) : sales price of one unit of i given an \( I_i(t) \); 
- \( s_i(t) \) : exit time \( (s_i(t) = t + D_i(I_i(t))) \).

**Time variables:**
- \([0, T_{\infty}] \) : analysis period.

**Remarks:**
- The objective of the company is to maximize its profits. That is, by subtracting production costs and inventory

\[
\frac{\partial U_i(t)}{\partial t} - c_i(t) u_i(t) - h_i(t) I_i(t) = 0, \quad t \geq 0, \quad i \in \{1, ..., n\}
\]

\[
\frac{\partial I_i(t)}{\partial t} = u_i(t) - v_i(t), \quad t \geq 0, \quad i \in \{1, ..., n\}
\]

**Objective Function:**

\[
\text{Max} \sum_{i=1}^{n} \int_{0}^{T_{\infty}} [p_i(t)] v_i(t) - c_i(t) u_i(t) - h_i(t) I_i(t) dt
\]

\[
\text{s.t.} \quad \frac{dI_i(t)}{dt} = u_i(t) - v_i(t), \quad t \geq 0, \quad i \in \{1, ..., n\}
\]

\[
V_i(t) = \int_{\omega \in W} u_i(\omega) d\omega, \quad \forall i \in \{1, ..., n\}, W = \{\omega: s_i(\omega) \leq t\}
\]

\[
U_i(0) = 0, \quad V_i(0) = 0, \quad I_i(0) = 0, \quad \forall i \in \{1, ..., n\}
\]

\[
\sum_{i=1}^{n} u_i(t) \leq CFR(t),
\]

\[
u_i(\cdot) \geq 0, \quad \forall i \in \{1, ..., n\}, \quad CFR(\cdot) \geq 0.
\]
costs from sales.

- The link dynamics (18) express the change in inventory at time $t$ as the difference between the production and the sales flow rates.
- The flow propagation equations (19) describe the flow progression over time. Note that a production flow entering link $i$ at time $t$ will be sold at time $s_i(t) = t + D_i(I_i(t))$. Therefore, by time $t$, the cumulative sales flow of link $i$ should be equal to the integral of all production inflow rates which would have entered link $i$ at some earlier time $\omega$ and would have been sold by time $t$.

- Furthermore, if the product exit time functions $s_i(\cdot)$ are continuous and satisfy the strict First-In-First-Out (FIFO) property, then the flow propagation equations (19) can be equivalently rewritten as
  \[ V_i(t) = \int_0^{s_i^{-1}(t)} u_4(\omega) d\omega, \quad \forall i \in \{1, \ldots, n\}. \]  
  (22)

Notice that $s_i^{-1}(t)$ is the time at which a unit of product $i$ needs to be produced so that it is sold at time $t$. Furthermore, under the strict FIFO condition, a unit of product $i$, entering the queue at time $t$, will be sold only after the units of product $i$, that entered the queue before it, are all sold. In mathematical terms, this is equivalent to the product exit time functions $s_i(\cdot)$ being strictly increasing. As a result, defining the production time $s_i^{-1}(t)$ makes sense.

Furthermore, notice $V_i(t)$ represents the cumulative demand. Equation (19) links the demand to the sojourn time. It replaces the demand-price relationship used in traditional models in the literature.

- Constraint (20) assumes that at each time $t$, the total production flow rate is no more than the total capacity flow rate $CFR(\cdot)$.
- It is not necessary to assume that $I_i(0) = 0$. Instead, we could assume $I_i(0) = I_0 > 0$. However, we consider zero-level inventories at $t = 0$ for simplicity of notation. Hence, in the beginning, we start producing before the demand arrives. As a result, we build inventory, which is characteristic of make-to-stock systems.
- In general, the DPM Model is a continuous-time nonlinear optimization problem. The non-linearity of the model comes from the unit price as a function of the inventory, as well as the integral equation (19). In this formulation, the known variables are the product sojourn time functions $D_i(\cdot)$, the production and inventory costs $C_i(\cdot)$ and $h_i(\cdot)$, and the total capacity flow rate function $CFR(\cdot)$.

The unknown variables we wish to determine are $u_i(t)$, $v_i(t)$, $U_i(t)$, $V_i(t)$ and $I_i(t)$. Notice that integral equation (19), which connects the production to the sales schedules through the delays incurred in the system due to price and inventory, makes this problem hard to solve.

- Notice that the model is general enough to account for the case where the FIFO property, defined above, is not necessarily verified (notice that Equation (19) does not assume that the FIFO property holds).

In Kachani and Perakis [12], we investigate when the FIFO property holds. We examine conditions on the product sojourn time functions $D_i(\cdot)$ and on the production flow rates $u_i(\cdot)$. We also establish the existence result below. This result illustrates that under general assumptions, the DPM Model possesses an optimal solution.

**Theorem 1:** [12] Assume that the following conditions hold:

- (E1) The price inventory functions $p_i(I_i)$ are continuously differentiable and bounded from above by scalars $p_i^\max$.
- (E2) The product sojourn time functions $D_i(\cdot)$ are continuously differentiable, and there exist two non-negative constants $B_{ii}$ and $B_{ti}$ such that for every inventory level $I_i$, $0 \leq B_{ii} \leq D_i(I_i) < B_{ii}$.
- (E3) The shared capacity flow rate function $CFR(\cdot)$ is Lebesgue integrable, non-negative and does not exceed $\min_{1, \ldots, n} \frac{1}{B_{ii}}$.

Then, the Dynamic Pricing Model has an optimal solution.

**Remark:** This theorem suggests that the maximum variation of the delay in terms of the inventory connects with the total production rate for all products. As a result, when the shared capacity (which is an upper bound on the total production rate) is small (or large) and the maximum variation of the delay in terms of the inventory is large (or small) then the Dynamic Pricing Model has a solution.

**C. Connection between Product Delay Functions in Dynamic Pricing and Travel Time Functions in Transportation**

A key feature that allows us to apply the Dynamic Pricing fluid model is that we consider a product delay function $D_i(I_i(t)) = T_i(p_i(I_i(t)))$, $I_i(t)$ that models the delay of price and level of inventory in affecting demand.

In Kachani and Perakis [12], we propose a methodology to estimate such a function in practice. This approach is based on a regression analysis of historical data. In more precise terms, we consider a parametric family of product delay functions $D_i(I_i(\cdot))$. For every time $t$, we collect the corresponding past time $t'$ such that the cumulative sales $V_i(t)$ equate the past cumulative productions $U_i(t')$. By using the relationship $V_i(t) = U_i(s_i^{-1}(t))$, we regress $s_i^{-1}(t)$ and $t'$ for every initial $t$. This enables us to estimate the parameters of the product delay functions $D_i(I_i(\cdot))$. Such methodology leads to a variety of product delay functions, the most common of which is a linear inventory delay function relationship of the type:

\[ D_i(I_i(t)) = D_i(0) + \tau \cdot \frac{I_i(t)}{C_i}. \]  
  (23)

Note that $D_i(0)$ denotes the free flow delay, $C_i$ the product capacity, and $\frac{\tau}{C_i}$ the marginal increase in delay due to an increase of one unit of inventory.

In this paper, we propose an alternate analytical approach for determining a product delay function $D_i(I_i(\cdot))$. We interpret this delay function as the travel time of a unit of product $i$ to traverse link $i$. By making an analogy...
Lemma 2: Under the assumption that \( u_i(t + T_i) = A_i(t) + B_i(T_i) \), product delay functions can be expressed by Equation (23).

Proof: In the analysis of first-order separable models for dynamic transportation networks, we derived several travel time functions. One example of such functions is:

\[
T_i(L_i, t) = \frac{L_i}{u_i^{\text{max}}} + \frac{A_i(t) L_i}{(u_i^{\text{max}})^2 k_i^{\text{max}}}. \tag{24}
\]

Further, one scenario we can consider is that for subsequent times in a small interval \([t, t + \Delta t]\), drivers assume that the entrance link flow rates can be approximated by an average of the past entrance flow rates on a time period \([t - h, t]\). This translates into \( A_i(t) = \frac{1}{h} \int_{t-h}^{t} f_i(w) dw \). In order to use these results, we need to make an analogy between a transportation network and an inventory system. In this context, \( u_i^{\text{max}} \) is the maximum speed for a product \( i \) to be sold. Moreover, \( D_i(0) \) is the free flow delay time, the length \( L_i \) of link \( i \) corresponds to \( D_i(0) u_i^{\text{max}} \) and, finally, the maximum density \( k_i^{\text{max}} \) corresponds to \( C_i \), where \( C_i \) is the product capacity. In this context, the production flow rate \( u_i(t) \) represents the link flow rate \( f_i(0,t) \).

This analogy allows us to determine delay times by viewing them as traffic times. In particular, by taking \( h = D_i(s_i^{-1}(t)) = t - s_i^{-1}(t) \), we obtain \( A_i(t) = \frac{1}{D_i(s_i^{-1}(t))} \int_{s_i^{-1}(t)}^{t} u_i(w) dw = \frac{L_i(t)}{D_i(s_i^{-1}(t))} \). The time travel function (24) can be rewritten as:

\[
D_i(I_i(t)) = D_i(0) + \frac{(D_i(0))^2}{D_i(s_i^{-1}(t))} \frac{I_i(t)}{C_i}, \quad \text{where} \quad \tau = \frac{(D_i(0))^2}{D_i(s_i^{-1}(t))}.
\]

The above travel time function corresponds to the linear link performance functions of the type

\[
D_i(I_i(t)) = D_i(0) + \tau \frac{I_i(t)}{C_i}, \quad \text{where} \quad \tau = \frac{(D_i(0))^2}{D_i(s_i^{-1}(t))}.
\]

Q.E.D. This is an example of a delay function we use in the analysis of the DPM Model.

D. A Connection between the Dynamic Pricing Fluid Model and the Hydrodynamic Fluid Model

In this subsection we explore a connection between the Dynamic Pricing Fluid Model and the Dynamic Travel Time Model (Model 1). In particular, we explore the relationship between the conservation law (2) and the equation expressing the link dynamics (18) in the Dynamic Pricing Model. We establish that the latter model is an aggregate approximation of Model 1. As a result, Model 1 provides a more accurate description of the dynamics at every point of the network links.

In order to provide more details, we first consider the following correspondence. For every link \( i \in I \) in the network, we consider a path that consists of an infinite number of serially connected links. That is, every location \( x_i \) on link \( i \) corresponds to a new link \( l_i = [x_i, x_i + \Delta x_i] \), where \( \Delta x_i \) is infinitesimal, that is, \( \Delta x_i \to 0 \). The next proposition establishes that an infinitesimal view of the Dynamic Pricing Model at every point of a link is equivalent to the hydrodynamic model, (i.e. Model 1).

Proposition 1: The conservation law (2) on link \( i \) is equivalent to the link dynamics equation (18) on every link \( l_i = [x_i, x_i + \Delta x_i] \), with \( \Delta x_i \to 0 \).

Proof: We first observe that the partial link load \( X_i \) on link \( l_i = [x_i, x_i + \Delta x_i] \) represents the total number of cars on link \( l_i \). As a result, a key observation is that

\[
X_i(t) = \int_{x_i}^{x_i + \Delta x_i} k_i(w, t) dw,
\]

where \( k_i \) represents the density function on link \( i \).

Furthermore, the link-dynamics equation (18) on link \( l_i \) becomes

\[
\frac{dX_i(t)}{dt} = u_i(t) - v_i(t) = f_i(x_i, t) - f_i(x_i + \Delta x_i, t).
\]

As a result, applying equation (25) on every link \( l_i \) and combining it with (25), gives rise to

\[
\frac{\partial}{\partial t} \int_{x_i}^{x_i + \Delta x_i} k_i(w, t) dw = f_i(x_i, t) - f_i(x_i + \Delta x_i, t). \tag{25}
\]

\[
\lim_{\Delta x_i \to 0} \frac{\frac{\partial}{\partial t} \int_{x_i}^{x_i + \Delta x_i} k_i(w, t) dw}{\Delta x_i} = \lim_{\Delta x_i \to 0} \frac{f_i(x_i, t) - f_i(x_i + \Delta x_i, t)}{-\Delta x_i}.
\]

Noticing that \( \frac{\partial}{\partial x_i} \int_{x_i}^{x_i + \Delta x_i} k_i(w, t) dw \approx -k(x_i, t) \) and letting \( \Delta x_i \to 0 \) gives rise to the conservation law (2),

\[
\frac{\partial f_i(x_i, t)}{\partial x_i} + \frac{\partial k_i(x_i, t)}{\partial t} = 0, \quad \text{for all} \quad i \in I.
\]

Q.E.D.

V. Conclusions

In this paper, we examined two classes of fluid models. The first class was based on the laws of hydrodynamic theory while the second was based on the Dynamic Network Loading Model. The first class of models allowed us to propose several families of travel time functions for dynamic transportation networks. These travel times correspond to what practitioners are using. Furthermore, through some numerical examples we established some insights on these functions.

The second fluid modeling approach was based on a more aggregate approach. We were able to show that the two fluid models were closely connected and establish that the fluid model based on hydrodynamic theory provides a more accurate description of the dynamics. Moreover, the second fluid modeling approach connected with a continuous-time dynamic pricing/inventory control model that incorporates the delay of price and level of inventory in affecting demand. Using the first approach we were also able
to propose delay functions that were useful in the dynamic pricing/inventory control model.

This research has the potential to impact significantly transportation planning as well as inventory control and manufacturing. In the area of transportation, we believe that our results could play an important role in the development of ITS. Furthermore, our results in dynamic pricing also lay the foundations for the use of the delay of price in affecting demand and fluid dynamics models in supply chain and inventory management systems.

REFERENCES


