Fast Polyhedral Adaptive Conjoint Estimation

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Fast Polyhedral Adaptive Conjoint Estimation

Abstract

We propose and test a new adaptive conjoint analysis method that draws on recent polyhedral "interior-point" developments in mathematical programming. The method is designed to offer accurate estimates after relatively few questions in problems involving many parameters. Each respondent's questions are adapted based upon prior answers by that respondent. The method requires computer support but can operate in both Internet and off-line environments with no noticeable delay between questions.

We use Monte Carlo simulations to compare the performance of the method against a broad array of relevant benchmarks. While no method dominates in all situations, polyhedral algorithms appear to hold significant potential when (a) metric profile comparisons are more accurate than the self-explicated importance measures used in benchmark methods, (b) when respondent wear out is a concern, and (c) when product development and/or marketing teams wish to screen many features quickly. We also test hybrid methods that combine polyhedral algorithms with existing conjoint analysis methods. We close with suggestions on how polyhedral methods can be used to address other marketing problems.

Polyhedral Methods for Conjoint Analysis

In this paper we propose and test a new adaptive conjoint data collection and estimation method that attempts to reduce respondent burden while simultaneously improving accuracy. For each respondent the method dynamically adapts the design of the next question using that respondent's answers to previous questions. Because the method makes full use of high-speed computations and adaptive, customized local web pages, it is ideally suited for Internet panels. We interpret the problem of selecting questions and estimating parameters as a mathematical program and estimate the solution to the program using recent developments based on the interior points of polyhedra. These techniques provide the potential for accurate estimates of partial utilities from fewer questions than required by extant methods.

Adapting question design within a respondent, using that respondent's answers to previous questions, is a difficult dynamic optimization problem. Adaptation *within* respondents should be distinguished from techniques that adapt *across* respondents. Sawtooth Software's Adaptive Conjoint Analysis (ACA) is the only published method of which we know that attempts to solve this problem (Johnson 1987, 1991). In contrast, aggregate customization methods, such as the Huber and Zwerina (1996), Arora and Huber (2001), and Sandor and Wedel (2001) algorithms, adapt designs across respondents based on either pretests or Bayesian priors.

Our goals are two-fold. First, we investigate whether polyhedral methods have the potential to enhance the effectiveness of existing conjoint methods. We do not propose to replace the existing methods, but, rather, to provide new capabilities that complement these methods. Second, by focusing on widely studied marketing problems we hope to illustrate the recent advances in mathematical programming and encourage their applications in the marketing literature.

Because the method is new and adopts a different estimation philosophy, we use Monte Carlo experiments to explore the properties of the proposed polyhedral methods. The Monte Carlo experiments explore the conditions under which polyhedral methods are likely to do better or worse than extant methods. We have reason for optimism. Conjoint analysis methods such as Linmap (Srinivasan and Shocker 1973a, 1973b) successfully use classical linear programming to obtain estimates by placing constraints on the feasible set of parameters. The Monte Carlo analysis explores seven issues:

1. **Accuracy vs. the number of questions**

   Because web-based conjoint surveys place a premium on a small number of questions, we investigate how rapidly the estimates converge to their true values as the number of questions increases. The simulations highlight situations in which the polyhedral methods obtain the same accuracy with fewer questions than some benchmark methods. Reasonable estimates with smaller numbers of questions are important for the new, highly iterative product development processes. These dispersed processes gather information on customer preferences more often, require feedback more quickly, and often deal with large numbers of product features.

2. **Self-explicated questions**

   While the polyhedral method does not depend upon self-explicated questions, some benchmark methods do. We explore how the relative performance of these benchmarks depends on the accuracy of the self-explicated answers.

3. **Biases**

   We explore whether the polyhedral algorithm and/or the benchmarks introduce biases into the estimates.

4. **Question selection vs. partworth estimation**

   Because polyhedral methods can be used for both question selection and partworth estimation, we test each component (question selection and estimation) versus traditional procedures.

5. **Hybrid methods**

   One of our goals is to investigate whether polyhedral methods have the potential to enhance existing conjoint methods. To explore this issue we evaluate several hybrid methods in which we combine polyhedral question selection or estimation with existing techniques.

6. **Respondent wear out or learning**

   We explore what happens if responses either degrade (wear out) or improve (learning) as the number of questions increases.

7. **Individual vs. population estimates**

   The polyhedral method (and many conjoint analysis methods) seek to provide estimates of customer preferences that vary by respondent. However, such heterogeneous estimates should not compromise the accuracy of population estimates. We compare the methods on their abilities to estimate population averages.

The paper is structured as follows. We begin by distinguishing two types of conjoint tasks: metric-paired-comparison tasks and stated-choice tasks. We then describe a polyhedral method for metric paired-comparison tasks. In later discussion we describe how this method can be adapted to accommodate stated-choice tasks. Detailed mathematics are provided in the Appendix. We follow this description of the methods with the design and results from the Monte Carlo experiments. We then briefly describe an empirical field test and close with a discussion of the applicability of polyhedral methods to conjoint analysis and other marketing problems.

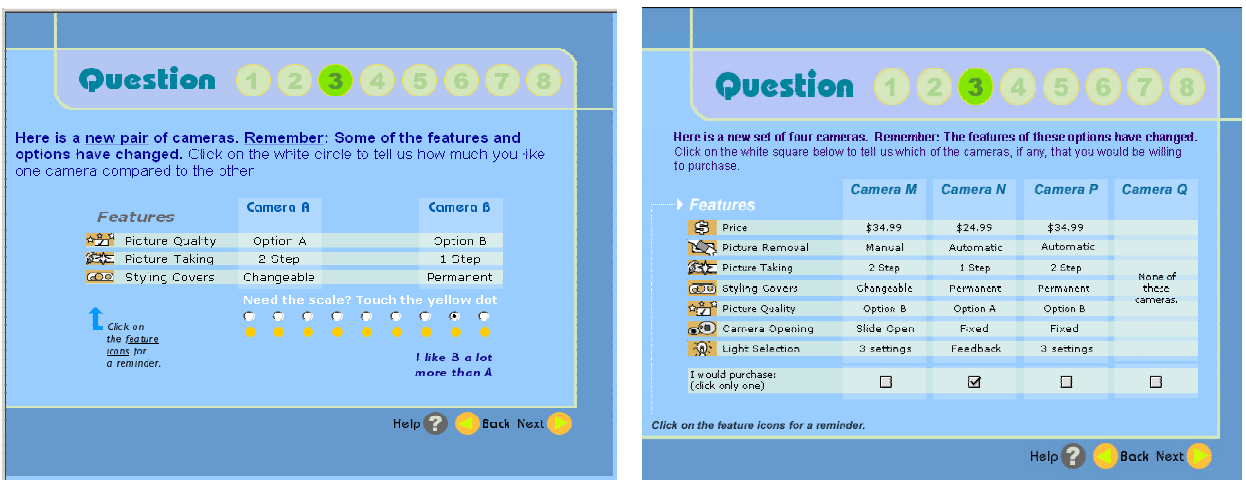
Alternative Data-Collection Formats for Conjoint Analysis

Polyhedral methods are new to conjoint analysis and show potential for many of the varied formats of data collection (respondent tasks) that are used in conjoint analysis. To date, we have found that polyhedral methods show particular promise for two well-studied respondent tasks.

In the first task subjects are presented with two product profiles and asked to provide a metric rating of the extent to which they prefer one product over the other ("metric-paired comparison" questions). In the second task subjects are presented with multiple product profiles and asked to pick which product profile they prefer ("stated-choice" questions, sometimes also called "Choice Based Conjoint" or "CBC"). We illustrate these two tasks in Figure 1, which is a modification of a commercial web-based conjoint questionnaire that was used to design a new version of Polaroid's I-Zone camera. The metric-paired comparison task is depicted in Figure 1a and the stated-choice task is illustrated in Figure 1b.[1]

Figure 1

Examples of Two Question Formats for I-Zone Camera Redesign



(a) Metric Paired Comparison

(b) Stated Choice

Adaptive Conjoint Analysis (ACA) uses metric-paired-comparison questions. Given ACA is the only other published method (of which we know) for adapting questions within a respondent; we also focus on metric paired-comparison tasks for an initial exploration of the potential of polyhedral methods. Metric paired-comparison questions (1) are common in computer-aided interviewing, (2) have proven reliable in previous studies (Reibstein, Bateson, and Boulding 1988; Urban and Katz 1983), (3) have been shown to provide interval-scaled paired-comparison data that has strong transitivity properties (Hauser and Shugan 1980), and (4) enjoy wide use in practice and in the literature that is exceeded only by the full profile task (see Cattin and Wittink 1982; and Wittink and Cattin 1989 for applications surveys). Further, there is growing evidence that carefully collected metric data provide valid and reliable parameter estimates (Carmone, Green, and Jain 1978; Currim, Weinberg, and Wittink 1981; Hauser and Shugan 1980; Hauser and Urban 1979; Huber 1975; Leigh, MacKay, and Summers 1984; Malhotra 1986; Srinivasan and Park 1997; and Wittink and Cattin 1981).

[1] The I-Zone application was based on metric paired-comparison data. Figure 1b is an adaptation, in the I-Zone format, of the stated-choice questions on pages 260 and 284 of Louviere, Hensher, and Swait (2000).

Stated-choice conjoint tasks have also achieved widespread acceptance in both the academic literature and industry practice (cf. Louviere, Hensher and Swait 2000). For this reason, we briefly describe how polyhedral methods can be adapted to address stated-choice conjoint tasks. We suggest one feasible algorithm as a prototype. (Code is available on our website). Although the polyhedral algorithms share many features, there are important differences between the metric-paired comparison and stated-choice tasks. Many interesting challenges have yet to be resolved, hence research on polyhedral algorithms for stated-choice tasks must be considered on-going. A complete treatment of the use of polyhedral methods in stated-choice tasks is beyond the scope of the current paper. However, our initial simulations suggest that the prototype algorithm is feasible in the sense that it can adapt stated-choice questions within respondents with no noticeable delay between questions. Initial simulations suggest that, like for the metric-paired comparison algorithm, adaptive polyhedral question selection and estimation improves extant choice-based methods (e.g., multinomial logit, Hierarchical Bayes) in some situations, but not in others.

Information and Polyhedral Feasible Sets

We now describe the polyhedral question selection and partworth estimation procedures for metric-paired-comparison questions. We begin with a conceptual description that highlights the geometry of the parameter space and then introduce the interior-point methods based on the "analytic center" of a polyhedron. We illustrate the concepts with a 3-parameter problem because 3-dimensional spaces are easy to visualize and explain. The methods generalize easily to realistic problems that contain ten, twenty, or even one hundred parameters. Indeed, relative to existing methods, the polyhedral methods are most useful for large numbers of parameters. By a parameter, we refer to a partworth that needs to be estimated. For example, twenty features with two levels each require twenty parameters because we can set to zero the partworth of the least preferred feature.[2] Similarly, ten three-level features also require twenty parameters. Interactions among features require still more parameters.

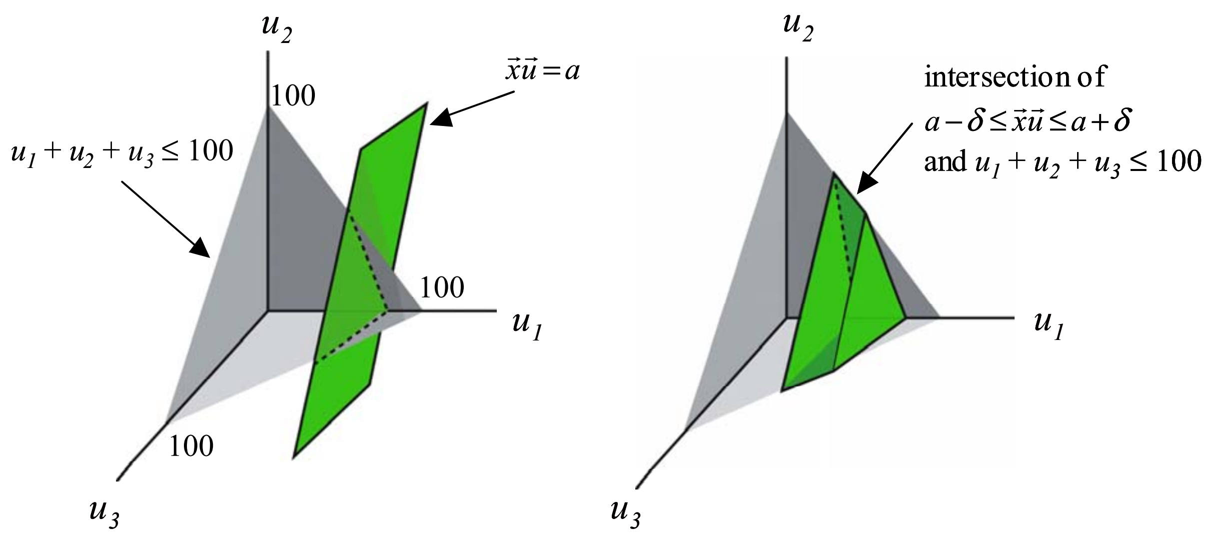
Suppose that we have three features of an instant camera – picture quality (illustrated with two options viewable on the web), picture taking (2-step vs. 1-step), and styling covers (changeable vs. permanent). If we scale the least desirable level of each feature to zero we have three non-negative parameters to estimate, u_1, u_2, and u_3, reflecting the additional utility (partworth) associated with the most desirable level of each feature.[3] In Figure 1a the sum of the partworths of Camera A minus the sum of the partworths of Camera B can be at most equal to the maximum scale difference – in this case the value of

[2] Technically, we lose a degree of freedom because the utility of a fully-featured product can be set arbitrarily. However, when we use metric data we regain that degree of freedom when we rescale utility to the implicit scale of the respondents' answers.

nine scale points. Thus, without loss of generality, in order to visualize the algorithm, we impose a constraint that the sum of the parameters does not exceed some large number. In this case, prior to any data collection, the feasible region for the parameters is the 3-dimensional bounded polyhedron in Figure 2a.

Figure 2

Respondent's Answers Affect the Feasible Region



(a) Metric rating without error (b) Metric rating with error

Suppose that we ask the respondent to evaluate a pair of profiles that vary on one or more features and the respondent says (without error) (1) that he or she prefers profile C_1 to profile C_2 and (2) provides a rating, a, to indicate the strength of his or her preference. This introduces an equality constraint that the utility associated with profile C_1 exceeds the utility of C_2 by an amount equal to the rating. If we define $\vec{u} = (u_1, u_2, u_3)^T$ as the 3×1 vector of parameters, $\vec{z}_\ell$ as the 1×3 vector of product features for the left profile, and $\vec{z}_r$ as the 1×3 vector of product features for the right profile, then, for additive utility, this equality constraint can be written as $\vec{z}_\ell \vec{u} - \vec{z}_r \vec{u} = a$. We can use geometry to characterize what we have learned from this question and answer.

Specifically, we define $\vec{x} = \vec{z}_\ell - \vec{z}_r$ such that $\vec{x}$ is a 1×3 vector describing the difference between the two profiles in the question. Then, $\vec{x}\vec{u} = a$ defines a hyperplane through the polyhedron in Figure 2a. The only feasible values of $\vec{u}$ are those that are in the intersection of this hyperplane and the polyhedron. The new feasible set is also a polyhedron, but it is reduced by one dimension (2-dimensions rather than 3-

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[3] In this example, we assume preferential independence which implies an additive utility function. We can handle interactions by relabeling features. For example, a 2x2 interaction between two features is equivalent to one four-

dimensions). Because smaller polyhedra mean fewer parameter values are feasible, questions that reduce the size of the initial polyhedron as fast as possible lead to more precise estimates of the parameters.

However, in any real problem we expect the respondent's answer to contain error. We can model this error as a probability density function over the parameter space (as in standard statistical inference). Alternatively, we can incorporate imprecision in a response by treating the equality constraint $\vec{x}\vec{u} = a$ as a set of two inequality constraints: $a - \delta \leq \vec{x}\vec{u} \leq a + \delta$. In this case, the hyperplane defined by the question-answer pair has "width." The intersection of the initial polyhedron and the "fat" hyperplane is now a three-dimensional polyhedron as illustrated in Figure 2b.

When we ask more questions we constrain the parameter space further. Each question, if asked carefully, will result in a hyperplane that intersects a polyhedron resulting in a smaller polyhedron – a "thin" region in Figure 2a or a "fat" region in Figure 2b. Each new question-answer pair slices the polyhedron in Figure 2a or 2b yielding more precise estimates of the parameter vector $\vec{u}$.

We incorporate prior information about the parameters by imposing constraints on the parameter space. For example, if u_m and u_h are the medium and high levels, respectively, of a feature, then we impose the constraint $u_m \leq u_h$ on the polyhedron. Previous research suggests that these types of constraints enhance estimation (Johnson 1999; Srinivasan and Shocker 1973a, 1973b). We now examine question selection for metric paired-comparison data by dealing first with the case in which subjects respond without error (Figure 2a). We then describe how to modify the algorithm to handle error (e.g., Figure 2b).

Question Selection

The question selection task describes the design of the profiles that respondents are asked to compare. Questions are more informative if the answers allow us to identify more quickly the correct answer. For this reason, we select the respondent's next question in a manner that is likely to reduce the size of the feasible set (for that respondent) as fast as possible.

Consider for a moment a 20-dimensional problem (without errors in the answers). As in Figure 2a, a question-based constraint reduces the dimensionality by one. That is, the first question reduces a 20-dimensional set to a 19-dimensional set; the next question reduces this set to an 18-dimensional set and so on until the twelfth question which reduces a 9-dimensional set to an 8-dimensional set (8 dimensions = 20 parameters – 12 questions). Without further restriction, the feasible parameters are generally not unique – any point in the 8-dimensional polyhedron is still feasible. However, the 8-dimensional set might be quite small and we might have a very good idea of the partworths. For example, the first twelve questions might be enough to tell us that some features, say picture quality, styling covers, and battery

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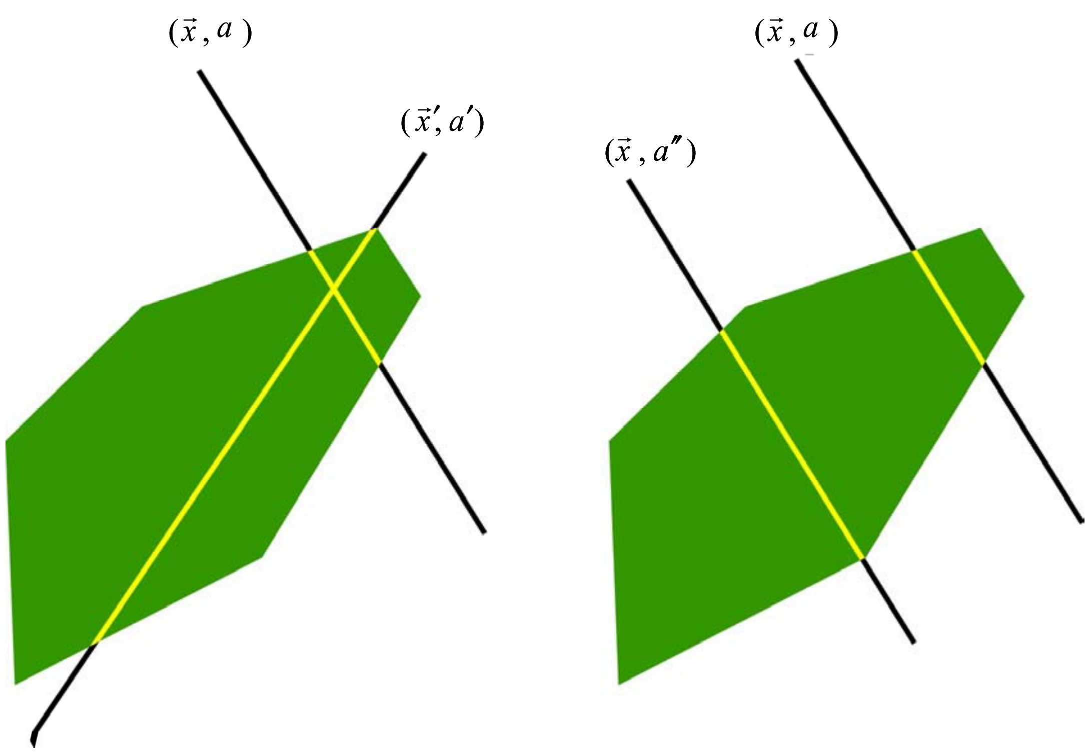
level feature. We hold this convention throughout the paper. We simplified Figure 1a to illustrate the problem in three dimensions. The I-Zone application included price and three other features as in Figure 1b.

life, have large partworths and some features, say folding capability, light selection, and film ejection method, have very small partworths. If this holds across respondents then, during an early phase of a product development process, the product development team might feel they have enough information to focus on the key features.

Although the polyhedral algorithm is most effective in high dimensional spaces, it is hard to visualize 20-dimensional polyhedra. Instead, we illustrate the polyhedral question-selection criteria in a situation where the remaining feasible set is easy to visualize. Specifically, by generalizing our notation slightly to q questions and p parameters, we define $\vec{a}$ as the $q\times 1$ vector of answers and X as the $q\times p$ matrix with rows equal to $\vec{x}$ for each question. (Recall that $\vec{x}$ is a $1\times p$ vector.) Then the respondent's answers to the first q questions define a (p-q)-dimensional hyperplane given by the equation $X\vec{u}=\vec{a}$. This hyperplane intersects the initial p-dimensional polyhedron to give us a (p-q)-dimensional polyhedron. In the example of p=20 parameters and q=18 questions, the result is a 2-dimensional polyhedron that is easy to visualize. One such 2-dimensional polyhedron is illustrated in Figure 3a.

Figure 3

Choice of Question (2-dimensional slice)



(a) Two question-answer pairs (b) One question, two potential answers

Our task is to select questions such that we reduce the 2-dimensional polyhedron as fast as possible. Mathematically, we select a new question vector, $\vec{x}$, and the respondent answers this question with a new rating, a. We add the new question vector as the last row of the question matrix and we add the new answer as the last row of the answer vector. While everything is really happening in p-dimensional space, the net result is that the new hyperplane will intersect the 2-dimensional polyhedron in a line segment (i.e., a 1-dimensional polyhedron). The slope of the line will be determined by $\vec{x}$ and the intercept by a. We illustrate two potential question-answer pairs in Figure 3a. The slope of the line is determined by the question, the specific line by the answer, and the remaining feasible set by the line segment within the polyhedron. In Figure 3a one of the question-answer pairs ($\vec{x}, a$) reduces the feasible set more rapidly than the other question-answer pair ($\vec{x}', a'$). Figure 3b repeats a question-answer pair ($\vec{x}, a$) and illustrates an alternative answer to the same question ($\vec{x}, a''$).

If the polyhedron is elongated as in Figure 3, then, in most cases, questions that imply line segments perpendicular to the longest "axis" of the polyhedron are questions that result in the smallest remaining feasible sets. Also, because the longest "axis" is in some sense a bigger target, it is more likely that the respondent's answer will select a hyperplane that intersects the polyhedron. From analytic geometry we know that hyperplanes (line segments in Figure 3) are perpendicular to their defining vectors ($\vec{x}$), thus, we can reduce the feasible set as fast as possible (and make it more likely that answers are feasible) if we choose question vectors that are parallel to the longest "axis" of the polyhedron. For example, both line segments based on $\vec{x}$ in Figure 3b are shorter than the line segment based on $\vec{x}'$ in Figure 3a.

If we can develop an algorithm that works in any p-dimensional space, then we can generalize this intuition to any question, q, such that $q \leq p$. (We address later the cases where the respondent's answers contain error and where $q > p$.) After receiving answers to the first q questions, we could find the longest vector of the (p-q)-dimensional polyhedron of feasible parameter values. We could then ask the question based on a vector that is parallel to this "axis." The respondent's answer creates a hyperplane that intersects the polyhedron to produce a new polyhedron. Later in the paper we use Monte Carlo simulation to determine if and when this question-selection method produces reasonable estimates of the unknown parameters. We then review an empirical study based on this polyhedral method.

Intermediate Estimates of Partworths and Updates to those Estimates

Polyhedral geometry also gives us a means to estimate the parameter vector, $\vec{u}$, when $q \leq p$. Recall that, after question q, any point in the remaining polyhedron is consistent with the answers the respondent has provided. If we impose a diffuse prior that any feasible point is equally likely, then we would like to select the point that minimizes the expected absolute error. This point is the center of the feasible polyhedron, or more precisely, the polyhedron's center of gravity. The smaller the feasible set, either due to better question selection or more questions (higher q), the more precise the estimate. If there were no respondent errors, then the estimate would converge to its true value when $q=p$ (the feasible set becomes a single point, with zero dimensionality). For $q>p$ the same point would remain feasible. (As we discuss below, this changes when responses contain error.) This technique of estimating partworths from the center of a feasible polyhedron is related to that proposed by Srinivasan and Shocker (1973b, p. 350) who suggest using a linear program to find the "innermost" point that maximizes the minimum distance from the hyperplanes that bound the feasible set.

Philosophically, the proposed polyhedral method makes maximum use of the information in the constraints and then takes a central estimate based on what is still feasible. Carefully chosen questions shrink the feasible set rapidly. We then use a centrality estimate that has proven to be a surprisingly good approximation in a variety of engineering problems including, for example, finding the center of gravity of a solid. More generally, the centrality estimate is similar in some respects to the proven robustness of linear models, and in some cases, to the robustness of equally-weighted models (Dawes and Corrigan 1974; Einhorn 1971, Huber 1975; Moore and Semenik 1988; Srinivasan and Park 1997).

Interior-point Algorithms and the Analytic Center of a Polyhedron

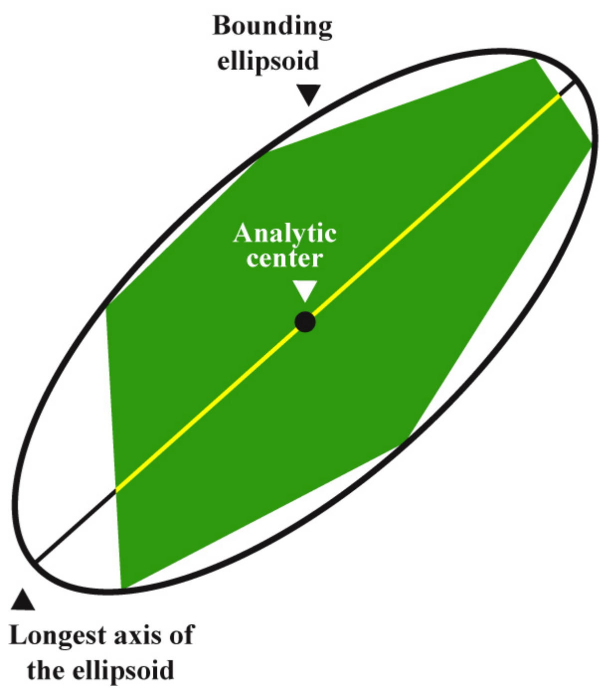
To select questions and obtain intermediate estimates the proposed heuristics require that we solve two non-trivial mathematical programs. First, we must find the longest "axis" of a polyhedron (to select the next question) and second, we must find the polyhedron's center of gravity (to provide a current estimate). If we were to define the longest "axis" of a polyhedron as the longest line segment in the polyhedron, then one method to find the longest "axis" would be to enumerate the vertices of the polyhedron and compute the distances between the vertices. However, solving this problem requires checking every extreme point, which is computationally intractable (Gritzmann and Klee 1993). In practice, solving the problem would impose noticeable delays between questions. Also, the longest line segment in a polyhedron may not capture the concept of a longest "axis." Finding the center of gravity of the polyhedron is even more difficult and computationally demanding.

Fortunately, recent work in the mathematical programming literature has led to extremely fast algorithms based on projections within the interior of polyhedrons (much of this work started with Karmarkar 1984). Interior-point algorithms are now used routinely to solve large problems and have spawned many theoretical and applied generalizations. One such generalization uses bounding ellipsoids. In 1985, Sonnevend demonstrated that the shape of a bounded polyhedron can be approximated by proportional ellipsoids, centered at the "analytic center" of the polyhedron. The analytic center is the point in the polyhedron that maximizes the geometric mean of the distances to the boundaries of the polyhedron. It is a central point that approximates the center of gravity of the polyhedron, and finds practical use in engineering and optimization. Furthermore, the axes of the ellipsoids are well-defined and intuitively capture the concept of an "axis" of a polyhedron. For more details see Freund (1993), Nesterov and Nemirovskii (1994), Sonnevend (1985a, 1985b), and Vaidja (1989).

We illustrate the proposed process in Figure 4, using the same two-dimensional polyhedron depicted in Figure 3. The algorithm proceeds in four steps. The mathematics are in the Appendix; we provide the intuition here. We first find a point in the interior of the polyhedron. This is a simple linear programming (LP) problem and runs quickly. Then, following Freund (1993) we use Newton's method to make the point more central. This is a well-formed problem and converges quickly to yield the analytic center as illustrated by the black dot in Figure 4. We next find a bounding ellipsoid based on a formula that depends on the analytic center and the question-matrix, *X*. We then find the longest axis of the ellipsoid (diagonal line in Figure 4) with a quadratic program that has a closed-form solution. The next question, $\vec{x}$, is based on the vector most nearly parallel to this axis.

Figure 4

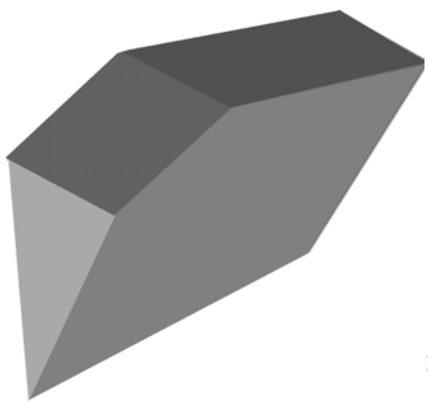
Bounding Ellipsoid and the Analytic Center (2-dimensions)



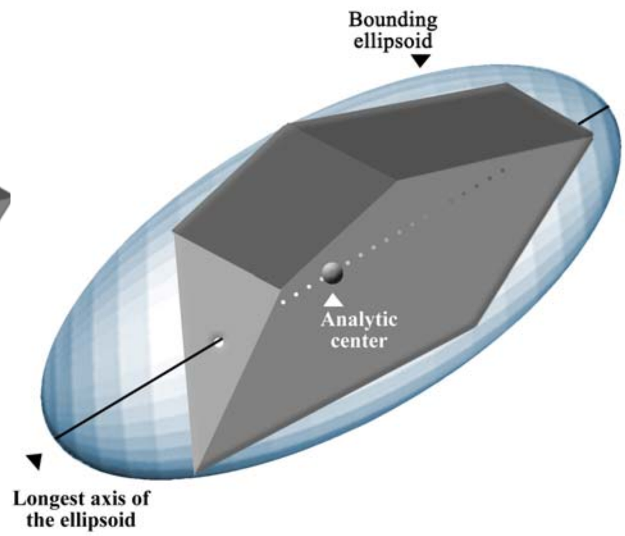
Analytically, this algorithm works well in higher dimensional spaces. For example, Figure 5 illustrates the algorithm when $(p - q) = 3$, that is, when we are trying to reduce a 3-dimensional feasible set to a 2-dimensional feasible set. Figure 5a illustrates a polyhedron based on the first q questions. Figure 5b illustrates a bounding 3-dimensional ellipsoid, the longest axis of that ellipsoid, and the analytic center. The longest axis defines the question that is asked next which, in turn, defines the slope of the hyperplanes that intersect the polyhedron. One such hyperplane is shown in Figure 5c. The respondent's answer selects the specific hyperplane; the intersection of the selected hyperplane and the 3-dimensional polyhedron is a new 2-dimensional polyhedron, such as that in Figure 4. This process applies (in higher dimensions) from the first question to the p^{th} question. For example, the first question implies a hyperplane that cuts the first p-dimensional polyhedron such that the intersection yields a $(p - 1)$-dimensional polyhedron.

Figure 5

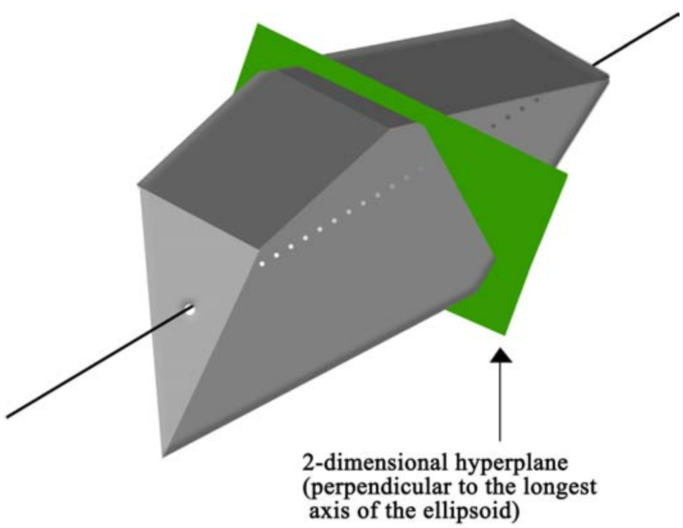
Question Selection with a 3-Dimensional Polyhedron



(a) Polyhedron in 3 dimensions



(b) Bounding ellipsoid, analytic center, and longest axis



(c) Example hyperplane determined by question vector and respondent's answer

The polyhedral algorithm runs extremely fast. We have implemented the algorithm to select questions for a web-based conjoint analysis application. Based on an example with ten two-level features, respondents notice no delay in question selection nor any difference in speed versus a fixed design. For a demonstration see the website listed in the acknowledgements section of this paper. Because there is no guarantee that the polyhedral algorithm will work well with the conjoint task, we use Monte Carlo simulation to examine how well the analytic center approximates the true parameters and how quickly ellipsoid-based questions reduce the feasible set of parameters.

Inconsistent Responses and Error-modeling with Polyhedral Estimation

Figures 2, 3, 4, and 5 illustrate the geometry when respondents answer without error. However, real respondents are unlikely to be perfectly consistent. It is more likely that, for some $q < p$, the respondent's answers will be inconsistent and the polyhedron will become empty. That is, we will no longer be able to find any parameters, $\vec{u}$, that satisfy the equations that define the polyhedron, $X\vec{u} = \vec{a}$. Thus, for real applications, we extend the polyhedral algorithm to address response errors. Specifically, we adjust the polyhedron in a minimal way to ensure that some parameter values are still feasible. We do this by modeling errors, $\vec{\delta}$, in the respondent's answers such that $\vec{a} - \vec{\delta} \le X\vec{u} \le \vec{a} + \vec{\delta}$. Review Figure 2b. We then choose the minimum errors such that these constraints are satisfied. The Appendix provides the mathematical program (OPT4) that we use to estimate $\vec{u}$ and $\vec{\delta}$. The algorithm is easily modified to incorporate alternative error formulations, such as least-squares or minimum sum of absolute deviations, rather than this "minimax" criterion.[4] Exploratory simulations suggest that algorithm is robust with respect to the choice of error criterion. This same modification covers estimation for the case of $q > p$.

To implement this policy we use a two-stage algorithm. In the first stage we treat the responses as if they occurred without error – the feasible polyhedron shrinks rapidly and the analytic center is a working estimate of the true parameters. However, as soon as the feasible set becomes empty, we adjust the constraints by adding or subtracting "errors," where we choose the minimum errors, $\|\vec{\delta}\|$, for which the feasible set is non-empty. The analytic center of the new polyhedron becomes the working estimate and $\vec{\delta}$ becomes an index of response error. As with all of our heuristics, the accuracy of our error-modeling method is tested with simulation.

Addressing Other Practical Implementation Issues

In order to apply polyhedral estimation to metric paired-comparison data we have to address several implementation issues. We note that other solutions to these problems may yield more or less accu-

[4] Technically, the minimax criterion is called the "∞-norm." To handle least-squares errors we use the "2-norm" and to handle average absolute errors we use the "1-norm." Either is a simple modification to OPT4 in the Appendix.

rate parameter estimates, and so the performance of the polyhedral method in the Monte Carlo simulations is a lower bound on the performance of this class of polyhedral methods.

Product profiles with discrete features. In most conjoint analysis problems, the features are specified at discrete levels as in Figure 1. This constrains the elements of the $\vec{x}$ vector to be 1, –1, or 0, depending on whether the left profile, the right profile, neither profile, or both profiles have the "high" feature. In this case we choose the vector that is most nearly parallel to the longest axis of the ellipsoid. Because we can always recode multi-level features or interacting features as binary features, the geometric insights still hold even if we otherwise simplify the algorithm.

Restrictions on question design. Experience suggests that for a p-dimensional problem we may wish to vary fewer than p features in any paired-comparison question. For example, Sawtooth Software (1996, p. 7) suggests that: "Most respondents can handle three attributes after they've become familiar with the task. Experience tells us that there does not seem to be much benefit from using more than three attributes." We incorporate this constraint by restricting the set of questions over which we search when finding a question-vector that is parallel to the longest axis of the ellipse.

First question. Unless we have prior information before any question is asked, the initial polyhedron of feasible utilities is defined by the boundary constraints. Because the boundary constraints are symmetric, the polyhedron is also symmetric and the polyhedral method offers little guidance in the choice of a respondent's first question. We choose the first question for each respondent so that it helps improve estimates of the population means by balancing the frequency with which each attribute level appears in the set of questions answered by all respondents. In particular, for the first question presented to each respondent we choose feature levels that appeared infrequently in the questions answered by previous respondents.

Question selection when the parameter set becomes infeasible. Polyhedral parameter estimation is well-defined when the parameter set becomes infeasible, but question selection is not. Thus, we use the ACA question-selection heuristic when the parameter set is infeasible.[5] This provides a lower bound on what might be achieved with improved infeasible-set question selection.

Programming. The optimization algorithms used for the simulations are written in MatLab and are available at the website listed in the acknowledgements section of this paper. We also provide the simulation code and demonstrations of web-based applications. All code is open-source.

Polyhedral Methods for Stated-Choice Tasks (CBC)

Recall that we earlier distinguished between two types of conjoint tasks: metric-paired-comparison and stated-choice tasks (see Figure 1). In this paper we focus on the application of polyhedral

[5] We describe the ACA question-selection heuristic in a later section.

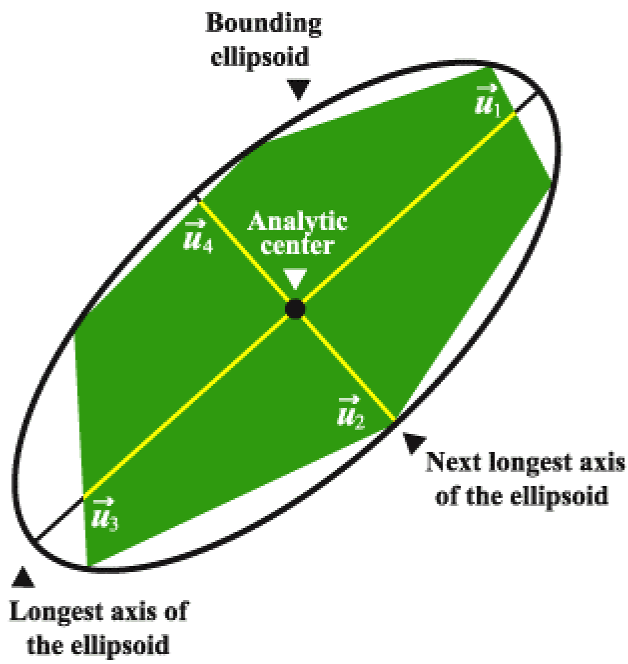
methods to metric-paired-comparison tasks. However, because the stated-choice task has also received considerable attention in the literature and is widely used in practice, it is helpful to describe how the polyhedral methods may be applied to this task. We present here one prototype algorithm. Many challenges remain, so we delay a complete treatment of this topic to a subsequent paper. One challenge will be to identify appropriate benchmarks against which to compare a polyhedral stated-choice method. Although ACA offers an established benchmark for adapting the design of metric-paired comparison questions *within* respondents, there is no such adaptive benchmark available for stated-choice questions. Several methods have been proposed for adapting choice questions *across* respondents (e.g., Anonymous 2001; Arora and Huber 2001; Huber and Zwerina 1996, Sandor and Wedel 2001) and could offer appropriate benchmarks.

There is an important difference between metric-paired-comparison tasks and stated-choice tasks. Responses to a metric-paired-comparison question yield an equality constraint ($X\vec{u} = \vec{a}$) on the feasible region, by describing how much the subject prefers one profile to another. In contrast, when responding to a stated-choice question, subjects describe which profile they prefer, but they do not describe how much they prefer one profile to the others. This response defines a set of inequality constraints, under which the utility of the chosen profile is higher than the utility of the profiles that were not chosen. Modifying the earlier notation, we define $\vec{z}_j$ as the vector of feature levels for the j^{th} profile in the choice set. In the case where there are only two profiles, j=1 and j=2, if a subject chooses profile j=1 this choice implies the following inequality constraint: $\vec{z}_1\vec{u} \geq \vec{z}_2\vec{u}$ (assuming no response error and ignoring the null option). With metric data, each question reduces the dimensionality of the feasible parameter space; with stated choice data the inequality constraints exclude sections of the space but the dimensionality is unchanged. If customers choose from J product profiles, then it is possible to design the profiles so that the subject's answer reduces the feasible region to one of J mutually exclusive, fully-exhaustive, unique subspaces. Intuitively, this means that the feasible region is reduced in size by each answer to a sub-space that is, on average, $1/J$ the size of the previous feasible region. Although algorithms can be developed for any number of profiles in the choice set (including null options), we illustrate the method for four non-null profiles. Extensions are covered in the Appendix.

Suppose that the respondent has already answered a number of stated-choice questions and suppose we want to select the next choice set of four profiles such that the respondent's answer will shrink the feasible polyhedron rapidly. In two dimensions, the current polyhedron might look like that in Figure 6. We begin by finding extreme estimates of the parameters, $\vec{u}_j$. To do this, we again approximate the polyhedron with an ellipse and find the two longest axes. The longest axes identify feasible parameter values that are maximally different. The intersection of these axes and the polyhedron gives the extreme values of the parameters that can be used to select profiles. Following Elrod, Louviere, and Davey (1992), Green, Helsen and Shandler (1988), and Johnson, Meyer and Ghosh (1989), we select representative Pareto profiles corresponding to each $\vec{u}_j$. This assures that we select profiles such that the respondent's stated choice indicates a region of the polyhedron. Specifically, for each *j*, we select the profile the respondent would have chosen, subject to an imposed budget constraint, had $\vec{u}_j$ been that respondent's partworths. This is a "knapsack" problem that is easily solved with a dynamic program. Inequality constraints (not shown in Figure 6) define the set of parameters consistent with each choice. Notice that unlike the metric paired-comparison polyhedral algorithm, there are always parameter values consistent with the respondent's stated choices. Thus, the feasible parameter space shrinks rapidly, but never vanishes.

Figure 6

One Polyhedral Method for Choice-Based Conjoint Analysis



Estimation proceeds as in the metric paired-comparison algorithm. The analytic center of the region indicated by the respondent's choice provides an estimate of the partworths. Detailed issues such as discrete features, first-question selection, external constraints, and programming are handled in an analogous manner to the metric paired-comparison algorithm. We have generated code for this adaptive polyhedral CBC algorithm. It is feasible and adapts questions with no noticeable delay. Estimates are similar to those obtained by logit analysis and by Hierarchical Bayes analysis when either is based on fixed designs or the swapping/relabeling designs of Arora and Huber (2001) and Huber and Zwerina (1996).

However, we caution the reader that these polyhedral CBC algorithms are still under development and require significant testing and improvement before we can identify the situations where they complement existing question-selection and estimation methods and where they do not.

Monte Carlo Experiments

The polyhedral methods for question selection and partworth estimation are new and untested. Although interior-point algorithms and the centrality criterion have been successful in many engineering problems, we are unaware of any other application to conjoint analysis (or any other marketing problem). Thus, we turn to Monte Carlo experiments to identify circumstances in which polyhedral methods may contribute to the effectiveness of current conjoint methods.

When evaluating new methods there are at least four validity criteria: estimation accuracy, internal validity, convergent validity, and external validity. In estimation accuracy, we want to know that the method itself, when faced with data that contains errors, can recover true parameters. In some cases, including OLS or logit, this is examined analytically by showing that if the errors satisfy certain distributional assumptions then the parameter estimates are consistent (McFadden 1974). In more complex cases, when analytic derivations are infeasible, Monte Carlo simulation demonstrates the ability of the method to recover true parameters. Internal validity refers to the ability of the method to predict respondent answers to new profiles, but with the same question format. Methods based on metric paired-comparisons and stated choice routinely do well on this measure.[6] Convergent validity refers to the ability of two or more methods to provide the same estimates. For example, Louviere, Hensher and Swait (2000) review sixteen empirical studies in marketing, transportation, and environmental valuation in which stated-choice models (CBC) provide estimates similar to those obtained by revealed preference choice models. External validity refers to the ability of conjoint models to predict actual behavior. For example, McFadden (2000) cites a case study for the Bay Area Rapid Transit in which a revealed preference models predicted market shares well; Tybout and Hauser (1981) report a quasi-experiment testing revealed-preference predictions for bus service improvements by the town of Evanston, IL. Leigh, MacKay and Summers (1984), Montgomery and Wittink (1980); Wittink and Montgomery (1979), and Wright and Kriewall

[6] Examples (and related tests) include Acito and Jain (1980), Akaah and Korgaonkar (1983), Bateson, Reibstein and Boulding (1987), Bucklin and Srinivasan (1991), Carmon, Green, and Jain (1978), Cattin, Hermet and Pioche (1982), Elrod, Louviere and Davey (1992), Green, Goldberg and Wiley (1982), Green, Goldberg and Montemayor (1981), Green and Helsen (1989), Green, Helsen and Shandler (1988), Green and Krieger (1985), Green, Krieger and Agarwal (1991), Green, Krieger and Bansal (1988), Haaijer, Wedel, Vriens and Wansbeek (1998), Hagerty (1985), Huber (1975), Huber, Wittink, Fiedler and Miller (1993), Hauser and Koppelman (1979), Hauser and Urban (1977), Hauser, Tybout and Koppelman (1981), Jain, Acito, Malhotra and Mahajan (1979), Johnson, Meyer and Ghosh (1989), Johnson (1999), Lenk, DeSarbo, Green and Young (1996), Louviere, Hensher, and Swait (2000), Malhotra (1986), Moore (1980), Moore, Louviere, and Verma (1999), Moore and Semenik (1988), Orme, Alpert and Christensen (1998), Parker and Srinivasan (1976), Reibstein, Bateson and Boulding (1988), Segal (1982), Srinivasan (1988), Srinivasan and Park (1997), and Tybout and Hauser (1981).

(1980) demonstrate external validity for full-profile conjoint analysis, Srinivasan (1988) for a form based on self-explicated importances, and Srinivasan and Park (1997) for a hybrid.

Monte Carlo simulations offer at least two advantages for the initial test of a new method. First, they can be repeated readily by other researchers. This facilitates comparison of different techniques in a range of contexts. By varying parameters we can evaluate modifications of the techniques and hybrid combinations. We can also evaluate performance based on the varying characteristics of the respondents, including the heterogeneity and reliability of their responses. Second, simulations resolve the issue of identifying the correct answer. In studies involving actual customers, the true partial utilities are unobserved. In simulations the true partial utilities are constructed so that we can compare how well alternative methods identify the true utilities from noisy responses. In this manner, Monte Carlo simulations provide a baseline from which to explore a wide range of behavioral responses to survey questions. We illustrate the use of baselines by examining both wear out and learning. Such baselines can also be generated to study order effects, routinization, carry-over effects, preference reversals, and correlated errors (Alreck and Settle 1995; Bickart 1993; Feldman and Lynch 1988; Nowlis and Simonson 1997; Simmons, Bickart and Lynch 1993; Tourangeau, Rips and Rasinski 2000).

Many papers have used the relative strengths of Monte Carlo experiments to study conjoint techniques, providing insights on interactions, robustness, continuity, attribute correlation, segmentation, new estimation methods, new data collection methods, post analysis with Hierarchical Bayes methods, and comparisons of ACA, CBC, and other conjoint methods.[7] Although we focus on specific benchmarks, there are many comparisons in the literature of these methods to other methods. (See reviews and citations in Green 1984; Green and Srinivasan 1978, 1990.)

We test the metric-paired-comparison algorithm against several benchmarks and focus on the seven issues identified in the introduction to this paper: (1) relative accuracy vs. the number of questions, (2) relative performance as the accuracy of self-explicated and paired-comparison data vary, (3) biases, (4) question selection vs. estimation, (5) hybrid methods, (6) respondent wear out and learning, and (7) relative performance on individual vs. population estimates. We begin by describing the design of the Monte Carlo experiments and then provide the results and interpretations.

Design of the Experiments for Metric-Paired Comparison Conjoint Analysis Methods

We focus on a design problem involving ten features, where a product development team is interested in learning the incremental utility contributed by each feature. We follow convention and scale to zero the partworth of the low level of a feature and, without loss of generality, bound it by 100. This re-

[7] See Carmone and Green (1981), Carmone, Green, and Jain (1978), Cattin and Punj (1984), Jedidi, Kohli, and DeSarbo (1996), Johnson (1987), Johnson, Meyer, and Ghose (1989), Lenk, DeSarbo, Green, and Young (1996), Malhotra (1986), Pekelman and Sen (1979), Vriens, Wedel, and Wilms (1996), and Wittink and Cattin (1981).

sults in a total of ten parameters to estimate (p = 10). We feel that this p is sufficient to illustrate the qualitative comparisons. We anticipate that the polyhedral methods are particularly well-suited to solving problems in which there are a large number of parameters to estimate relative to the number of responses from each individual ($q < p$). However, we would also like to investigate how well the methods perform under typical situations when the number of questions exceeds the number of parameters ($q > p$). In particular, when implementing ACA, Sawtooth Software (1996, p. 3) recommends that the total number of questions be approximately three times the number of parameters. For p = 10, this means 10 self-explicated and 20 paired-comparison questions. (We describe ACA's self-explicated questions in the next subsection.) Thus, we examine estimates of partworths for all q up to and including 20 paired-comparison questions.

We simulate each respondent's partworths by drawing independently and randomly from a uniform distribution ranging from zero to 100. We explored the sensitivity of the findings to this specification by testing different methods of drawing partworths, including beta distributions that tend to yield more similar partworths (inverted-U shape distributions) or more diverse partworths (U-shaped distributions). This sensitivity analysis yielded similar patterns of results, suggesting that the qualitative insights are not sensitive to the choice of partworth distribution.

To simulate the response to each paired-comparison question, we calculate the true utility difference between each pair of product profiles by multiplying the design vector by the vector of true partworths: $\vec{x}\vec{u}$. We assume that the respondents' answers to the metric-paired-comparison questions equal the true utility difference plus a zero-mean normal response error with variance σ_{pc}^2 . The assumption of normally distributed error is common in the literature and appears to be a reasonable assumption about response errors. (Wittink and Cattin 1981 report no systematic effects due to the type of error distribution assumed.) For each comparison, we simulate 1,000 respondents.

Benchmark Methods for Metric-Paired Comparison Question Selection

We compare the metric-paired-comparison polyhedral method against four benchmarks – Sawtooth Software's Adaptive Conjoint Analysis algorithm (ACA) and three Fixed algorithms that use the same design for every respondent. The industry and academic standard for within-respondent adaptive questioning is Adaptive Conjoint Analysis (ACA). Indeed, this appears to be the appropriate benchmark for adapting conjoint questions for each respondent based on that respondent's answers to previous questions. For example, in 1991 Green, Krieger and Agarwal (p. 215) stated that "in the short span of five years, Sawtooth Software's Adaptive Conjoint Analysis has become one of the industry's most popular software packages for collecting and analyzing conjoint data," and go on to cite a number of academic papers on ACA. Although accuracy claims vary, ACA appears to predict reasonably well in many situations (Johnson 1991; Orme 1999).

The benchmark ACA algorithm includes five sections:

1. **Unacceptability task**

   The respondent is asked to indicate unacceptable levels, which are subsequently deleted from the survey tasks. However, this step is often skipped because respondents can be too quick to eliminate levels (Sawtooth 1996).

2. **Ranking of levels within attributes**

   If the rank-order preference for levels of a feature is unknown a priori (e.g., color), the respondent ranks the levels of a feature.

3. **Self-explicated task**

   The respondent states the relative importance (on a 4-point scale) of improving the product from one feature level to another (e.g., adding automatic film ejection to an instant camera).

4. **Paired-comparison task**

   The respondent states his or her preference for pairs of partial profiles in which two or three features vary (and all else is assumed equal). This is the adaptive stage because the specific pairs are chosen by an heuristic algorithm designed to increase the incremental information yielded by the next response. In particular, based on "current" estimates of partworths, the respondent is shown pairs of profiles that are nearly equal in utility. Constraints ensure the overall design is nearly orthogonal (features and levels are presented independently) and balanced (features and levels appear with near equal frequency).

5. **Calibration concepts**

   Full profiles are presented to the respondents who evaluate them on a purchase intention scale.

For ACA we use the self-explicated (SE) and paired-comparison (PC) stages in the algorithm (Sawtooth Software 1996). (We assume rankings of levels within attributes are known without error.) Estimates of the partworths are obtained after each paired-comparison question by minimizing a least squares criterion. Specifically, an ordinary-least-squares (OLS) criterion is used in which the updated partworths are selected to minimize two sum-of-squares components – one based on the self-explicated partworths and one based on the paired-comparison questions. The code was written using Sawtooth Software's documentation together with e-mail interactions with the company's representatives. We then confirmed the accuracy of the code by asking Sawtooth Software to re-estimate partworths for a small sample of data.

Over time Sawtooth Software has modified the ACA estimation procedures. For example, recent versions of ACA allow for Hierarchical Bayes estimation. Hierarchical Bayes estimation uses data from the population to constrain the distribution of partworths across respondents and, in doing so, estimates the posterior mean of respondent-level partworths with an algorithm based on Gibbs sampling and the Metropolis Hastings Algorithm (Allenby and Rossi 1999; Arora, Allenby and Ginter 1998; Johnson 1999; Lenk, et. al. 1996; Liechty, Ramaswamy and Cohen 2001; Sawtooth Software 1999). In separate analyses later in the paper we compare the accuracy of ACA's Hierarchical Bayes estimation with the standard ACA estimates and the estimates from the polyhedral algorithm.

To simulate the SE data we assume that respondents' answers are unbiased but imprecise. In particular, we simulate response error in the SE questions by adding to the vector of true partworths, $\vec{u}$, a vector of independent identically-distributed normal error terms with variance σ_{se}^2.

We report three non-adaptive (fixed) question selection benchmarks. The benchmark we expect to perform best is a fixed-efficient-design algorithm, where, for a given q, we select the design with the highest D-efficiency (Kuhfield 1999; Kuhfield, Tobias, and Garratt 1994; Sawtooth 1999). This selection is as if the designer knew a priori how many questions would be asked. Because the questions are customized based on the total number of questions, we call this the "Custom Fixed" benchmark. Following Lenk, et. al. (1996) we also evaluate a benchmark in which questions are selected randomly for each respondent from a twenty-question D-efficient design. Because the questions are all drawn from the same fixed design, we call this the "Single Fixed" benchmark. Our third non-adaptive benchmark is a design in which the questions are chosen randomly subject to the constraints that at most three features vary, that neither of the pairs dominates, and that the design matrix is full rank. We call this the "Random" benchmark. We estimate the partworths for all three fixed designs using least-squares, hence we report results for these benchmarks only for $q \geq p$.

All benchmark methods use the PC questions, but ACA requires additional SE questions. If the SE questions are extremely accurate, then little information will be added by PC questions and ACA will dominate. Indeed, accuracy might even degrade for ACA as the number of PC questions grows (Johnson 1987). On the other hand, if the SE questions are very noisy, then, as q increases, ACA's accuracy will depend primarily on the PC questions. These two situations bound empirical experience, thus we report results for two conditions – highly accurate SE questions and noisy SE questions. To facilitate comparisons among methods, we hold constant the noise in the PC questions.

To test relative performance we plot the mean absolute accuracy of the parameter estimates (true vs. estimated values averaged across parameters and respondents). We chose to report mean absolute error (MAE) rather than root mean squared error (RMSE) because the former is less sensitive to outliers and is more robust over a variety of induced error distributions (Hoaglin, Mosteller and Tukey 1983; Tukey 1960). However, as a practical matter, the qualitative implications of our simulations are the same for both error measures. Indeed, except for a scale change, Figure 7 (and other figures) are almost identical for both MAE and RMSE.[8] Because both MAE and RMSE measure the combined impact of bias and reliability, we examine bias separately.

Results of the Monte Carlo Experiments for Metric-Paired-Comparison Data

Our goal is to illustrate the potential of the polyhedral methods and, in particular, to find situations where they add incremental value to the suite of conjoint analysis methods. We also seek to identify situations where extant methods are superior. As in any simulation analysis we cannot vary all parameters of the problem, thus, in these simulations we vary those parameters that best illustrate the differences among the methods. The simulation code is available on our website so that other researchers might investigate other parameters.

We select a moderate error in the paired-comparison questions. In particular, we select $\sigma_{pc} = 30$. This is 5% of the range of the answers to the PC questions and 30% of their maximum standard deviation (9% of their variance).[9] We compared several of the findings under more extreme errors and observed similar qualitative insights.

Figure 7a compares the polyhedral algorithm to our three non-adaptive benchmarks. Consider first the benchmark ("Custom Fixed") that chooses a different efficient design for every q. For $q < 15$ the polyhedral algorithm yields lower estimation error than these efficient fixed designs. However, as more degrees of freedom are added to the least-squares estimates, about 50% more than the number of parameters (p=10), we begin to see the advantage of orthogonality and balance in question design – the goal of efficiency. However, even after twenty questions the performance of the polyhedral algorithm is almost as effective as the most efficient fixed design. This is reassuring, indicating that the polyhedral algorithm's focus on rapid estimates from relatively few questions comes at little loss in accuracy when respondents answer more questions. Another way to look at Figure 7a is horizontally; in many cases of moderate *q,* the polyhedral algorithm can achieve the same accuracy as the most efficient fixed design, but with fewer questions. This is particularly relevant in a web-based context. In Figure 7a we also see the advantage of choosing a different efficient design for each q (Custom Fixed vs. Single Fixed). Finally, both the polyhedral algorithm and the efficient designs outperform randomly selected questions.[10]

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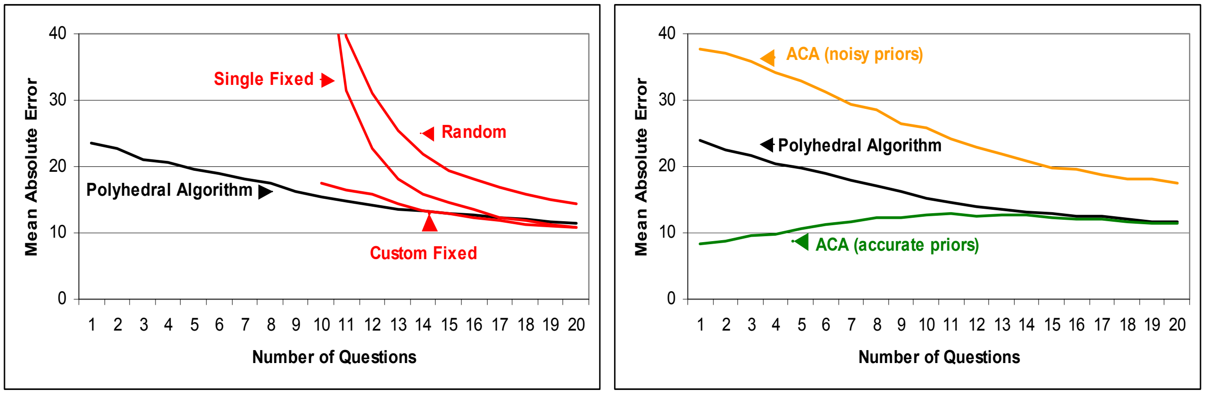
[8] For a standard normal distribution, MAE = $(2/\pi)^{1/2}$ RMSE.

[9] The maximum standard deviation is 100 because the PC responses are a sum of at most three features (±) each uniformly distributed on [0,100]. Our review of the conjoint simulation literature suggests that the median error percentage reported in that literature is 29%. Johnson (1987, p. 4) suggests that, with a 25% simulated error, ACA estimation error "increases only moderately" relative to estimates based on no response error. Some interpretations depend on the choice of error percentage – for example, all methods do uniformly better for low error variances than for high error variances. We leave to future papers the complete investigation of error-variance sensitivity.

[10] For each of comparison with the benchmarks in Figure 7b we maintain a consistent scale. For q=10 the MAEs are 58.6 for Single Fixed and 59.3 for Random.

Figure 7

Comparison of Polyhedral Methods to ACA and Fixed Designs



(a) Comparison to fixed designs (b) Comparison to ACA

Figure 7b compares the polyhedral algorithm to the two ACA benchmarks. In one benchmark we add very little error ($\sigma_{se} = 10$) to the SE responses making them three times as accurate as the PC questions ($\sigma_{pc} = 30$). In the second benchmark we make the SE questions relatively noisy ($\sigma_{se} = 50$). We expect that these benchmarks should bound empirical situations. We label the benchmark methods: "ACA (accurate priors)" and "ACA (noisy priors)."

As expected, the accuracy of the SE responses determines the precision of the ACA predictions. The polyhedral algorithm outperforms the ACA method when the SE responses are noisy but does not perform as well when respondents are able to give highly accurate self-explicated responses. Interestingly, the accuracy of the ACA method initially worsens when the priors are highly accurate (see also Johnson 1987). Not until q exceeds p does the efficiency of least-squares estimation begin to reduce this error. Once sufficient questions are asked, the information in the PC responses begins to outweigh measurement error and the overall accuracy of ACA improves. However, despite ACA's ability to exploit accurate SE responses, the polyhedral algorithm (without SE questions) begins to approach ACA's accuracy soon after q exceeds p. This ability to eliminate SE questions can be important in web-based interviewing if the SE questions add significantly to respondent wear out.

For noisy SE responses, ACA's accuracy never approaches that of the polyhedral algorithm, even when $q=2p$, the number of questions suggested by Sawtooth. Summarizing, Figure 7b suggests that ACA is the better choice if the SE responses are highly accurate (and easy to obtain). The polyhedral algorithm is likely a better choice when SE responses are noisy or difficult to obtain. The selection of algorithms depends upon the researcher's expectations about the context of the application. For example, for product categories in which customers often make purchasing decisions about features separately, perhaps by purchasing from a menu of features, we might expect more accurate SE responses. In contrast, if the features are typically bundled together, so that customers have little experience in evaluating the importance of the individual features, the accuracy of the SE responses may be lower. Relative accuracy of the two sets of questions may also be affected by the frequency with which customers purchase in the category and their consequent familiarity with product features.

Exploration of Potential Biases in the Estimates

We consider two types of bias: relative bias and scale bias. Relative bias describes a situation in which the bias for one or more coefficients is systematically different than the bias for other coefficients. Relative bias is important from a managerial perspective; if the partworth estimates for one set of attributes are systematically higher than the estimates for another set of attributes, then managers may misallocate resources towards the first set of attributes. On the other hand, if all coefficients are biased by the same amount then the managerial recommendations will often be unaffected (Louviere, Hensher and Swait 2000, p. 360). In our simulations the attributes are simulated in an identical fashion, and so systematic differences in bias across types of attributes cannot arise.[11]

Scale bias focuses on the presence of bias in any coefficient estimate. We measure scale bias by calculating the average difference between the estimated and true coefficients. The absence of relative bias allows us to average across all of the coefficients. For Fixed designs in which OLS is used for parameter estimation, the estimates are unbiased, hence, we expect no scale bias (Judge, et. al. 1985, p. 14). Indeed, for the Fixed designs, after 20 questions there is no significant difference between the sum of the true parameters and the sum of the estimated parameters ($t = 0.3$).[12] However, for adaptive designs the question matrix, X, is endogenous because each row of X depends upon the respondent's prior answers, including any errors in those answers. This is a classic econometric endogeneity problem and OLS estimates under such conditions are, in general, biased (Judge, et. al. 1985, p. 571). This turns out to be the case for ACA when its estimates are based on OLS. As q grows, the biases grow. For example, there are significant positive biases for $q = 10$ ($t = 12.8$), $q = 15$ ($t = 16.8$), and $q = 20$ ($t = 21.2$). At $q = 20$, endogeneity leads to approximately an 8% scale bias.

We do not have a formal theory of endogeneity for analytic-center estimation, but we can explore potential scale biases in the simulation data. To do so, we examine the polyhedral estimates that are based only on analytic-center question-selection and estimation. That is, we examine polyhedral estimates obtained just before the polyhedron becomes empty. For such estimates there is no significant bias ($t = 1.2$). For the full algorithm, which reverts to the ACA question design heuristics when the feasible set is empty, the scale bias grows to approximately 6% at $q = 20$. In summary, there are no observed scale biases for

[11] In practice product features carry extrinsic characteristics, such as ethereal vs. concrete, that distinguish one feature from another – future empirical tests might explore the differential impact of such extrinsic characteristics.

[12] Note that the Standard Fixed and Custom Fixed designs are the same when q=20.

analytic-center question-selection and estimation, but for ACA question selection, endogeneity introduces a scale bias to the estimates.

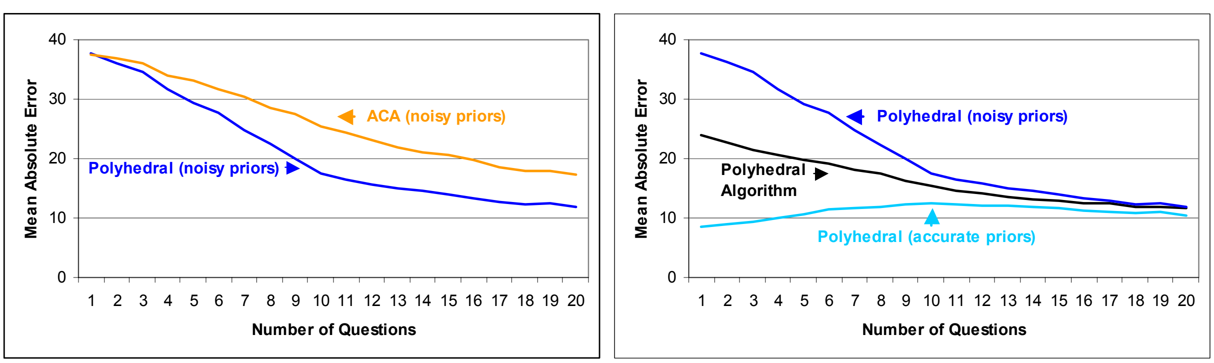
Question Selection vs. Estimation with Alternative Methods

We now examine whether the polyhedral algorithm and/or the benchmarks can be improved. For example, Figure 7b suggests that if SE responses can be obtained easily and accurately, then they have the potential to improve the accuracy of adaptive conjoint methods. This is consistent with the conjoint literature, which suggests that both SE and PC questions add incremental information (Green, Goldberg, and Montemayor 1981; Griffin and Hauser 1993; Huber, et. al. 1993, Johnson 1999; Leigh, MacKay, and Summers 1984). This evidence raises the possibility that the precision of polyhedral methods can also be improved by incorporating SE responses.

To examine the effectiveness of including SE responses in polyhedral algorithms and to isolate the polyhedral question-selection method, we test a hybrid method that combines the polyhedral question selection method with the ACA OLS estimation method. That is, we use ACA's estimation procedure to incorporate self-explicated responses, but replace ACA's question-selection procedure (for the metric-paired-comparison questions) with polyhedral question selection. Figure 8a compares the two question-selection methods holding constant the estimation procedures (and the noise level of the priors). This figure suggests that polyhedral question selection has the potential to improve ACA. We observe a similar pattern of results when comparing the methods under more accurate priors.

Figure 8

Including Self-Explicated Responses in Polyhedral Algorithms



(a) Comparison of algorithms, noisy priors (b) Alternative Estimation Methods

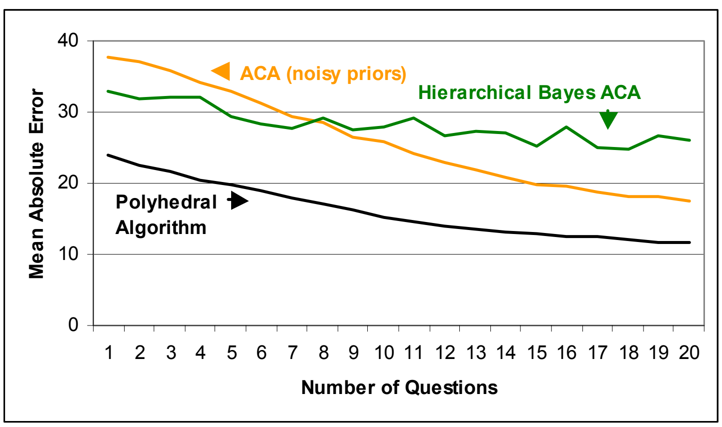
Figure 8b compares polyhedral estimates without priors (black line from Figure 7b) to estimates based on accurate and noisy priors (holding the noise in the PC responses constant). This comparison holds the question design constant and varies the estimation method. As in Figure 7b, the choice of method depends upon the accuracy of the SE responses. We see that the polyhedral method without priors dominates the polyhedral method with noisy priors, although just slightly for large q. However, if SE responses are accurate and easy to obtain, then combining ACA estimation with polyhedral question selection yields more accurate forecasts than either ACA alone or the polyhedral algorithm alone. As the SE responses become noisy, then the hybrid method becomes increasingly less accurate until, at a moderate noise level, it is better to ignore the SE responses altogether and use polyhedral question selection with analytic-center estimation.

Hierarchical Bayes Estimation

We now explore an alternative estimation technique for ACA, Hierarchical Bayes estimation. This estimation technique has attracted widespread interest in the academic literature in recent years (e.g., Allenby and Rossi 1999; Arora, Allenby and Ginter 1998; Johnson 1999; Lenk, et. al. 1996; Liechty, Ramaswamy and Cohen 2001; Sawtooth Software 1999) and is now available for applications of ACA. Recall that Hierarchical Bayes estimation uses information from the population of respondents in an effort to improve the estimates for each individual. In Figure 9 we evaluate the accuracy of this modification by comparing it to estimates from the ACA estimates in Figure 7b.[13] In both cases we use noisy estimates for the self-explicated priors. As a basis for comparison we also include the polyhedral estimates from earlier figures.

Figure 9

ACA Hierarchical Bayes



Hierarchical Bayes improves the accuracy of the ACA estimates for low q. As the number of questions increases, the standard ACA estimates outperform the Hierarchical Bayes measures. We interpret this finding as evidence that Hierarchical Bayes' use of population data is an effective means of

[13] Due to the intensive computational and computer-memory demands of Hierarchical Bayes estimation, we simulate 200 respondents (rather than 1,000) for each of q = 1 to 20. The smaller sample size causes the Hierarchical Bayes curve to be less smooth.

moderating the error in the individual responses. However, once there are sufficient individual responses, so that the information in these responses outweighs the error, there is less need to moderate the individual responses. As with Figure 7 and 8, the absolute accuracy of Hierarchical Bayes ACA depends upon the accuracy of the SE responses.

Modeling Respondent Wear Out and/or Learning

The literature includes a broad range of potential variations in response errors reflecting differences in the way that human subjects interact with research instruments. Monte Carlo experiments provide an ideal mechanism for investigating the impact of these variations. Space restrictions limit our investigation in this paper to just two sources of variation: wear out and learning. We encourage readers interested in investigating other sources of variation to download code from the website listed in the acknowledgments to this paper.

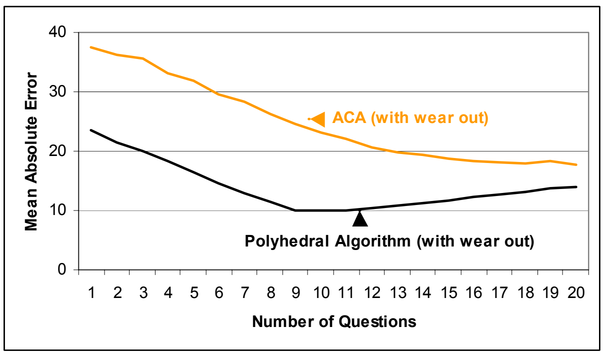
Wear out may lead to increased response errors if respondents tire as q increases, causing them to pay less attention to later PC questions. The literature offers some support for this prediction (see for example, Alreck and Settle 1995; Black, et. al. 2000; Couper 2000; De Angelis 2001; and Lenk et. al. 1996 at 173). Alternatively, other researchers have observed a learning or priming phenomenon, in which the initial questions help respondents clarify their values, increasing the accuracy of the later questions (Green, Krieger and Agarwal 1991; Huber, et. al. 1993; Johnson 1991). The literature advances several theories that support this training hypothesis, including task learning, self-preference learning, memory accessibility, and context effects (Bickart 1993; Feldman and Lynch 1988; Nowlis and Simonson 1997; Simmons, Bickart and Lynch 1993; Tourangeau, Rips and Rasinski 2000). While the magnitude and direction of wear out and/or learning is an empirical question, simulation can assess the impact of each, should either occur. Our goal is to demonstrate the phenomenon and to investigate how it affects each method. We hope also to motivate empirical investigations into the shape of the wear out and/or learning functions.

We could find little precedent for quantification of the wear out or learning functions in the literature, so we assume simple linear growth and decay functions. In particular, if ε denotes a draw of normal measurement error for the PC questions, then, in our wear-out analysis, we select $E_W(q)= \varepsilon q/10$. Dividing by 10 matches the wear out error to the prior analysis at $q = p$ yielding an average error that is roughly equivalent. To model learning, we reverse the function and select $E_L(q)= \varepsilon(20-q)/10$. For ease of comparison we leave the variance of the error terms unchanged from the previous figures and assume that the variance of response error in the SE questions is constant for all questions.

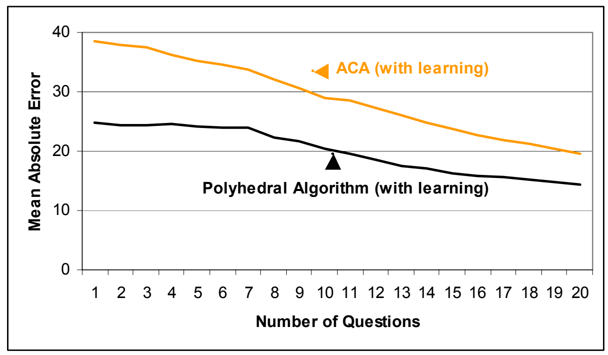
Figure 10a summarizes the simulation of respondent wear out. Initially, as more PC questions are answered, estimation accuracy improves. The new information improves the estimates even though the information becomes increasingly noisy. After approximately 10-12 questions the added noise overwhelms the added information and the estimates begin to degrade yielding a U-shaped function of *q*. The rate of degradation for the ACA benchmarks is slower – the U-shape begins to appear around question 18. The slower degradation can be explained, in part, because ACA uses the SE responses to reduce reliance on the increasingly inaccurate PC questions. The hybrid method of Figure 8 declines at a rate similar to ACA, reflecting the inclusion of SE responses in the hybrid method. This interpretation is supported by the results of further simulations not reported in Figure 10a. Efficient fixed designs, which do not use SE responses, decline at a rate similar to the polyhedral method.[14]

Figure 10

Modeling Respondent Wear Out and/or Learning



(a) Effect of Wear Out



(b) Effect of Learning

Figure 10b summarizes the simulation of respondent learning. In this case the initial questions are less accurate so, for low *q,* MAE is higher. However, MAE declines as the respondents learn and response error decreases. The exact amount and direction of wear out and/or learning and rate at which it grows or decays is an empirical question. Indeed, more complex respondent reactions are also possible. For example, respondents might learn for low *q* and then begin to wear out as *q* gets large. In either case, analyses such as those in Figure 10 suggest that wear out and learning can have important impacts on the accuracy of question-selection and estimation methods. Initial experience with web-based conjoint analysis methods suggests that wear out appears to occur for large *q* (Chan 1999; and McArdle 2000) and that learning appears to occur for small *q* (Anonymous 2002).

Estimates of Mean Partworths for the Population

One advantage of the polyhedral method is that it can estimate partworths for each respondent with a relatively few questions per respondent. Such respondent-specific partworths are valuable for

[14] We invite the reader to explore this wear out phenomenon with alternative levels of noise in the SE responses. We have found that the results vary with noise level in a manner analogous to the discussions of Figure 7 and 8. As

product-line decisions and/or segmentation.[15] However, if the population is homogeneous, then the product development team may seek to estimate average partworths to represent the population's preferences. If this is the case, aggregate methods can obtain excellent estimates with few questions per respondent.

To investigate this issue we draw ten sets of population means and simulate 200 respondents for each set of population means. Data are pooled within each population of 200 respondents and OLS is used to estimate a representative set of partworths for that population. We first draw the partworth means for each population from independent uniform distributions on [25, 75]. We then simulate the partworths for each of 200 respondents in the population by adding heterogeneity terms to the vector of population means. In separate simulations we compare uniform and triangle distributions for these heterogeneity terms and for each distribution we consider two ranges: a relatively homogeneous range of [-10, 10] and a relatively heterogeneous range of [-25, 25]. Given the vectors of partworths for each individual, we proceed as before, adding measurement error to construct the PC and SE responses. We report the average MAE in the forecast population means, averaged across ten parameters times ten populations. The findings after twenty questions for each individual are summarized in Table 1 below. (Both efficient Fixed algorithms use the same designs for $q = 20$.)

As expected all methods perform well. The magnitudes of the MAE are lower for population-level estimates than for all respondent-level estimates reported above, reflecting the larger sample of data used to estimate the partworths. Although we might improve all methods by selecting questions that vary optimally across respondents, it is reassuring that even without such modifications, all of the techniques yield accurate estimates of the population partworths, especially for relatively homogeneous populations.

When the partworths are relatively homogeneous within a population, the polyhedral and fixed methods yield slightly more accurate parameter estimates than ACA, but all perform well. When the partworths are relatively heterogeneous, the efficient fixed design, which is optimized to OLS, outperforms both adaptive methods. However, in this latter case, population means have relatively less managerial meaning.

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the SE responses become more accurate, ACA and the hybrid method perform relatively better.

[15] Currim (1981), Green and Helsen (1989), Green and Krieger (1989), Hagerty (1985), Hauser and Gaskin (1984), Page and Rosenbaum (1987), Vriens, Wedel, and Wilms (1996), and Zufryden (1979) all provide examples of using respondent-specific partworths to identify segments with managerial meaning. Furthermore, Green and Helsen (1989), Moore (1980), Moore and Semenik (1988), and Wittink and Montgomery (1979) all provide evidence that respondent-specific partworths predict better than population means.

Table 1
MAE of Population Mean Estimates

Individual Heterogeneity		**Mean Absolute Error**		
Distribution	Range	Polyhedral	ACA	Fixed
Relatively homogeneous				
Uniform	[-10, 10]	0.94	1.22	0.82
Triangle	[-10, 10]	0.76	0.88	0.78
Relatively heterogeneous				
Uniform	[-25, 25]	4.30	6.88	0.84
Triangle	[-25, 25]	2.54	3.70	0.78

Summary of the Results of the Monte Carlo Experiments

Our simulations suggest that no method dominates in all situations, but that there are a range of relevant situations where the polyhedral algorithm or the ACA-polyhedral hybrid is a useful addition to the suite of conjoint analysis methods available to a product developer or a market researcher. If SE responses can be obtained accurately (relative to PC responses) and with little respondent wear out, then either ACA or the ACA-polyhedral hybrid method is likely to be most accurate. If new PC question formats can be developed that engage respondents with visual, interactive media, such that respondents are willing and able to answer sufficient questions ($q > 1.5p$ in Figure 7a), then customized efficient fixed designs might perform better than adaptive methods such as ACA, the polyhedral algorithm, or a hybrid.

The real advantage of the polyhedral methods for metric paired-comparison data comes when the researcher is seeking respondent-level partworths and is limited to relatively few questions ($q < p$), when wear out is a significant concern, and/or when SE responses are noisy relative to PC responses. We believe that these situations are becoming increasingly relevant to conjoint analysis applications, especially in the context of web-based conjoint analysis and for applications in which the product-development team cycles through iterative designs many times before the product is launched (Black, et. al. 2000; Buckman 2000; Cusumano and Yoffie 1998; Cusumano and Selby 1995; Dahan and Hauser 2002; Dahan and Srinivasan 2000; Marketing News 2000; McGrath 1996; Smith and Reinertsen 1998; Ulrich and Eppinger 2000).

Finally, the relative accuracy of SE vs. PC responses, and hence the choice of conjoint analysis method, is likely to depend upon context. For complex products, for products where industrial design is important, or for products with a high emotional content, it might be easier for a respondent to make a holistic judgment by providing metric evaluations of pairs of products than it would be for the respondent to evaluate the products feature by feature. Web-based methods in which realistic, but virtual, products are presented to respondents, might also enhance the ability of respondents to make holistic judgments.[16]

An Initial Empirical Test of Metric Paired-Comparison Polyhedral Methods

In a separate paper we (and our colleagues) report the results of a field test investigating the predictive accuracy of the polyhedral method in an actual purchase situation (Anonymous 2002). In particular, we compare how well different conjoint methods predicted demand for an innovative "messenger-style" laptop computer bag that was under development by Timbuk2, Inc (www.Timbuk2.com). The features of the bag included size, color, type of sleeve closure, and whether or not it had a logo, handle, PDA holder, cell phone holder, mesh pocket, and protective boot. These features plus price represented a 10-parameter conjoint problem. The 330 respondents were randomly assigned to one of three question-selection methods, corresponding to the polyhedral method, ACA, and an efficient Fixed design. Each respondent answered sixteen metric-paired-comparison questions and, in the ACA cell, ten self-explicated questions. After the sixteen pairs, they answered four hold-out pairs – an internal validity test. Following a filler task designed to cleanse short-term memory, each respondent was presented with a choice among five laptop computer bags – an external validity task. The choice was real – the respondents received the bag plus any change (in cash) from $100.[17] The five bags were randomly chosen from an efficient design of sixteen bags and the price was chosen so that all choice sets were Pareto.

Table 2 reports the results for both the holdout pairs and the choice of actual bags (plus cash). These results are consistent with the simulations in Figure 7 and suggest that the metric-paired-comparison polyhedral algorithm has the potential to do at least as well as the benchmarks for metric paired-comparison data.

[16] The website listed in the acknowledgements section of this paper provides links to web-based conjoint analysis demonstrations for cameras, laptop computer bags, cross-over vehicles, ski resorts, bicycle pumps, and copying machines.
[17] In addition, respondents' ranked the five bags based on the premise that their first choice might not be available and that they would be given their most preferred laptop bag (plus cash) from those that were manufactured. Results were qualitatively the same for both first choice and rank order choice. Table 2 reports the rank-order results, which have greater statistical power.

Table 2
Initial Empirical Application of the Metric-Paired-Comparison Polyhedral Algorithm: Correlation Between Predicted and Observed Choices

	ACA	Fixed	Polyhedral
Internal Validity Task (Holdout Pairs)			
Estimates using 8 questions *	0.53	–	0.71
Estimates using 16 questions *,+	0.61	0.73	0.81
External Validity Task (Bag Choice)			
Estimates using 8 questions *	0.45	–	0.59
Estimates using 16 questions *,o	0.52	0.54	0.68

* Significant difference ($p<0.05$) between the polyhedral algorithm and ACA.
o Significant difference ($p<0.05$) between the polyhedral algorithm and Fixed.
\+ Significant difference ($p<0.05$) between Fixed and ACA.

Summary, Conclusions, and Further Research

We have proposed a new conjoint analysis method for metric-paired-comparison questions. The method is designed to identify, using relatively few questions, features that have the most influence on customer preferences. The algorithm uses advanced interior-point mathematical programming methods to adaptively select questions that constrain the set of feasible parameters. The method uses centrality concepts and ellipsoid shape approximations. We tested the method using a series of Monte Carlo simulations. The findings confirm that the polyhedral algorithm is particularly suited to contexts where researchers are limited to asking relatively few questions compared to the number of parameters. By isolating the impact of the question design component, we found that the relative accuracy of the method is due, at least in part, to the design of the questions. Our simulations suggest that hybrid polyhedral question-selection methods could be used to enhance existing estimation methods.

To evaluate the contribution of the proposed polyhedral methods we had to make several practical implementation decisions. Examples include the choice of the first question and the procedures to design questions and estimate partworths when responses are inconsistent (the feasible set is empty). Improvements in the resolution of these issues could lead to improvements in the accuracy of the method. There are several other issues that we would like to identify as topics for future research. First, the choice of initial constraints may be important. Tight constraints may be more accurate, but could introduce ceiling effects that affect the precision of the estimates. Second, the algorithm is myopic in the sense that it only looks one step ahead to choose the next question. Improvements may be obtained from less myopic algorithms that look more than one step ahead. Third, the estimation procedures do not currently adjust for the impact that constraints have on the resulting estimates. Alternative empty-feasible-set question-selection algorithms could improve performance. We anticipate that further research on these, and other as-yet-undiscovered issues, might yield more accurate predictions. In this respect, the performance of the specific polyhedral methods evaluated in the Monte Carlo simulations should be interpreted as a lower bound on the potential performance of this class of methods. Like many new technologies, we expect its performance to improve with use and evolution (Bower and Christensen 1995; Christensen 1998). Under this interpretation, the performance of the current version of the method is gratifying and suggests that this class of methods is worth investigating.

More generally, we believe that the polyhedral methods used in this study are just one of a range of recent developments in the mathematical programming and optimization literatures that can contribute to our understanding of marketing problems. For example, we have proposed one feasible polyhedral algorithm for adaptive choice-based conjoint analysis using stated-choice data. This algorithm obtains reasonable estimates rapidly, but requires further development before it is ready for simulation and field testing. In addition, we are aware of other researchers who are developing adaptive algorithms for full-profile conjoint-analysis data collection using new methods in combinatorial optimization and we are aware of researchers exploring the use of support vector machines for conjoint estimation.

Because the new mathematical programming methods obtain near optimal solutions extremely fast, they might be also used to select promotion variables dynamically as respondents navigate a website. Alternatively, they might be used to design ongoing experiments in which parameters of a website are varied in an optimal manner trading off current effectiveness for long-term improvement.

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Appendix: Mathematics of Fast Polyhedral Adaptive Conjoint Estimations

Consider the case of p parameters and q questions where $q \le p$. Let u_j be the j^{th} parameter of the respondent's partworth function and let $\vec{u}$ be the $p\times 1$ vector of parameters. Without loss of generality we assume binary features such that u_j is the high level of the j^{th} feature. For more levels we simply recode the $\vec{u}$ vector and impose constraints such as $u_m \le u_h$. We handle such inequality constraints by adding slack variables, $v_{hm} \ge 0$, such that $u_h = u_m + v_{hm}$. Let r be the number of externally imposed constraints, of which $r' \le r$ are inequality constraints.

Notation for Adaptive Metric-Paired-Comparison Questions

Let $\vec{z}_{i\ell}$ be the $1\times p$ vector describing the left-hand profile in the i^{th} paired-comparison question and let $\vec{z}_{ir}$ be the $1\times p$ vector describing the right-hand profile. The elements of these vectors are binary indicators taking on the values 0 or 1. Let X be the $q\times p$ matrix of $\vec{x}_i = \vec{z}_{i\ell} - \vec{z}_{ir}$ for $i = 1$ *to* q. Let a_i be the respondent's answer to the i^{th} question and let $\vec{a}$ be the $q\times 1$ vector of answers for $i = 1$ *to* q. Then, if there were no errors, the respondent's answers imply $X\vec{u} = \vec{a}$. To handle additional constraints, we augment these equations such that X becomes a $(q+r)\mathrm{x}(p+r')$ matrix, $\vec{a}$ becomes a $(q+r)\mathrm{x}1$ vector, and $\vec{u}$ becomes a $(p+r')\mathrm{x}1$ vector. These augmented relationships form a polyhedron, $\mathscr{P} = \{\vec{u} \in \Re^{p+r'} \mid X\vec{u} = \vec{a},\ \vec{u} \ge 0\}$. We begin by assuming that $\mathscr{P}$ is non-empty, that X is full-rank, and that no j exists such that $u_j=0$ for all $\vec{u}$ in $\mathscr{P}$. We later indicate how to handle these cases.

Finding an Interior Point of the Polyhedron

To begin the algorithm we first find a feasible interior point of $\mathscr{P}$ by solving a linear program, LP1 (Freund, Roundy and Todd 1985). Let $\vec{e}$ be a $(p+r')\times 1$ vector of 1's and let $\vec{0}$ be a $(p+r')\times 1$ vector of 0's; the y_j's and θ are parameters of LP1 and $\vec{y}$ is the $(p+r')\times 1$ vector of the y_j's. (When clear in context, inequalities applied to vectors apply for each element.) LP1 is given by:

$$\text{(LP1)} \qquad \max \sum_{j=1}^{p+r'} y_j\,, \qquad \text{subject to: } X\vec{u} = \theta\vec{a}\,, \quad \theta \ge 1, \quad \vec{u} \ge \vec{y} \ge \vec{0}\,, \quad \vec{y} \le \vec{e}$$

If $(\vec{u}^*, \vec{y}^*, \theta^*)$ solves LP1, then $\theta^{*-1}\vec{u}^*$ is an interior point of $\mathscr{P}$ whenever $\vec{y}^* > \vec{0}$. If there are some y_j's equal to 0, then there are some j's for which $u_j=0$ for all $\vec{u} \in \mathscr{P}$. If LP1 is infeasible, then $\mathscr{P}$ is empty. We address these cases later in this appendix.

Finding the Analytic Center

The analytic center is the point in $\mathscr{P}$ that maximizes the geometric mean of the distances from the point to the faces of $\mathscr{P}$. We find the analytic center by solving OPT1.

$$\text{(OPT1)} \qquad \max \sum_{j=1}^{p+r'} \ln(u_j)\,, \qquad \text{subject to: } X\vec{u} = \vec{a}\,, \quad \vec{u} > \vec{0}$$

Freund (1993) proves with projective methods that a form of Newton's method will converge rapidly for OPT1. To implement Newton's method we begin with the feasible point from LP1 and improve it with a scalar, α, and a direction, $\vec{d}$, such that $\vec{u} + \alpha\vec{d}$ is close to the optimal solution of OPT1. ($\vec{d}$ is a $(p+r')\times 1$ vector of d_j's.) We then iterate subject to a stopping rule.

We first approximate the objective function with a quadratic expansion in the neighborhood of $\vec{u}$.

(A1) $$\sum_{j=1}^{p+r'} \ln(u_j + d_j) \approx \sum_{j=1}^{p+r'} \ln(u_j) + \sum_{j=1}^{p+r'} \left(\frac{d_j}{u_j} - \frac{d_j^2}{2u_j^2} \right)$$

If we define U as a $(p+r')\times(p+r')$ diagonal matrix of the u_j*'s*, then the optimal direction solves OPT2:

(OPT2) $$\max\ \vec{e}^T U^{-1} \vec{d} - (\tfrac{1}{2}) \vec{d}^T U^{-2} \vec{d} \qquad \text{subject to:} \quad X\vec{d} = \vec{0}$$

Newton's method solves OPT1 quickly by exploiting an analytic solution to OPT2. To see this, consider first the Karush-Kuhn-Tucker (KKT) conditions for OPT2. If $\vec{z}$ is a $(p+r')\times 1$ vector parameter of the KKT conditions that is unconstrained in sign then the KKT conditions are written as:

(A2) $$U^{-2}\vec{d} - U^{-1}\vec{e} = X^T \vec{z}$$

(A3) $$X\vec{d} = \vec{0}$$

Multiplying A2 on the left by XU^2, gives $X\vec{d} - XU\vec{e} = XU^2X^T\vec{z}$. Applying A3 to this equation gives: $-XU\vec{e} = XU^2X^T\vec{z}$. Since $U\vec{e} = \vec{u}$ and since $X\vec{u} = \vec{a}$, we have $-\vec{a} = XU^2X^T\vec{z}$. Because X is full rank and U is positive, we invert XU^2X^T to obtain $\vec{z} = -(XU^2X^T)^{-1}\vec{a}$. Now replace $\vec{z}$ in A2 by this expression and multiply by U^2 to obtain $\vec{d} = \vec{u} - U^2X^T(XU^2X^T)^{-1}\vec{a}$.

According to Newton's method, the new estimate of the analytic center, $\vec{u}'$, is given by $\vec{u}' = \vec{u} + \alpha\vec{d} = U(\vec{e} + \alpha U^{-1}\vec{d})$. There are two cases for α. If $\left\| U^{-1}\vec{d} \right\| < \tfrac{1}{4}$, then we use $\alpha = 1$ because $\vec{u}$ is already close to optimal and $\vec{e} + \alpha U^{-1}\vec{d} > \vec{0}$. Otherwise, we compute α with a line search.

Special Cases

If X is not full rank, XU^2X^T might not invert. We can either select questions such that X is full rank or we can make it so by removing redundant rows. Suppose that $\vec{x}_k$ is a row of X such that $\vec{x}_k^T = \sum_{i=1, i\neq k}^{q+r} \beta_i \vec{x}_i^T$. Then if $a_k = \sum_{i=1, i\neq k}^{q+r} \beta_i a_i$, we remove $\vec{x}_k$. If $a_k \neq \sum_{i=1, i\neq k}^{q+r} \beta_i a_i$, then $\mathscr{P}$ is empty and we employ OPT4 described later in this appendix.

If in LP1 we detect cases where some y_j*'s* = 0, then there are some *j's* for which u_j=0 for all $\vec{u} \in \mathscr{P}$. In the later case, we can still find the analytic center of the remaining polyhedron by removing those *j's* and setting $u_j = 0$ for those indices. If $\mathscr{P}$ is empty we employ OPT4.

Finding the Ellipsoid and its Longest Axis

If $\vec{\bar{u}}$ is the analytic center and $\bar{U}$ is the corresponding diagonal matrix, then Sonnevend (1985a, 1985b) demonstrates that $\mathscr{E} \subseteq \mathscr{P} \subseteq \mathscr{E}_{p+r'}$ where, $\mathscr{E} = \{\vec{u} \mid X\vec{u} = \vec{a},\ \sqrt{(\vec{u} - \vec{\bar{u}})^T \bar{U}^{-2} (\vec{u} - \vec{\bar{u}})} \leq 1\}$ and $\mathscr{E}_{p+r'}$ is constructed proportional to $\mathscr{E}$ by replacing 1 with $(p+r')$. Because we are interested only in the direction of the longest axis of the ellipsoids we can work with the simpler of the proportional ellipsoids, $\mathscr{E}$. Let $\vec{g} = \vec{u} - \vec{\bar{u}}$, then the longest axis will be a solution to OPT3.

(OPT3) $$\max\ \vec{g}^T\vec{g} \qquad \text{subject to:} \quad \vec{g}^T \bar{U}^{-2} \vec{g} \leq 1, \quad X\vec{g} = \vec{0}$$

OPT3 has an easy-to-compute solution based on the eigenstructure of a matrix. To see this we begin with the KKT conditions (where ϕ and γ are parameters of the conditions).

(A4) $\vec{g} = \phi \overline{U}^{-2} \vec{g} + X^T \vec{\gamma}$

(A5) $\phi(\vec{g}^T \overline{U}^{-2} \vec{g} - 1) = 0$

(A6) $\vec{g}^T \overline{U}^{-2} \vec{g} \le 1, \quad X\vec{g} = \vec{0}, \quad \phi \ge 0$

It is clear that $\vec{g}^T \overline{U}^{-2} \vec{g} = 1$ at optimal, else we could multiply $\vec{g}$ by a scalar greater than 1 and still have $\vec{g}$ feasible. It is likewise clear that ϕ is strictly positive, else we obtain a contradiction by left-multiplying A4 by $\vec{g}^T$ and using $X\vec{g} = \vec{0}$ to obtain $\vec{g}^T \vec{g} = 0$ which contradicts $\vec{g}^T \overline{U}^{-2} \vec{g} = 1$. Thus, the solution to OPT3 must satisfy $\vec{g} = \phi \overline{U}^{-2} \vec{g} + X^T \vec{\gamma}$, $\vec{g}^T \overline{U}^{-2} \vec{g} = 1$, $X\vec{g} = \vec{0}$, and $\phi > 0$. We rewrite A4-A6 by letting I be the identify matrix and defining $\eta = 1/\phi$ and $\vec{\omega} = -\vec{\gamma}/\phi$.

(A7) $(\overline{U}^{-2} - \eta I)\vec{g} = X^T \vec{\omega}$

(A8) $\vec{g}^T \overline{U}^{-2} \vec{g} = 1$

(A9) $X\vec{g} = \vec{0}, \quad \phi > 0$

We left-multiply A7 by X and use A9 to obtain $X\overline{U}^{-2}\vec{g} = XX^T\vec{\omega}$. Since X is full rank, XX^T is invertible and we obtain $\vec{\omega} = (XX^T)^{-1} X\overline{U}^{-2}\vec{g}$ which we substitute into A7 to obtain $(\overline{U}^{-2} - X^T(XX^T)^{-1}X\overline{U}^{-2})\vec{g} = \eta\vec{g}$. Thus, the solution to OPT3 must be an eigenvector of the matrix, $M \equiv (\overline{U}^{-2} - X^T(XX^T)^{-1}X\overline{U}^{-2})$. To find out which eigenvector, we left-multiply A7 by $\vec{g}^T$ and use A8 and A9 to obtain $\eta\vec{g}^T\vec{g} = 1$, or $\vec{g}^T\vec{g} = 1/\eta$ where $\eta > 0$. Thus, to solve OPT3 we maximize $1/\eta$ by selecting the smallest positive eigenvalue of M. The direction of the longest axis is then given by the associated eigenvector of M. We then choose the next question such that $\vec{x}_{q+1}$ is most nearly collinear to this eigenvector subject any constraints imposed by the questionnaire design. (For example, in our simulation we require that the elements of $\vec{x}_{q+1}$ be –1, 0, or 1.) The answer to $\vec{x}_{q+1}$ defines a hyperplane orthogonal to $\vec{x}_{q+1}$.

We need only establish that the eigenvalues of M are real. To do this we recognize that $M = P\overline{U}^{-2}$ where $P = (I - X^T(XX^T)^{-1}X)$ is symmetric, i.e., $P=P^T$. Then if η is an eigenvalue of M, $\det(P\overline{U}^{-2} - \eta I) = 0$, which implies that $\det[\overline{U}(\overline{U}^{-1}P\overline{U}^{-1} - \eta I)\overline{U}^{-1}] = 0$. This implies that η is an eigenvalue of $\overline{U}^{-1}P\overline{U}^{-1}$, which is symmetric. Thus, η is real (Hadley 1961, 240).

Adjusting the Polyhedron so that it is non-Empty

$\mathscr{P}$ will remain non-empty as long as respondents' answers are consistent. However, in any real situation there is likely to be $q < p$ such that $\mathscr{P}$ is empty. To continue the polyhedral algorithm, we adjust $\mathscr{P}$ so that it is non-empty. We do this by replacing the equality constraint, $X\vec{u} = \vec{a}$, with two inequality constraints, $X\vec{u} \le \vec{a} + \vec{\delta}$ and $X\vec{u} \ge \vec{a} - \vec{\delta}$, where $\vec{\delta}$ is a $q\times 1$ vector of errors, δ_i, defined only for the question-answer imposed constraints. We solve the following optimization problem. Our current implementation uses the ∞-norm where we minimize the maximum δ_i, but other norms are possible. The advantage of using the ∞-norm is that (OPT 4) is solvable as a linear program.

(OPT4) $\min \left\|\vec{\delta}\right\|$ subject to: $X\vec{u} \le \vec{a} + \vec{\delta}, \quad X\vec{u} \ge \vec{a} - \vec{\delta}, \quad \vec{u} \ge \vec{0}$, At some point such that $q > p$, extant algorithms will outperform OPT4 and we can switch to those algorithms. Alternatively, a researcher might choose to switch to constrained regression (norm-2) or mean-absolute error (norm-1) when $q > p$. Other options include replacing some, but not all, of the equality constraints with inequality constraints. We leave these extensions to future research.

Modifications for Stated-Choice Data (Adaptive Choice-Based Conjoint Analysis)

Redefine $\vec{z}_{ij}$ as the jth profile in the ith choice set where j=1 to 4. Retain the notation of r externally imposed constraints of which r' are inequality constraints. Without loss of generality, define the index of the profile chosen by the respondent as 1, then the respondent's choice implies three inequality constraints: $(\vec{z}_{i1} - \vec{z}_{i2})\vec{u} \geq 0$, $(\vec{z}_{i1} - \vec{z}_{i3})\vec{u} \geq 0$, $(\vec{z}_{i1} - \vec{z}_{i4})\vec{u} \geq 0$. For every stated-choice question we gain three inequality constraints and need to add three slack variables, thus redefine the augmented $\vec{u}$ to be a $(p+3q+r')$x1 vector, the augmented X to be a $(3q+r)$x$(p+3q+r')$ matrix, and $\vec{a}$ to be a $(3q+r)$x1 vector of zeros. We proceed as before to find an interior point of $\mathscr{P}$, find its analytic center, approximate it with an ellipsoid, and find the two longest axes. For the choice-based algorithm, we find it convenient to constraint the partworths so that they sum to 100. This does not constrain the conjoint problem since it is only the relative partworths that matter in choice. Let $\vec{g}_1$ and $\vec{g}_2$ be these longest axes and, as before, let $\vec{u}'$ be the analytic center.

To find the extreme estimates of the parameters, $\vec{u}_{ij}$, we solve for the points where $\vec{u}_{ij} = \vec{u}' + \alpha_1\vec{g}_1$, $\vec{u}_{ij} = \vec{u}' - \alpha_2\vec{g}_1$, $\vec{u}_{ij} = \vec{u}' + \alpha_3\vec{g}_2$, and $\vec{u}_{ij} = \vec{u}' - \alpha_4\vec{g}_2$ intersect $\mathscr{P}$. For each α we do this by increasing α until the first constraint in $\mathscr{P}$ is violated. To find the profiles in the choice set we select, as researcher determined parameters, feature costs, $\vec{c}$, and a budget, M. Without such constraints, the best profile is trivially the profile will all features set to their high levels. Subject to this budget constraint, we solve the following knapsack problem with dynamic programming.

(OPT5) $\quad \max \ \vec{z}_{ij}\vec{u}_{ij} \quad$ subject to: $\quad \vec{z}_{ij}\vec{c} \leq M$, elements of $\vec{z}_{ij} \in \{0,1\}$

In the algorithms we have implemented to date, we set $\vec{c} = \vec{u}'$ and draw M from a uniform distribution on [0, 50], redrawing M (up to thirty times) until all four profiles are distinct. If distinct profiles cannot be identified, then it is likely that $\mathscr{P}$ has shrunk sufficiently for the managerial problem. To extend the algorithm to an even number of profiles ($2n$), select the n longest axes. For an odd number of profiles simply drop the last selected profile. For null profiles, extend the constraints accordingly.