The Heat Kernel for Manifolds with Conic Singularities

by

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Abstract

For N a compact manifold without boundary and $g(r)$ a smooth family of metrics on N, let $C(N)$ be the manifold with boundary

$$C(N) = N \times [0, \infty) \ni (x, r)$$

with metric $dr^2 + r^2 g(r)$. A manifold with conic singularities is a metric space which is a smooth Riemannian manifold outside of a subset of isolated singularities, each isomorphic in some neighborhood to a metric cone of type $C(N)$. The Laplacian on k-forms on $C(N)$ can be written as

$$H_{A(r)} = -\partial_r^2 + \frac{A(r)}{r^2}$$

with $A(r)$ a smooth family of second order operators on the cross-sectional manifold N. For certain operators $A(r)$ I show that the Laplacian on $C(N)$ is not essentially self-adjoint and that the self-adjoint extensions of $H_{A(r)}$ are parameterized by the Lagrangian Grassmannian of a symplectic vector space. I define a manifold with boundary on which the heat kernels for these extensions are smooth functions conormal to the boundary faces, and give a complete asymptotic expansion at the boundary faces. For $\dim N = 0$ or $A(r) \equiv A$ constant in r, I show that the heat kernels for the various self-adjoint extensions can be written in terms of the heat kernel for the Friedrichs extension of the operator, and when $\dim N = 0$ and with $A(r) \equiv \kappa > -1/4$ a constant, κ and a constant parameterizing the self-adjoint extension can be recovered from the coefficients in expansion of the trace of the heat kernel at $t = 0$.

Thesis Supervisor: Richard Melrose

Title: Professor of Mathematics

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Contents

Introduction

This thesis is concerned with the study of the heat kernels of operators defined on manifolds with conic singularities. These are metric spaces which have a finite singular set with dense compliment equal to a smooth Riemannian manifold, and with each singular point having a neighborhood isometric to a metric cone.

The metric cone $C(N)$ for N a compact manifold without boundary is defined to be the manifold with boundary $C(N) = N \times [0, \infty) \ni (x, r)$ with metric

$$dr^2 + r^2 g(r)$$

for $g(r)$ a smooth family of metrics on N. I describe the heat kernels for the Laplacian Δ_g on $C(N)$ with the cross-sectional metric $g(r)$ constant in r when $\dim N > 0$. The Laplacian Δ_g is defined on the space of smooth compactly supported forms vanishing to infinite order at the boundary $N \times \{0\}$. The heat kernel $E(r, r', x, x', t)$ for a self-adjoint extension of the Laplacian on $C(N)$ is a distribution on the manifold with corners

$$X = [0, \infty)^2 \times N \times N \times [0, \infty) \ni (r, r', x, x', t)$$

satisfying

$$(\partial_t - \Delta_g)E(r, r', x, x', t) = 0, \qquad E(r, r, x, x, 0) = \delta_{(r,x)}(r', x').$$

The heat space $\tilde{X}$ for $C(N)$ will be a manifold with corners constructed from X by a parabolic blow-up, first along the submanifold $(0, 0, x, x, 0)$ in X corresponding to the singularity in the operator, and then along the diagonal

$$\Delta_0 = \{(r, r.x, x, 0) \in X; r \in \mathbb{R}^+, x \in N\}$$

corresponding to the singularity in the initial data. I show that $E(r, r', x, x', t)$ is a function conormal to the boundary faces of the heat space $\tilde{X}$ and classify the terms in the asymptotic expansions at the boundary faces.

Denote by $\Lambda^k(N)$ the space of k-forms on N. Under the rescaling

$$(\Lambda^k(N), \Lambda^{k-1}(N)) \;\rightarrow\; (r^{k-\frac{n}{2}}\Lambda^k(N), r^{k-\frac{n}{2}-1}\Lambda^{k-1}(N))$$

the Laplacian on k-forms on $C(N)$ becomes an operator of the form

$$H_A = -\partial_r^2 + \frac{A(r)}{r^2}$$

where $A(r)$ is a family of second order elliptic operators on $C^\infty(N;\Lambda^k \oplus \Lambda^{k-1})$ depending smoothly on r [8]. Let $\psi(x)$ be an eigen form for $A(r)$ with eigen value κ, then there is a real valued function $\nu(\kappa)$ such that

$$\psi^+(r,x) = r^{\frac{1}{2}+\nu(\kappa)}\psi(x) \qquad \psi^-(r,x) = \begin{cases} r^{\frac{1}{2}-\nu(\kappa)}\psi(x) & \nu(\kappa) > 0 \\ r^{\frac{1}{2}}\ln(r)\psi(x) & \nu(\kappa) = 0 \end{cases}$$

are generalized homogeneous k-forms for H_A.

If $1 > \nu(\kappa) \geq 0$, $\psi^+(r,x)$ and $\psi^-(r,x)$ are square integrable and H_A is not essentially self-adjoint. This can only occur for form values near the middle dimension $m/2$ of the cone $C(N)$. If there are eigen k-forms for $A(r)$ with $1 > \nu(\kappa_i) \geq 0$ for eigen values $\{\kappa_i\}_{i=1}^p$, the self-adjoint extensions of H_A are fixed by including in the domain a p-dimensional subspace of the vector space spanned by $\{\psi_i^\pm\}_{i=1}^p$. I show that for operators H_A that are not essentially self-adjoint, the p-dimensional subspace that fixes the self-adjoint extension determines the powers and coefficients in the expansion of the heat kernel at the front face on the blown-up space $\tilde{X}$.

The simplest example of an operator of type H_A is when N is zero dimensional and $A(r) = \kappa(r)$ is a compactly supported real valued function on the positive real line,

$$H_\kappa = \partial_r^2 - \kappa(r)r^{-2} \qquad \kappa(0) \geq -1/4.$$

When $\kappa(0) < -1/4$ the operator is no longer semi-bounded. Using separation of variables, the heat kernels for the various self-adjoint extensions of this differential operator are used to construct the heat kernels of self-adjoint extensions of the operator H_A with $\dim N > 0$.

For $3/4 > \kappa(0) \geq -1/4$ there is a two dimensional vector space of generalized harmonic eigen-functions for H_κ spanned by functions $\{\phi_+(r), \phi_-(r)\}$. Inclusion of a vector from this space in the domain of the operator fixes the self-adjoint extension, and these extensions are parameterized by $\Theta \in \mathbb{PR}$ corresponding to the vector $\Theta\phi_+(r) + \phi_-(r)$.

The asymptotics for the heat kernel trace for the Friedrichs extension of the differential operator $H_A = \partial_r - \kappa r^{-2}$ with $\kappa \geq 3/4$ a constant is treated by Callias in [1]. For κ in this range, the operator is essentially self-adjoint, and Callias computes the modified trace

$$tr(e^{-tH_\kappa} - e^{-tH})(0) = \sqrt{\kappa + 1/4}$$

where $H = \partial_r^2$, as well as the asymptotic expansion as $t \downarrow 0$. There results are partially extended in [2] [3] to the the variable coefficient operators of type H_κ where $\kappa(r)$ is a smooth function satisfying certain regularity conditions. I show that for $\Theta \neq +$, the modified trace of the heat kernel $E_\Theta(x, x', t)$, for H_κ a differential operator with κ constant defined on D_Θ, can be written in terms of the modified trace of the Friedrich's heat kernel $E_+(x, x', t)$,

$$tr(E_\Theta - e^{-tH})(t) = tr(E_+ - e^{-tH})(t) + \nu + \Theta\frac{\nu 4^{\nu-\frac{1}{2}}\sqrt{\pi}}{\Gamma(3/2-\nu)}t^\nu + O(t^{2\nu})$$

In [6] [7] [5] Cheeger treats the homogeneous extensions of the Laplacian on cones when the metric $g(r)$ is constant for small r and describes the generalized harmonic eigen forms and the range of values for a given form dimension for which the Laplacian on $C(N)$ is not essentially self-adjoint.

In [8] Brüning and Seeley compute the asymptotics of $tr(e^{-tH_A} - e^{-tH})(t)$ as $t \downarrow 0$ for H_A the Friedrichs extension of the operator on $C(N)$ with $A(r)$ satisfying certain ellipticity and smoothness conditions, and compute the leading term in the contribution to the trace expansion of the singular term for A constant in r.

Our approach differs from the previous ones in that we do not proceed by constructing the resolvant, but instead exploit the homogeneity of the heat kernel for the Friedrichs extension of the constant coefficient operator to get a description of the boundary asymptotics on the blown-up space $\tilde{X}$, then construct a map from solutions to the Friedrich's problem to solutions for other self-adjoint extensions of the operator, and thus give a description of the heat kernels for general self-adjoint boundary conditions.

In section 1 the self-adjoint extensions for the differential operator H_κ on $[0, \infty)$ are described and a basic existence proof is given for solutions to the heat equation for the operator on these domains.

For $1 > \nu \geq 0$ there are at most two domains on which the heat kernel for H_κ is homogeneous of degree -1 under the parabolic scaling

$$\mu_s(r, r', t) = (sr, sr', s^2t), \qquad s \in (0, \infty).$$

which I call the positive and negative extensions. In section 2 the blown-up space $\tilde{X}$ is described and the heat kernels for homogeneous extensions of the model operator with $\kappa \geq -1/4$ a constant are shown to lift to be functions conormal to the boundary faces of $\tilde{X}$. Lifted to $\tilde{X}$, the scaling μ_s becomes a homogeneous dependence on the radial variable in the blow up of the corner $r = r' = t = 0$. This simplifies the description of the homogeneous kernels by separating the singularity in the initial data from the singularity introduced by the perturbed metric.

In section 3 a solution to the forcing problem

$$F(h)(r,t) \sim_{r=0} h(t)\phi_-(r) + O(\phi_+(r))$$
$$(\partial_t - H_\kappa)F(h)(r,t) = 0, \qquad F(h)(r,0) = 0$$

for $h(t)$ a smooth function on $[0,\infty)$ is constructed for H_κ with κ constant using the positive heat kernel, and from this a map is constructed from solutions to the heat equation in the domain of the positive extension to solutions in non-homogeneous domains. This gives a map from the positive heat kernel to the heat kernels for Θ extensions and a description of the asymptotic properties of these extensions lifted to $\tilde{X}$.

In section 4 the maps constructed in section 3 are used along with the resolvant construction of the Friedrichs heat kernel given in [1] to compute the leading two coefficients in the contribution from the singularity to the trace of the heat kernel for Θ boundary conditions, appropriately modified to be a trace class operator. These two invariants are sufficient to determine the extension Θ of the operator and the constant κ.

In sections 5 and 6 an operator calculus on $\tilde{X}$ is used to construct the heat kernel for the self-adjoint extensions of the variable coefficient differential operator H_κ from those of the constant coefficient operator with $\kappa = \kappa(0)$. The construction show that the heat kernels for the variable coefficient operator can also be described as functions conormal to the boundary faces of $\tilde{X}$.

In the last two sections I describe the possible self-adjoint extensions for the operator H_A with A an operator constant in r on a compact manifold without boundary N, with $\dim N > 0$, and show that the heat kernels for these extensions lift to be functions conormal to $\tilde{X}^m$, the heat space for $C(N)$, with an expansion at the front face determined by the choice of extension.

In the published version of this thesis I will extend the results in sections 5 and 6 for the variable coefficient differential operator H_κ to the higher dimensional case covered in sections 7 and 8; I will describe an operator calculus for $\tilde{X}^m$ and construct the heat kernel asymptotics on $\tilde{X}^m$ for the operator H_A with $A(r)$ a family of operators depending smoothly on r and satisfying conditions such as those given in [8] by modifying the heat kernel for the operator H_A with $A = A(0)$ a constant in r.

Let $\kappa(x)$ be a smooth, real valued function compactly supported on the positive real line $\mathbb{R}^+ = [0, \infty)$ with $\kappa(0) \geq -1/4$. The differential operator

$$H_\kappa = -\partial_x^2 + \kappa(x)x^{-2}$$

is symmetric on $\dot{C}_c^\infty(\mathbb{R}^+)$, the space of smooth functions compactly supported on $\mathbb{R}^+$ and vanishing to infinite order at the boundary $x = 0$. Let $\mathcal{L}^2(\mathbb{R}^+)$ be the space of the square integrable functions on the positive real line with inner product $< \cdot, \cdot >$ and norm $\| \cdot \|$, so for functions $f(x)$, $g(x)$ on $\mathbb{R}^+$,

$$< f, g >= \int_0^\infty f(x)g(x)dx, \qquad \|f\|^2 = \int_0^\infty |f(x)|^2 dx$$

Self-adjoint extensions of H_κ in $\mathcal{L}^2(\mathbb{R}^+)$ are determined by the boundary behavior of functions included in the domain.

Definition 1.1. The operator norm $\| \cdot \|_\kappa$ on $L^2(\mathbb{R}^+)$, which I will call the κ-norm, is given by

$$\| \cdot \|_\kappa = \| \cdot \| + \|H_\kappa \cdot \|$$

The domain D_{min} of the minimal extension of the operator H_κ is the closure of $\dot{C}_c^\infty(\mathbb{R}^+)$ under the κ-norm,

$$D_{min} = \{u(x) \in \mathcal{L}^2(\mathbb{R}^+) : \exists \{u_n\} \in \dot{C}_c^\infty(\mathbb{R}^+) \text{ such that } u_n \xrightarrow{\kappa} u\}$$

The domain D_{max} of the maximal extension of H_κ consists of square integrable functions $u(x)$ on $\mathbb{R}^+$ such that $H_\kappa u(x)$ defined as a distribution is also square integrable.

$$D_{max} = \{u(x) \in \mathcal{L}^2(\mathbb{R}^+); < u, H_\kappa v > \ \leq \ C\|v\|_{\mathcal{L}^2} \text{ for all } v(x) \in C_0^\infty(\mathbb{R}^+)\}$$

Proposition 1.1. *H_κ is symmetric on D_{min}.*

PROOF: For $u(x)$, $v(x)$ in $\dot{C}_c^\infty(\mathbb{R}^+)$, $\partial_x^2 u(x)$ and $r^{-2}u(x)$ are in $\mathcal{L}^2(\mathbb{R}^+)$ and integration by parts is valid, so

$$\begin{aligned} < H_\kappa u, v > &= - < \partial_x^2 u, v > + < \kappa \cdot u, v > \\ &= \{u(x)\partial_x v(x) - \partial_x u(x)v(x)\}|_0^\infty + < u, H_\kappa v > \ = \ < u, H_\kappa v > \end{aligned}$$

since functions in $\dot{C}_c^\infty(\mathbb{R}^+)$ are compactly supported and vanish to infinite order at the boundary $x = 0$. For $u(x)$, $v(x)$ in D_{min}, there are sequences of functions $\{u_n\}$, $\{v_n\}$ in $\dot{C}_c^\infty(\mathbb{R}^+)$ such that u_n converges to u and v_n to v in κ-norm.

$$< u_n, H_\kappa v_n >= \frac{1}{2}\{\|u_n + H_\kappa v_n\|^2 - \|u_n\|^2 - \|H_\kappa v_n\|^2\}$$

and the right hand side converges, so

$$< u_n, H_\kappa v_n > \xrightarrow{\mathcal{L}^2} < u, H_\kappa v > \text{ and } < H_\kappa u_n, v_n > \xrightarrow{\mathcal{L}^2} < H_\kappa u, v >$$

therefore $< u, H_\kappa v >=< H_\kappa u, v >$ □

When $\kappa(0) < -1/4$, the Mellin transform can be used to show that H_κ is not semi-bounded. In this thesis only operators with $\kappa(0) \geq -1/4$ will be considered.

Definition 1.2. Let $\phi_\pm(x)$ be compactly supported functions smooth on $(0, \infty)$ such that $H_\kappa \phi_\pm(x) = 0$ for $x \in [0, c)$ for some $c > 0$. For $\nu = \sqrt{\kappa(0) + 1/4}$,

$$\phi_\pm(x) = \begin{cases} x^{\pm\nu+\frac{1}{2}}\omega_\pm(x) & 0 < \nu < 1 \\ x^{\frac{1}{2}}\omega_\pm(x) \ \text{ or } \ x^{\frac{1}{2}}\ln(x)\omega_\pm(x) & \nu = 0 \end{cases}$$

with $\omega_\pm(x)$ smooth compactly supported functions on $[0, \infty)$ and $\omega_\pm(0) = 1$. The domains D_Θ on which H_κ is self-adjoint are defined by

$$D_\Theta = cl\{u_+\phi_+(x) + u_-\phi_-(x) + v(x);\ \ v(x) \in \dot{C}_c^\infty(\mathbb{R}^+),\ \ u_+/u_- = \Theta\}$$
$$D_+ = D_{\pm\infty} = cl\{u_+\phi_+(x) + v(x);\ \ v(x) \in \dot{C}_c^\infty(\mathbb{R}^+)\ \ u_+ \in \mathbb{R}\}$$
$$D_- = D_0 = cl\{u_-\phi_-(x) + v(x);\ \ v(x) \in \dot{C}_c^\infty(\mathbb{R}^+)\ \ u_- \in \mathbb{R}\}$$

where the closure is in the κ-norm. The domains D_+ and D_- will be referred to as the domains of the positive and negative extensions of H_κ, respectively.

Proposition 1.2. *If $u(x)$ is in D_{max} then for $0 \leq \nu < 1$*

$$u(x) = \tilde{u}(x) + u_+\phi_+(x) + u_-\phi_-(x) \qquad \tilde{u}(x) \in D_{min}$$

PROOF: If $u(x)$ is in D_{max} then $H_\kappa u(x) = f(x)$ for some $f(x)$ in $\mathcal{L}^2(\mathbb{R}^+)$, and there is a sequence of functions $\{f_n\}$ in $\dot{C}_c^\infty(\mathbb{R}^+)$ converging to $f(x)$ in $\mathcal{L}^2(\mathbb{R}^+)$ norm. Since H_κ is elliptic away from $x = 0$, we may assume that $f(x)$ and $f_n(x)$ are supported in $[0, c)$ for $c > 0$ a constant independent of n.

Let $\phi(x) = x^{\frac{1}{2}-\nu}\omega(x)$ be a function compactly supported on $[0, c')$ such that for $x \in [0, c)$ with $c < c'$, $H_\kappa \phi = 0$ and $\omega(x) \neq 0$. We construct functions $h_n(x)$ such that

$$(\partial_x^2 - \kappa(x)x^{-2})h_n(x)\phi(x) = f_n(x)$$

for each n and $\tilde{u}_n(x) = h_n(x)\phi(x)$ is in $\dot{C}_c^\infty(\mathbb{R}^+)$ and the sequence converges in the $\mathcal{L}^2(\mathbb{R}^+)$ norm. Set

$$(\partial_x^2 - x^{-2}\kappa(x))h_n(x)\phi(x) = \partial_x^2(h_n)(x)\phi(x) + 2\partial_x(h_n)(x)\partial_x(\phi)(x) = f_n(x)$$

The integrating factor

$$g(x) = \exp\{-2\int_x^\infty \frac{\partial_t\phi(t)}{\phi(t)}dt\}$$

gives the solution

$$\tilde{u}_n(x) = h_n(x)\phi(x) = \phi(x)\int_0^x g^{-1}(y)\int_0^y g(z)\phi^{-1}(z)f_n(z)dzdy$$

Let ω_0 be a bound for $\omega(x)$ and $(\omega)^{-1}(x)$ on $[0, c)$.

$$|\partial_x\phi(x)/\phi(x)| = (\frac{1}{2} - \nu)x^{-1} + O(1)$$

at $x = 0$, so there is a constant $K > 0$ such that

$$g(x) \leq K' \exp\{2\int_x^{c'} \omega_0 t^{-1}dt\} \leq Kx^2$$

and likewise $g^{-1}(x) \leq Kx^{-2}$. Then

$$|\tilde{u}_n(x)| \leq x^{\frac{1}{2}-\nu}\omega_0^2\int_0^x y^{-2}\int_0^y z^{\nu+\frac{3}{2}}|f_n(z)|dzdy \leq x^2\omega_0^2||f_n||_{\mathcal{L}^2} \tag{1.1}$$

for $\tilde{u}_n(x)$ supported in $[0, c')$ and

$$||\tilde{u}_n||_{\mathcal{L}^2} \leq C||f_n||_{\mathcal{L}^2}$$

with $C > 0$ a constant. Since $f_n(x)$ is in $\dot{C}_c^\infty(\mathbb{R}^+)$, $\tilde{u}_n(x)$ vanishes to infinite order at $x = 0$ and $\{\tilde{u}_n\}$ converges in $\mathcal{L}^2(\mathbb{R}^+)$ norm to a function $\tilde{u}(x)$ in D_{min} such that

$$H_\kappa(u(x) - \tilde{u}(x)) = 0$$

near $x = 0$ and their difference is in the space spanned by solutions to H_κ. □

Proposition 1.3. *For $u(x)$ and $v(x)$ in D_{max},*

$$< H_\kappa u, v > - < u, H_\kappa v > \;=\; C_\nu(u_+v_- - u_-v_+) \qquad C_\nu = \begin{cases} 2\nu & 0<\nu<1 \\ 1 & \nu = 0 \end{cases}$$

PROOF: By Prop. 1.2,

$$u(x) = u_+\phi_+(x) + u_-\phi_-(x) + \tilde{u}(x) \qquad v(x) = v_+\phi_+(x) + v_-\phi_-(x) + \tilde{v}(x)$$

with $\tilde{u}(x)$ and $\tilde{v}(x)$ in D_{min}. Assume first that $\tilde{u}(x)$ and $\tilde{v}(x)$ are in $\dot{C}_c^\infty(\mathbb{R}^+)$, so $u(x)$ and $v(x)$ are smooth on $(0,\infty)$. Then

$$\begin{aligned} &< H_\kappa u, v > - < u, H_\kappa v >= \\ &\quad \lim_{\epsilon\to 0}\{\int_\epsilon^\infty (\partial_x^2(v)(x)u(x) - v(x)\partial_x^2(u)(x))dx\} = \lim_{\epsilon\to 0}\{u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon)\} \\ &\qquad = (u_+v_- - u_-v_+)\lim_{\epsilon\to 0}\{\phi_+'(\epsilon)\phi_-(\epsilon) - \phi_+(\epsilon)\phi_-'(\epsilon)\} = (u_+v_- - u_-v_+)C_\nu \end{aligned}$$

since $\lim_{\epsilon\to 0}\tilde{v}'(\epsilon)\phi_\pm(\epsilon) = \lim_{\epsilon\to 0}\tilde{v}(\epsilon)\phi_\pm'(\epsilon) = 0$.

If $\tilde{v}(x) \in D_{min}$ is not in $\dot{C}_c^\infty(\mathbb{R}^+)$ then there is a sequence $\{\tilde{v}_n\}$ in $\dot{C}_c^\infty(\mathbb{R}^+)$ converging to $\tilde{v}(x)$ in κ-norm. By the estimate 1.1, if $H_\kappa\tilde{v}_n(x) = f_n(x)$, then

$$\begin{array}{lll} |v_n(x)\partial_x(\phi_\pm)(x)| \le x^{\pm\nu+3/2}C'\|f_n\|_{\mathcal{L}^2}, & |\phi_\pm(x)\partial_x(v_n)(x)| \le x^{\pm\nu+3/2}C''\|f_n\|_{\mathcal{L}^2} & 0<\nu<1 \\ |v_n(x)\partial_x(\phi_-)(x)| \le x^{\frac{1}{2}}\ln(x)C'\|f_n\|_{\mathcal{L}^2}, & |\phi_-(x)\partial_x(v_n)(x)| \le x^{3/2}\ln(x)C''\|f_n\|_{\mathcal{L}^2} & \nu = 0 \end{array}$$

with $\pm\nu + 3/2 > 1/2$, and the integration by parts remains valid in the limit. □

Theorem 1.4. *For $1 > \nu \ge 0$, H_κ with domain D_Θ is self-adjoint for every Θ in $\mathbb{PR}$. For $\nu \ge 1$, H_κ is essentially self-adjoint on D_+.*

PROOF: By Prop. 1.2, if $u(x)$ is in D_{max} then

$$u(x) = u_+\phi_+(x) + u_-\phi_-(x) + \tilde{u}(x) \qquad \tilde{u}(x) \in D_{min}$$

Since D_{min} is in D_Θ for every Θ, we can assume that u_+ and u_- are not both zero. The theorem is proved by showing that $u(x)$ is not in D_Θ^* unless $u_+/u_- = \Theta$. For $w(x) = w_+\phi_+(x) + w_-\phi_-(x) + \tilde{w}(x)$ in D_Θ,

$$< H_\kappa w, u > - < w, H_\kappa u >= C_\nu(w_+u_- - w_-u_+).$$

The cross term is zero if and only if $w_+/w_= \ne u_+/u_-$. For $\nu \ge 1$, $\Theta = +$ is the only boundary condition in $\mathcal{L}^2_{loc}(\mathbb{R}^+)$, so $D_{min} = D_+ = D_{max}$. □

Lemma 1.5 ([8]). *For $\kappa(x) \equiv \kappa(0) \geq -1/4$, H_κ is non-negative on $\dot{C}_c^\infty(\mathbb{R}^+)$, compactly supported functions on $\mathbb{R}^+$ vanishing to infinite order at $x = 0$.*

PROOF: If $\kappa(0) \geq 0$ this is obvious. If $0 > \kappa(0) \geq -1/4$ then Hardy's inequality for $u(x)$ in $\dot{C}_c^\infty(\mathbb{R}^+)$,

$$\|u(x)/x\| \leq 2\|u'(x)\|$$

shows that

$$< -u'' + \kappa(0)x^{-2}u, u > = \|u'\|^2 + \kappa(0)\|u/x\|^2 \geq \|u'\|^2(1 + 4\kappa(0)) \geq 0$$

Hence the operator is bounded below. □

The following two theorems show the existence and basic properties of the heat kernel for self-adjoint extensions of the operator H_κ. The reader is referred to [12] for proofs.

Theorem 1.6. *(Spectral Theorem) Let A be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$ with domain $D(A)$. There exists a measure space $< M, \mu >$ with μ a finite measure, a unitary operator $U : \mathcal{H} \rightarrow \mathcal{L}^2(M, d\mu)$ and a real valued function $f(\cdot)$ on M which is finite a. e. such that*
a) ψ is in $D(A)$ if and only if $f(\cdot)U\psi(\cdot)$ is in $\mathcal{L}^2(M, d\mu)$.
b) If ϕ is in $U[D(A)]$ then $(UAU^{-1}\phi)(m) = f(m)\phi(m)$.

If A is a bounded operator, the heat operator is defined by convergence in norm of the sum

$$e^{-tA} = \sum_{n=0}^{\infty} \frac{(-t)^n A^n}{n!}.$$

For an operator A semi-bounded from below, the spectral theorem gives a natural way to define the heat operator e^{-tA} and prove many of its basic properties. Given a bounded Borel function $h(x)$ on $\mathbb{R}$, define

$$h(A) = U^{-1}T_{h(f)}U, \quad \text{where} \quad T_{h(f)}\phi = h(f(m))\phi \quad \text{for} \quad \phi \in \mathcal{L}^2(M, d\mu).$$

Theorem 1.7. *If $U(t) = e^{-tA}$ is defined using the functional calculus with A a self-adjoint operator semi-bounded from below, then*
a) $U(t)$ is a unitary operator and $U(t+s) = U(t)U(s)$ for all s, t in $[0, \infty)$.
b)If ϕ is in $\mathcal{H}$, and $t \rightarrow t_0$, then $U(t)\phi \rightarrow U(t_0)\phi$.
c)If ϕ is in $D(A)$, then $t^{-1}(U(t)\phi - \phi) \rightarrow A\phi$ as $t \rightarrow 0$.

d)If $\lim_{t\to 0} t^{-1}(U(t)\phi - \phi)$ *exists, then* ϕ *is in* $D(A)$.
e)If D *is core for* A, *then for all* t, $e^{tA} : D \to D$

This gives the existence of heat kernels for self-adjoint extensions of H_κ as well as some of their basic properties.

Theorem 1.8. *For each* $\phi(x)$ *in* D_Θ, *the domain of the self-adjoint extension* H_Θ *of* H_κ, *there is a unique function* $e^{-tH_\Theta}\phi(x) = u(x,t)$ *in* $\mathcal{L}^2(\mathbb{R}^+ \times \mathbb{R}^+)$, *smooth away from the boundary* $x = 0$ *and such that* $u(x,t)$ *is in* D_Θ *for fixed* t *in* $[0,\infty)$ *and is a solution to the heat equation*

$$(\partial_t - \partial_x^2 + \kappa(x)x^{-2})u(x,t) = 0, \qquad u(x,0) = \psi(x)$$

For fixed t, $e^{-tH_\Theta} : D_\Theta \to D_\Theta$ *and forms a semi-group of operators with Schwartz kernel* $E_\Theta(x,x',t)$ *in* $\mathcal{S}'(\mathcal{X})$, *where* $X = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \ni (x,x',t)$, *satisfying*

$$(\partial_t - \partial_x^2 + \kappa(x)x^{-2})E_\Theta(x,x',t) = 0, \qquad E_\Theta(x,x',0) = \delta(x-x')$$

$E_\Theta(x,x',t)$ *smooth in the interior and symmetric in* x *and* x'. *For* $t > 0$, $E_\Theta(x,x',t)$ *is in* D_Θ *in* x *and* x' *up to the edge* $x = x' = 0$.

PROOF: The existence of the semi-group e^{-tH_Θ} and the existence, uniqueness and symmetry in x and x' of the heat kernels $E_\Theta(x,x',t)$ follows from the spectral theorem. By the semi-group properties of the heat kernel,

$$e^{-tH_\Theta}(x,x') = \int_0^\infty e^{-\frac{t}{2}H_\Theta}(x,z)e^{-\frac{t}{2}H_\Theta}(z,x')dz.$$

Both $e^{-\frac{t}{2}H_\Theta}(x,z)$ and $e^{-\frac{t}{2}H_\Theta}(z,x')$ are in D_Θ in x and x', respectively, so for fixed $t > 0$, $e^{-tH_\Theta}(x,x')$ is in D_Θ in x and x' up to the edge $x = x' = 0$. □

Definition 1.3. Let μ_s be the 1-parameter group of $\mathbb{R}^+$ parabolic scalings on X

$$\mu_s : X \to X, \quad \mu_s(x,x',t) = (sx, sx', s^2t) \text{ for } s \in (0,\infty)$$

Proposition 1.9. *Let* $\kappa(x) \equiv \kappa > -1/4$ *be a constant. Then* $E_+(x,x',t)$ *and* $E_-(x,x',t)$, *the heat kernels for the positive and negative domains* D_+ *and* D_-, *are homogeneous of degree* -1 *under the action of* μ_s, *meaning*

$$\mu_s^* E_\pm(x,x',t) = E_\pm(sx,sx',s^2t) = s^{-1}E_\pm(x,x',t).$$

The heat kernels for the operator on the the domains D_Θ with $\Theta \neq \pm$ are not homogeneous. When $\kappa(0) = -1/4$, only $E_+(x, x', t)$ is homogeneous under the action of μ_s.

PROOF: The homogeneity of the heat kernel comes from the homogeneity of the operator and of the domain on which it is defined. Let $\psi(x)$ be a function in D_Θ. Since $\dot{C}_c^\infty(\mathbb{R}^+)$ is invariant under $\mathbb{R}^+$ scaling, assume that

$$\psi(x) = \tilde{\omega}_+(x)x^{\nu+\frac{1}{2}} + \tilde{\omega}_-(x)x^{-\nu+\frac{1}{2}} \text{ such that } \tilde{\omega}_+(0)/\tilde{\omega}_-(0) = \Theta.$$

Under the parabolic scaling

$$\mu_s^*\psi(x) = \tilde{\omega}_+(sx)(sx)^{\nu+\frac{1}{2}} + \tilde{\omega}_-(sx)(sx)^{-\nu+\frac{1}{2}}$$

$$\tilde{\omega}_+(0)s^{\nu+\frac{1}{2}}/\tilde{\omega}_-(0)s^{-\nu+\frac{1}{2}} = \Theta s^{2\nu}$$

so D_Θ is $\mathbb{R}^+$ invariant if and only if $\Theta = \pm$. For $\kappa(0) = -1/4$, the negative solution contains a log term, so

$$\psi(x) = \tilde{\omega}_+x^{\frac{1}{2}} + \tilde{\omega}_-x^{\frac{1}{2}}\ln x$$

$$\mu_s^*\psi(x) = \tilde{\omega}_+(sx)^{\frac{1}{2}} + \tilde{\omega}_-(sx)(sx)^{\frac{1}{2}}\ln sx = s^{\frac{1}{2}}\{\tilde{\omega}_+(sx)x^{\frac{1}{2}}\ln s + \tilde{\omega}_-(sx)x^{\frac{1}{2}}\ln x\}$$

$\tilde{\omega}_+(0)\ln s/\tilde{\omega}_-(0) = \Theta \ln s$ and the domain is only $\mathbb{R}^+$ invariant if $\tilde{\omega}_-(0) = 0$.

If $\psi(x)$ is in any $\mathbb{R}^+$ invariant domain D of a self-adjoint extension H of H_κ,

$$u(x,t) = e^{-tH}\psi(x), \qquad \mu_s^*u(x,t) = e^{-s^2tH}\psi(sx)$$

so $\mu_s^*u(x,t)$ is a solution to the heat equation smooth away from $x = 0$ and contained in D for fixed t, with initial data $\mu_x^*\psi(x)$. If $E(x, x', t)$ is the Schwartz kernel for H then

$$\mu_s^*u(x,t) = \int_0^\infty E(sx, x', s^2t)\psi(x')dx' = s\int_0^\infty \mu_s^*(E)(x, x', t)\mu_s^*(\psi)(x')dx'$$

Since the domain D is $\mathbb{R}^+$ invariant,

$$\mu_s^*u(x,t) = \int_0^\infty E(x, x', t)\mu_s^*(\psi)(x')dx'$$

and $\mu_s^*(E)(x, x', t) = s^{-1}E(x, x', t)$. $\square$

2. The heat kernel for the model operator in one dimension with homogeneous boundary conditions

Let $E_+(x,x',t)$ be the positive heat kernel for the model operator

$$H_\kappa = -\partial_x^2 + \kappa x^{-2} \qquad \text{with} \qquad \kappa \geq -1/4$$

with κ a constant and domain D_+ defined by 1.2. The short time asymptotics of $E_+(x,x',t)$ can be described by defining a manifold with corners $\tilde{X}$ to which the heat kernel lifts to be a polyhomogeneous function conormal to the boundary faces. The proof exploits the symmetry of the heat kernel for self-adjoint operators and the invariance under the action of μ_s defined by 1.3 of the positive heat kernel. For $1 > \nu > 0$, the construction of $E_-(x,x',t)$, the negative heat kernel for the domain D_-, is completely analogous to the construction of $E_+(x,x',t)$ and a detailed description will be given only for the latter. When $\nu = 0$, the negative heat kernel is no longer invariant under μ_s and this method has to be modified.

The heat space $X = [0,\infty)^3 \ni (x,x',t)$ will be blown up, first parabolically at the origin $\{0\} = \{(0,0,0) \in X\}$, then parabolically along the diagonal $\Delta_0 = \{(x,x,0) \in X\}$ producing a manifold with corners with five boundary faces. The reader is referred to [15] for a detailed description of the blow-up construction in the general case. Let

$$Q_p = \{\phi = (\phi_0, \phi_\prime, \phi') \in \mathbb{R}^3 : \phi_0, \phi_\prime, \phi' \geq 0,\ \phi_0^2 + \phi_\prime^4 + (\phi')^4 = 1\} \tag{2.1}$$

$$X_Q = [X \setminus \{0\}] \sqcup [Q_p] = [X; \{0\}, sp(dt)] \tag{2.2}$$

X_Q is the parabolic blow-up of the origin in X in the direction of $sp\{dt\} \subset N^*(\{0\})$, the span of dt in the conormal bundle to the origin in X, producing a fourth boundary face isomorphic to the parabolic octant Q_p. X_Q is supplied with a smooth structure by identifying points of Q_p with equivalence classes of parabolic curves $\chi(r)$ in X;

$$\chi : [0,1] \to X, \quad \chi(0) = \{0\} \subset X \quad \text{and} \quad \chi^*(t)(r) = O(r^2) \quad \text{as} \quad r \downarrow 0$$
$$\text{so} \qquad \chi(r) = (rx(r), rx'(r), r^2 t(r))$$

for $x(r)$, $x'(r)$ and $t(r)$ smooth functions on $[0,\infty)$, with the equivalence relation

$$\chi_1(r) \sim \chi_2(r) \quad \Leftrightarrow \quad \begin{cases} \chi_1^* t(r) - \chi_2^* t(r) = O(r^3) \\ \chi_1^* f(r) - \chi_2^* f(r) = O(r^2) \quad \forall\, f \in C^\infty(X) \end{cases}$$
$$\chi(sr) \sim \chi(r) \qquad s \in (0,\infty)$$

These conditions imply that

$$\chi_1(r) \sim \chi_2(r) \ \text{ if } \ (x_1(0), x_1'(0), t_1(0)) = \mu_s(x_2(0), x_2'(0), t_2(0))$$

for some $s > 0$. The equivalence relation defines the t−parabolic co-sphere bundle to $\{0\}$. For the blow-up of the diagonal in X_Q, let

$$S_p = \{\omega = (\omega_0, \omega') \in \mathbb{R}^2 : \omega_0 \geq 0,\ \omega_0^2 + (\omega')^4 = 1\}$$
$$\tilde{X} = [X_Q \setminus \tilde{\Delta}_0] \sqcup [S_p \times \mathbb{R}^+] = [X_Q; \tilde{\Delta}_0, sp(dt)]$$

where $\tilde{\Delta}_0$ is the closure of the set $\Delta_0 \setminus \{0\} \subset X$ lifted to to X_Q,

$$\tilde{\Delta}_0 = cl\{\beta^{-1}(\Delta_0 \setminus \{0\})\} \subset X_Q.$$

This intersects the front (blown-up) face of X_Q at the point $\{\phi' = \phi_{\prime}, \phi_0 = 0\}$ on Q_p. The fifth boundary face in $\tilde{X}$ produced by the blow-up of $\tilde{\Delta}_0$ is isomorphic to $S_p \times \mathbb{R}^+$, the positive real line crossed with the parabolic half-circle. As before, a smooth structure is given to $\tilde{X}$ by identifying points of $S_p \times \mathbb{R}^+$ with equivalence classes of parabolic curves in X_Q that define the t-parabolic co-sphere bundle of $S_p \times \mathbb{R}^+$.

Definition 2.1. $\qquad \tilde{X} = [\,X\,;\ \{0\}, dt\,; \Delta_0, dt\,]$

Denote by $\mathfrak{R}$ (right face) and $\mathfrak{L}$ (left face) the boundary faces of $\tilde{X}$ corresponding the lifts of $x = 0$ and $x' = 0$, $\mathfrak{O} \sim Q_p$ (origin face) the lift of the origin, $\mathfrak{D} \sim S_p \times \mathbb{R}^+$ (diagonal face) the lift of Δ_0 and $\mathfrak{T}$ (temporal face) the lift of $t = 0$ away from Δ_0, Denote by $\rho_{\mathfrak{F}}$ a boundary defining function corresponding to the face $\mathfrak{F}$.

There is a blow-down map $\beta : \tilde{X} \to X$ which gives an isomorphism on the interiors of $\tilde{X}$ and X and satisfies

$$\beta^*(x) = \rho_{\mathfrak{R}} \cdot \rho_{\mathfrak{O}}, \qquad \beta^*(x') = \rho_{\mathfrak{L}} \cdot \rho_{\mathfrak{O}}, \qquad \beta^*(t) = \rho_{\mathfrak{T}} \cdot \rho_{\mathfrak{D}}^2 \cdot \rho_{\mathfrak{O}}^2 \tag{2.3}$$

Lemma 2.1. *The scaling function μ_s on X pulls back to $\tilde{X}$ under β as a scaling in the defining function $\rho_{\mathfrak{O}}$ for the front face $\mathfrak{O}$.*

Proof: This follows from the action of β on the boundary defining functions for X. □

Definition 2.2. $\Phi^k(X)$ for k in $\mathbb{N}$ is the space of distributions $R(x,x',t)$ on X such that $\beta^*(R)(\tilde{X})$ is a function conormal to the boundary faces of $\tilde{X}$, has leading term at the front face $\mathfrak{O}$ of order $k-3$, vanishes to infinite order at $\mathfrak{T}$, has an integer expansion with leading term -1 at $\mathfrak{D}$.

Let $\dot{\Phi}^k(X)$ be the space of distributions $R(x,x',t)$ in $\Phi^k(X)$ such that $\beta^*(R)(\tilde{X})$ vanishes to infinite order at $\mathfrak{D}$. Let $\Phi^k_{phg}(X) \subset \Phi^k(X)$ and $\dot{\Phi}^k_{phg}(X) \subset \dot{\Phi}^k(X)$ be distributions that lift to $\tilde{X}$ to have classical polyhomogeneous conormal expansions at the boundary faces.

Theorem 2.2. *For $\kappa \geq -1/4$, the heat kernel $E_+(x,x',t)$ for the positive extension of H_κ has a boundary expansion*

$$\beta^*(E_+)(\tilde{X}) = \rho_{\mathfrak{D}}^{-1}\rho_{\mathfrak{O}}^{-1}\rho_{\mathfrak{T}}^{\infty}(\rho_{\mathfrak{R}}\rho_{\mathfrak{L}})^{\nu+\frac{1}{2}} \times \mathrm{C}^\infty(Q_p) \in \Phi^2_{phg}(X)$$

homogeneous of degree -1 in $\rho_{\mathfrak{O}}$

PROOF: The interiors of X and $\tilde{X}$ are isomorphic under β, so $\beta^*(E_+)(\tilde{X})$ is smooth in the interior of $\tilde{X}$ by Theorem 1.8. By Prop. 1.9, the positive heat kernel is homogeneous of degree -1 under the scaling μ_s for $s>0$, so $\beta^*(E_+)(\tilde{X})$ is homogeneous in $\rho_{\mathfrak{O}}$ of degree -1 and its properties are completely determined by restriction to a parabolic octant Q_p away from the origin $\{0\} = \{(0,0,0) \in X\}$.

From Theorem 1.8, the expansion of $E_+(X)$ at the faces $x=0$ and $x'=0$ for $t>0$ is given by inclusion in the domain D_+, so

$$E_+|_{t>0} = (xx')^{\nu+\frac{1}{2}} \times C^\infty(X).$$

The body of the proof consists of describing the asymptotic properties of the lift to $\tilde{X}$ of E_+ at $t=0$.

Definition 2.3. There is a map $\alpha : X \to X$ that is an isomorphism on $X|_{x'>\epsilon}$ for $\epsilon > 0$, and satisfying

$$\alpha(x) = yr, \qquad \alpha(x') = r, \qquad \alpha(t) = sr^2$$

where (y,s) are local coordinates for $x' > \epsilon > 0$ on a parabolic octant in X away from the origin $\{0\}$ and r is a radial variable.

With these coordinates,

$$\alpha^*(\partial_t + H_\kappa) = \alpha^*(\partial_t - \partial_x^2 + \kappa x^{-2}) = r^{-2}\{\partial_s - \partial_y^2 + \kappa y^{-2}\}$$
$$(\partial_s - \partial_y^2 + \kappa y^{-2})\alpha^*(E_+)(y, s, r) = 0$$
$$\alpha^*(E_+)(y, r, 0) = \delta(r(y-1)) = r^{-1}\delta(y-1).$$

Let $\psi(x) \in \mathrm{C}^\infty(\mathbb{R}^+)$ be a cut off function vanishing near $x = 0$ such that

$$\psi(x) \equiv 1 \quad \text{for} \quad x > \epsilon > 0$$

and let $E_\psi(x, x', t)$ on X be the heat kernel for the Friedrichs extension of

$$H_\psi = -\partial_x^2 + \kappa x^{-2}\psi(x)$$

defined on $\dot{C}_c^\infty(\mathbb{R}^+)$. Let $\Omega_{\mathfrak{D}}$ in $\tilde{X}$ be the lift under β of a neighborhood of the diagonal Δ_0 away from the boundary faces of $\tilde{X}$. Since H_ψ is the Laplacian plus a smooth perturbation, $\beta^*(E_\psi)(\tilde{X})$ restricted to $\Omega_{\mathfrak{D}}$ is a polyhomogeneous conormal function with an asymptotic expansion at the diagonal face $\mathfrak{D}$ with leading power -1, and vanishing to infinite order at the temporal face $\mathfrak{T}$ [14].

In the neighborhood $\Omega_{\mathfrak{D}}$ in $\tilde{X}$ there is a map

$$\beta_\alpha = \alpha^{-1} \circ \beta : \Omega_{\mathfrak{D}} \to \beta(\Omega_{\mathfrak{D}}) \subset X$$

giving an isomorphism on the interiors of these spaces and satisfying

$$\beta_\alpha^*(r) = \rho_{\mathfrak{D}}, \qquad \beta_\alpha^*(s) = \rho_{\mathfrak{T}}\rho_{\mathfrak{D}}^2.$$

Let $F(y, s) = E_\psi(y, 1, s)$ on $\mathbb{R}^+ \times \mathbb{R}^+$ be the restriction of the positive heat kernel for H_ψ to the submanifold $y' = 1$, then

$$\beta_\alpha^*(F)(\rho_{\mathfrak{T}}, \rho_{\mathfrak{D}}) = \rho_{\mathfrak{D}}^{-1}\rho_{\mathfrak{T}}^\infty \times \mathrm{C}^\infty(\Omega_{\mathfrak{D}}), \qquad F(y, 0) = \delta(y-1).$$

Let $\phi(y)$ in $\mathrm{C}^\infty(\mathbb{R}^+)$ be a cut-off function supported near $y = 0$, such that

$$\phi(y) \equiv 1 \quad \text{for} \quad y > \epsilon', \qquad \phi(y) \equiv 0 \quad \text{for} \quad 1 \gg \epsilon' > \epsilon > y$$

$\partial_y\phi(y)$ is supported in the interval (ϵ, ϵ'). Define

$$J(y, s) = (\partial_s - \partial_y^2 + \kappa y^{-2})\phi(y)F(y, s) =$$
$$- (\partial_y^2\phi(y) - \partial_y\phi(y))F(y, s) + \phi(y)(1 - \psi(y))y^{-2}\kappa F(y, s)$$

$J(y,s)$ is supported in (ϵ,ϵ') and is smooth and vanishes to infinite order at $s=0$, since the distribution $F(y,s)$ vanishes to infinite order at $s=0$ away from the singular submanifold $y=1$.

Let $E_+(y,y',s)$ be the positive heat kernel for H_κ on X with coordinates (y,y',s). By Theorem 1.8, $E_+(y,y',s)$ exists and is in D_+ in y for $s>0$. Define

$$H(y,s)=\int_0^s\int_0^\infty E_+(y,y',s-s')J(y',s')dy'ds'$$

then by Duhamel's principle

$$(\partial_s-\partial_y^2+\kappa y^{-2})H(y,s)=J(y,s).$$

Since $J(y,s)$ is smooth and compactly supported away from $y=0$, the y' integral is well-defined. $J(y,s)$ vanishes to infinite order at $s=0$ and convolution is smoothing, so $H(y,s)$ is smooth up to $s=0$ and vanishes to infinite order there. Since $E_+(y,y',s)$ is in D_+ in y, $H(y,s)$ is in D_+ up to $s=0$. Define

$$E(y,s)=\phi(y)F(y,s)-H(y,s).$$

By construction, $E(y,s)$ is in D_+ in y and

$$(\partial_s-\partial_y^2+\kappa y^{-2})E(y,s)=0 \qquad E(y,0)=\delta(y-1)$$

and vanishes to infinite order at $s=0$ away from $y=1$. Lifting $E(y,s)$ to the blown-up manifold $\tilde{X}$ bounded away from the faces $\mathfrak{L}$ and $\mathfrak{D}$,

$$\beta_S^*(E)(\tilde{X})=\rho_{\mathfrak{D}}^{-1}\rho_{\mathfrak{T}}^{\infty}\rho_{\mathfrak{R}}^{\nu+\frac{1}{2}}\times \mathrm{C}^\infty(\tilde{X}).$$

$r^{-1}E(y,s)$ satisfies the heat equation for $\alpha^*(H_\kappa)$ and has the same initial data at $t=0$ and is in the same domain in y as the lift under α of the positive heat kernel $E_+(x,x',t)$ restricted to $x'>\epsilon>0$, so by uniqueness of solutions to the heat equation

$$\alpha^*(E_+|_{x'>\epsilon})(y,s,r)=r^{-1}E(y,s)$$

and in a neighborhood of the diagonal in X which lifts under β to a neighborhood in $\tilde{X}$ bounded away from $\mathfrak{R}$ and $\mathfrak{L}$,

$$\beta^*(E_+)(\rho_{\mathfrak{D}},\rho_{\mathfrak{T}},\rho_{\mathfrak{D}})=\rho_{\mathfrak{D}}^{-1}\rho_{\mathfrak{D}}^{-1}\rho_{\mathfrak{T}}^{\infty}\times \mathrm{C}^\infty(\rho_{\mathfrak{D}},\rho_{\mathfrak{T}}).$$

In a neighborhood of the face $x=0$ in X which lifts under β to a neighborhood in $\tilde{X}$ bounded away from $\mathfrak{D}$ and $\mathfrak{L}$,

$$\beta^*(E_+|_{\Lambda_{x'}})(\rho_{\mathfrak{R}},\rho_{\mathfrak{T}},\rho_{\mathfrak{D}})=\rho_{\mathfrak{D}}^{-1}\rho_{\mathfrak{R}}^{\nu+\frac{1}{2}}\rho_{\mathfrak{T}}^{\infty}\times \mathrm{C}^\infty(\rho_{\mathfrak{R}},\rho_{\mathfrak{T}})$$

proving the theorem by symmetry. □

Theorem 2.3. *For $0 < \nu < 1$, the heat kernel $E_-(x,x',t)$ for the negative extension of H_κ has a boundary expansion*

$$\beta^*(E_-)(\tilde{X}) = \rho_{\mathfrak{D}}^{-1}\rho_{\mathfrak{D}}^{-1}\rho_{\mathfrak{T}}^{\infty}(\rho_{\mathfrak{R}},\rho_{\mathfrak{L}})^{-\nu+\frac{1}{2}} \times C^\infty(Q_p) \in \Phi^2_{phf}(X)$$

homogeneous of degree -1 in $\rho_{\mathfrak{D}}$.

The proof is completely analogous to the proof for the lift of the positive heat kernel. When $\kappa = -1/4$ the non-homogeneity of the boundary data causes the boundary expansions to be slightly more complicated. This case will be dealt with in the next section.

3. The heat kernel for the model operator with general self-adjoint boundary conditions

Theorem 3.1. *Denote by $E_\Theta(x,x',t)$ the heat kernel for the constant coefficient operator H_κ defined on the domain D_Θ for $\Theta \neq \pm$. For $0 \leq \nu < 1$ and $\Theta \neq +$ the heat kernel $E_\Theta(x,x',t)$ for self-adjoint extension of the operator H_κ with domain D_Θ defined by 1.2 is in D_Θ in x and x' and for any N in $\mathbb{N}$ has an expansion when $1 > \nu > 0$*

$$E_\Theta(X) = E_-(X) + \sum_{j=0}^{N} E^j_\Theta(X) + R^{N+1}_\Theta(X) \in \Phi^2_{phg}(X)$$

$$\textit{with} \qquad E^j_\Theta(X) \in \dot{\Phi}^{2\nu j+2}_{phg}(X) \qquad R^N_\Theta(X) \in \dot{\Phi}^{2\nu N+2}_{phg}(X)$$

and $E^j_\Theta(X)$ homogeneous in $\rho_{\mathfrak{D}}$ of degree $2\nu j - 1$. When $\nu = 0$

$$E_\Theta(X) = E_+(X) + R_\Theta(X) \in \Phi^2(X)$$

$$\textit{with} \qquad R_\Theta(X) \in \dot{\Phi}^2(X)$$

and $\mu^*_s(R_\Theta)(X) = s^{-1}R_\Theta(X) + O(s^{-1})$.

For $E_\Theta(x,x',t)$ not invariant under the action of μ_s, the boundary expansion of $\beta^*(E_\Theta)(\tilde{X})$ cannot be computed by the same methods as those used for $\beta^*(E_+)(\tilde{X})$, the heat kernels for the positive extension. The positive heat kernel for H_κ will be used to construct the solution $w(x,t)$ to the heat kernel for H_κ with domain D_Θ and initial data $\phi(x)$ in $\dot{C}^\infty_c(\mathbb{R}^+)$ from the positive solution $u(x,t)$ with the same initial data $\phi(x)$. By closure under the $\mathcal{L}^2(\mathbb{R}^+)$ norm, this construction gives solutions to the heat equation for general initial data in the domain D_Θ.

Lemma 3.2. *Let $\{\phi_n\}$ be a sequence of functions in $\dot{C}^\infty_c(\mathbb{R}^+)$ such that $\phi_n(x)$ converges to $\phi(x)$ in D_Θ in the $\mathcal{L}^2(\mathbb{R}^+)$ norm. If $E(x,x',t)$ is a distribution on X such that $E(\phi_n)(x,t)$ is a smooth function in D_Θ for fixed t and*

$$(\partial_t + H_\kappa)E(\phi_n)(x,t) = 0, \qquad E(\phi_n)(x,0) = \phi_n(x)$$

then $E(\phi_n)(x,t)$ converges in $\mathcal{L}^2(\mathbb{R}^+)$ to a smooth function $v(x,t)$ in D_Θ for fixed t such that

$$(\partial_t + H_\kappa)v(x,t) = 0 \;\; \textit{and} \;\; v(x,0) = \phi(x).$$

PROOF: This follows from the Spectral Theorem 1.6. □

Definition 3.1. For $0 < \nu < 1$, the leading functional coefficients at the face $x = 0$ of the positive and negative heat kernels are

$$NE_+(x', t) = \{x^{-\nu-\frac{1}{2}} E_+(x, x', t)\}|_{x=0}$$

$$NE_-(x', t) = \begin{cases} \{x^{\nu-\frac{1}{2}} E_-(x, x', t)\}|_{x=0} & 0 < \nu < 1 \\ \{x^{-\frac{1}{2}} / \ln(x) E_-(x, x', t)\}|_{x=0} & \nu = 0 \end{cases}$$

Definition 3.2. Define the operators N_+ on D_+ and N_- on D_- by

$$N_+(\phi_+)(t) = \int_0^\infty NE_+(x', t)\phi_+(x')dx', \qquad \phi_+(x) \in D_+$$

$$N_-(\phi_-)(t) = \int_0^\infty NE_-(x', t)\phi_-(x')dx', \qquad \phi_-(x) \in D_-$$

$N_\pm(\phi)(t)$ are the boundary data functions for positive or negative solutions to the heat equation with initial data $\phi(x)$ in $D_\pm$.

Definition 3.3. $F(h)(x, t)$ is a solution to the signaling problem with negative boundary data $h(t)$ in $C^\infty(\mathbb{R}^+)$ if

$$(\partial_t - \partial_x^2 + x^{-2}\kappa)F(h)(x, t) = 0, \qquad F(h)(x, 0) = 0$$

$$F(h)(x, t) \sim_{x=0} \begin{cases} h(t)x^{-\nu+\frac{1}{2}} + O(x^{\nu+\frac{1}{2}}) & 0 < \nu < 1 \\ h(t)x^{\frac{1}{2}} ln(x) + O(x^{\frac{1}{2}}) & \nu = 0 \end{cases}$$

By the uniqueness of solutions to the heat equation for self-adjoint extensions of H_κ, $F(h)(x, t)$ cannot have purely positive or negative boundary data. It will be shown that the solution to the signaling problem gives a map from $N_+(D_+)$, the space of positive boundary data functions, to $N_-(D_-)$, the space of negative boundary data functions.

Lemma 3.3. *The solution $F(h)(x, t)$ to the signaling problem with negative boundary data $h(t)$ in $C^\infty(0, \infty)$ is*

$$F(h)(x, t) = 2\nu h(t) *_t N'E_+(x, t) \qquad 0 < \nu < 1$$

$$F(h)(x, t) = h(t) *_t N'E_+(x, t) \qquad \nu = 0$$

where $N'E_+(x,t) = \{(x')^{-\nu-\frac{1}{2}}E_+(x,x',t)\}|_{x'=0}$.

PROOF: Let $\psi_-(x)$ be a compactly supported function, smooth up to the boundary $x=0$ and equal to the harmonic function $x^{-\nu+\frac{1}{2}}$ for H_κ in some neighborhood of the boundary,

$$\psi_-(x) \in \mathrm{C}_0^\infty(\mathbb{R}^+ \setminus 0) \text{ and } \psi_-(x) \sim_{x=0} \begin{cases} x^{-\nu+\frac{1}{2}} & 0<\nu<1 \\ x^{\frac{1}{2}}\ln(x) & \nu=0 \end{cases} \tag{3.1}$$

Since $\psi_-(x)$ is harmonic near the boundary, $H_\kappa(\psi_-)(x)$ is compactly supported away from the boundary. Assume that $h(t) = N_-(\phi)(t)$ is a boundary defining function for a negative solution to the heat equation with initial data $\phi(x)$ in $\dot{C}_c^\infty(\mathbb{R}^+)$. The function $h(t)$ vanishes at $t=0$, so $h(t)\psi_-(x)$ has negative boundary data and zero initial data. Applying H_κ to this function gives

$$(\partial_t - \partial_x^2 + \kappa x^{-2})h(t)\psi_-(x) = \partial_t h(t)\psi_-(x) + h(t)H_\kappa(\psi_-)(x).$$

Let H'_κ denote the operator $-\partial_{x'}^2 + \kappa(x')^2$. By Duhamel's principal,

$$\mathcal{D}(h)(x,t) = \partial_t h(t) *_t E_+(\psi_-)(x,t) + h(t) *_t E_+(H'_\kappa(\psi_-))(x,t)$$

solves the forcing problem for the error term and has positive boundary data at $x=0$. Although the Schwartz kernel $E_+(x,x',t)$ only gives smooth solutions to the heat equation for initial data in D_+, $E_+(\psi_-)(x,t)$ is well-defined since $\psi_-(x)$ is in $L^2(\mathbb{R}^+)$. $E_+(\psi_-)(x,t)$ satisfies the heat equation but is discontinuous at the initial face $t=0$, and the singularity in $E_+(\psi_-)(x,t)$ is smoothed by convolution with $\partial_t h(t)$. Since $H_\kappa(\psi_-)(x)$ is in $\dot{C}_c^\infty(\mathbb{R}^+)$, $E_+(H'_\kappa(\psi_-))(x,t)$ is smooth up to the boundary faces, therefore $\mathcal{D}(h)(t)$ is smooth away from the boundary, has positive boundary data, and vanishes to infinite order at $t=0$.

$$F(h)(x,t) = h(t)\psi_-(x) - \mathcal{D}(h)(x,t)$$

solves the signaling problem with negative boundary data $h(t)$. Integrating the first term in $\mathcal{D}(h)(x,t)$ by parts,

$$\begin{aligned}\partial_t h(t) *_t E_+(\psi_-)(x,t) &= \{h(t')\int_0^\infty E_+(x,x',t-t')\psi_-(x')dx'\}|_{t'=0}^t \\ &+ \int_0^t h(t')\int_0^\infty \{\partial_t E_+(x,x',t-t')\psi_-(x')h(t')\}dx'dt' \\ &= h(t)\psi_-(x) + h(t) *_t \partial_t E_+(\psi_-)(x,t)\end{aligned}$$

This leaves a convolution of $h(t)$ with the term

$$\partial_t E_+(\psi_-)(x,t) + E_+(H'_\kappa(\psi_-))(x,t)$$

which can be further simplified by using the boundary expansions of the heat kernel $E(x,x',t)$ and the function $\psi_-(x)$.

$$\partial_t E_+(\psi_-)(x,t) = \int_0^\infty \partial_t E_+(x,x',t)\psi_-(x')dx' = -\int_0^\infty H'_\kappa E_+(x,x',t)\psi_-(x')dx'$$

by the symmetry in the x and x' variables, so

$$\begin{aligned}\partial_t E_+(\psi_-)(x,t) + E_+(H'_\kappa(\psi_-))(x,t) &= -H'_\kappa(E)(\psi_-)(x,t) + E(H'_\kappa(\psi_-))(x,t)\\ &= \lim_{\epsilon\to 0}\int_\epsilon^\infty \{\partial^2_{x'}E_+(x,x',t)\psi_-(x') - E_+(x,x',t)\partial^2_{x'}\psi_-(x')\}dx'\\ &= \lim_{\epsilon\to 0}\{-E_+(x,x',t)\partial_{x'}\psi_-(x') + \partial_{x'}E_+(x,x',t)\psi_-(x')\}|_{x'=\epsilon}^\infty\end{aligned}$$

This cross term vanishes at infinity since $\psi_-(x)$ is compactly supported. Near $x'=0$,

$$\psi_-(x') = \begin{cases} (x')^{-\nu+\frac{1}{2}} & 0 \le \nu < 1 \\ (x')^{\frac{1}{2}}\ln(x) & \nu = 0 \end{cases}$$

$$\partial_{x'}\psi_-(x') = \begin{cases} (-\nu+\frac{1}{2})(x')^{-\nu-\frac{1}{2}} & 0\le\nu<1 \quad \nu\neq\frac{1}{2} \\ 0 & \nu=\frac{1}{2} \\ \frac{1}{2}(x')^{-\frac{1}{2}}\ln(x') + (x')^{-\frac{1}{2}} & \nu = 0\end{cases}$$

$$\partial_{x'}E_+(x,x',t) = (\nu+\frac{1}{2})(x')^{\nu-\frac{1}{2}}N'E_+(x,t) + O((x')^{\nu+\frac{1}{2}})$$

This gives the cross term

$$\partial_t E_+(\psi_-)(x,t) + E_+(H_\kappa\psi_-)(x,t) = \begin{cases} -2\nu N'E_+(x,t) & 0\le\nu<1 \\ -N'E_+(x,t) & \nu=0\end{cases}$$

□

Denote the Fourier transform on $\mathbb{R}$ by $\mathcal{F}$. For $g(t)$ in $\mathcal{L}^2(\mathbb{R}^+)$,

$$\mathcal{F}(g)(\zeta) = \int_0^\infty e^{-ix\zeta}g(t)\frac{dt}{2\pi}$$

Lemma 3.4. *For α a complex number with $\Re(\alpha) > -1$ the function*

$$t_+^\alpha = t^\alpha \quad \text{if } t>0 \qquad t_+^\alpha = 0 \quad \text{if } t<0$$

is locally integrable and defines a distribution with the properties

$$t\cdot t_+^\alpha = t_+^{\alpha+1} \text{ for } \Re(\alpha) > -1 \qquad \text{and} \qquad \partial_t t_+^\alpha = \alpha t_+^{\alpha-1} \text{ for } \Re(\alpha)>0.$$

Let $T_+^\alpha = t_+^\alpha/\Gamma(\alpha+1)$*, where* Γ *is the standard Gamma function, then* T_+^α *can be analytically continued to all* α *in* $\mathbb{C}$ *such that* $\partial_t T_+^\alpha = T_+^{\alpha-1}$*,* $T_+^0 = H(t)$ *the Heavyside function and* $T_+^{-k} = \delta_0^{(k-1)}$*,* k *in* $\mathbb{N}_0$*.* T_+^α *has Fourier transform*

$$\mathcal{F}(T_+^\alpha) \;=\; \exp\{-i\pi(\alpha+1)/2\}(\zeta - i0)^{-\alpha-1}$$

with ζ *the transform variable in* $\mathbb{R}$*.*

PROOF: A detailed explanation is given in [11]. □

Lemma 3.5. *there exists an operator* $K : N_+(D_+)(t) \to N_-(D_-)(t)$ *with convolution kernel* $K(t)$ *conormal to* $t = 0$ *and well defined inverse* G *and.*

$$\begin{array}{lll} K(t) = c_\nu t_+^{\nu-1} = c_\nu \mathcal{F}^{-1}((\zeta - i0)^{-\nu}\Gamma(\nu)e^{-i\pi\nu/2}) & 0<\nu<1 & c_\nu \in \mathbb{R}\setminus 0 \\ K(t) = \mathcal{F}^{-1}(2/\ln(\zeta - i0)) & \nu = 0 & \end{array}$$

PROOF: If $\phi(x)$ is in $\dot{C}_c^\infty(\mathbb{R}^+)$ then the restriction maps $N_\pm$ on $D_\pm$ give an isomorphism between boundary data functions for positive and negative solutions to the heat equation,

$$K : N_+(\phi)(t) \to N_-(\phi)(t).$$

By the $\mathcal{L}^2(\mathbb{R}^+)$ closure of $\dot{C}_c^\infty(\mathbb{R}^+)$, this isomorphism extends to boundary data functions for positive and negative solutions with initial data in the domains D_+ and D_-, respectively, and has a well-defined inverse

$$G : N_-(D_-)(t) \to N_+(D_+)(t).$$

By construction, if $v(x,t)$ is a negative solution with initial data $\phi(x)$ in $\dot{C}_c^\infty(\mathbb{R}^+)$ and boundary data function $h(t) = N_-(\phi)(t)$,

$$u(x,t) = v(x,t) - F(h)(x,t)$$

is a solution to the heat equation with purely positive boundary data and initial data $\phi(x)$ in $\dot{C}_c^\infty(\mathbb{R}^+)$.

The differential operator $\partial_t + H_\kappa$ is invariant under the $\mathbb{R}^+$ scaling μ_s, so $\mu_s^*(F(h))(x,t)$ is also a solution to the signaling problem if $F(h)(x,t)$ is. $F(h)(x,t)$ is in D_{max} for fixed t so

$$F(h)(x,t) = f(t)\phi_+(x) + h(t)\phi_-(x) + v(x) \qquad v(x) \in D_{min}.$$

Since D_{min} is invariant under scaling by $s > 0$ I will suppress the term $v(x)$. For $0 < \nu < 1$,

$$F(h)(x,t) \sim_{x=0} f(t)x^{\nu+\frac{1}{2}} + h(t)x^{-\nu+\frac{1}{2}}$$

$$\mu_s^*(F(h))(x,t) \sim_{x=0} f_s(t)x^{\nu+\frac{1}{2}} + h_s(t)x^{-\nu+\frac{1}{2}}, \qquad s \in (0,\infty)$$

$$f_s(t) = s^{\nu+\frac{1}{2}}f(s^2t), \qquad h_s(t) = s^{-\nu+\frac{1}{2}}h(s^2t)$$

The boundary data functions for solutions to the heat equation are supported on the positive real line so $K(t) \equiv 0$ for $t < 0$ and

$$s^{\nu-\frac{1}{2}}h_s(t) = h(s^2t) = K(f)(s^2t) = \int_0^\infty K(s^2t - t')f(t')dt'$$

$$= s^2\int_0^\infty K(s^2(t-t'))f(s^2t')dt' = s^{-\nu+\frac{3}{2}}\int_0^\infty K(s^2(t-t'))f_s(t')dt'$$

and by definition of $h_s(t)$

$$h_s(t) = \int_0^\infty K(t-t')f_s(t')dt' = s^{-2\nu+2}\int_0^\infty K(s^2(t-t'))f_s(t')dt'$$

so $K(t) = s^{1-\nu}K(st)$.

When $\nu = 0$ it is easiest to first consider $G(t)$, the map from negative to positive boundary data. For $g(t)$ a function on $\mathbb{R}^+$ and s in $(0,\infty)$ set $g_s(t) = g(s^2t)$.

$$F(h)(x,t) \sim_{x=0} h(t)x^{\frac{1}{2}}\ln(x) + h(t)x^{\frac{1}{2}}$$

$$\mu_s(F(h))(x,t) \sim_{x=0} h(s^2t)(sx)^{\frac{1}{2}}\ln(sx) + f(s^2x)(xs)^{\frac{1}{2}}$$

$$= h(s^2t)s^{\frac{1}{2}}x^{\frac{1}{2}}\ln(x) + s^{\frac{1}{2}}\{\ln(s)h(s^2t) + f(s^2t)\}x^{\frac{1}{2}}$$

therefore

$$G(h_s)(t) = \ln(s)h_s(t) + f_s(t)$$

$$G(h_s)(t) = \int_0^t G(t-t')h(s^2t')dt'$$

$$= \ln(s)\int_0^t \delta(t-t')h(s^2t')dt' + \int_0^{s^2t} G(s^2t-t')h(t')dt'$$

$$= \ln(s)\int_0^t \delta(t-t')h_s(t')dt' + s^2\int_0^t G_s(t-t')h_s(t')dt'$$

This gives an equation for $G(t)$

$$\ln(s)\delta(t) = G(t) - s^2G_s(t), \qquad s \in (0,\infty).$$

Since $G(t) = s^2 G_s(t)$ for $t > 0$, $G(t) \sim t^{-1}$ as $t \to \infty$ and $G(t)$ is not in $\mathcal{L}^2(\mathbb{R}^+)$. Let

$$\mathcal{F}(G)(\zeta - i\epsilon) = \int_0^\infty e^{-it(\zeta - i\epsilon)} G(t)dt \qquad \text{for } \epsilon > 0$$

This is a well defined function on $\mathbb{R}$ and

$$\mathcal{F}(G_s)(\zeta - i\epsilon) = \int_0^\infty e^{-it(\zeta-i\epsilon)} G(s^2 t)dt = s^{-2}\mathcal{F}(G)(\frac{\zeta - i\epsilon}{s^2})$$

$$\ln(s) = \mathcal{F}(G)(\zeta - i\epsilon) - \mathcal{F}(G)(\frac{\zeta - i\epsilon}{s^2})$$

$$\Rightarrow \qquad \mathcal{F}(G) = lim_{\epsilon \to 0}\mathcal{F}(G)(\zeta - i\epsilon) = \frac{1}{2}\ln(\zeta - i0)$$

$$K(t) = \mathcal{F}^{-1}(\frac{2}{\ln(\zeta - i0)})$$

□

Corollary 3.6. *For* $0 < \nu < 1$,

$$E_-(x, x', t) - E_+(x, x', t) = 2\nu N' E_+ *_t K *_t *N E_+(x, x', t)$$

Proof: Let $\phi(x)$ in $\dot{C}_c^\infty(\mathbb{R}^+)$ be the initial data function for $u(x,t)$ a positive solution to the heat equation with boundary data function $f(t) = N_+(\phi)(t)$.

$$v(x,t) = u(x,t) + F(h)(x,t) = u(x,t) + 2\nu K(f)(t) *_t N' E_+(x,t)$$

is a solution to the heat equation with initial data $\phi(x)$ and negative boundary data $K(f)(t)$. Since $\phi(x)$ is compactly supported,

$$\begin{aligned} K(f)(t) *_t N'E_+(x,t) &= NE_+(\phi)(t) *_t K(t) *_t N'E_+(x,t) \\ &= \int_0^\infty NE_+(x', t) *_t K(t) *_t N'E_+(x,t)\phi(x')dx' \end{aligned}$$

The corollary follows from the $\mathcal{L}^2(\mathbb{R}^+)$ closure of $\dot{C}_c^\infty(\mathbb{R}^+)$. □

Lemma 3.7. *For each* $\Theta \neq +$ *in* $\mathbb{PR}$ *there is a convolution map with kernel* $K_\Theta(t)$ *conormal to* $t = 0$ *and supported on* $t \geq 0$

$$K_\Theta(t) = \mathcal{F}^{-1}((\mathcal{F}(G) + \Theta)^{-1})(t)$$

such that

$$E_\Theta(x, x', t) = E_+(x, x', t) + \begin{cases} 2\nu N' E_+ *_t K_\Theta *_t NE_+(x, x', t) & 0 < \nu < 1 \\ N' E_+ *_t K_\Theta *_t NE_+(x, x', t) & \nu = 0 \end{cases}$$

PROOF: For $t > 0$ the Θ heat kernel $E_\Theta(x, x', t)$ is in D_Θ and conormal to $x = 0$ by the Spectral Theorem 1.6, therefore the restriction

$$NE_\Theta(x', t) = \{x^{\nu-\frac{1}{2}} E_\Theta(x, x', t)\}|_{x=0}$$

is well-defined. By inclusion in D_Θ, the Taylor series expansion of $E_\Theta(x, x', t)$ at $x = 0$ has a term of order $\nu + 1/2$ with coefficient $\Theta N E_\Theta(x', t)$. If $w(x, t)$ is a solution to the heat equation for H_Θ then with initial data $\phi(x)$ in D_Θ, define the boundary data function for $w(x, t)$ by

$$h(t) = N_\Theta(\phi)(t) = \int_0^\infty NE_\Theta(x', t)\phi(x')dx'.$$

If $u(x, t)$ is a positive solution with initial data $\phi(x)$ in $\dot{C}_c^\infty(\mathbb{R}^+)$ and boundary data $f(t) = N_+(\phi)(t)$, the solution to the signaling problem will be used to find a $h(t)$ in $C^\infty(0, \infty)$ such that

$$w(x, t) = u(x, t) + F(h)(x, t)$$

is a Θ solution with initial data $\phi(x)$ and boundary data function $h(t) = N_\Theta(\phi)(t)$. $G(h)(t)$ gives the positive data for $F(h)(x, t)$, so this amounts to solving the convolution equation for $h(t)$

$$f(t) - G(t) *_t h(t) = h(t)\Theta.$$

Using the Fourier transform,

$$\mathcal{F}(f)(\zeta) = \mathcal{F}(h)(\zeta)\{\mathcal{F}(G)(\zeta) + \Theta\}$$

so

$$h(t) = f(t) *_t \mathcal{F}^{-1}((\mathcal{F}(G)(\zeta) + \Theta)^{-1}) = K_\Theta(f)(t)$$

$$w(x, t) = \Theta h(t)\phi_+(x) + h(t)\phi_-(x) + v(x), \qquad v(x) \in D_{min}$$

For $0 < \nu < 1$, K_Θ is the inverse Fourier transform of the distribution

$$\mathcal{F}(K_\Theta)(\zeta) = \{\mathcal{F}(G)(\zeta) + \Theta\}^{-1} = \Theta\{(c_\nu \Gamma(\nu))^{-1} e^{i\pi\nu/2}(\zeta - i0)^\nu + \Theta\}^{-1}.$$

For $\nu = 0$,

$$\mathcal{F}(K_\Theta)(\zeta) = \{\frac{1}{2}\ln(\zeta - i0) + \Theta\}^{-1} = \ln(C_\Theta(\zeta - i0))^{-1} \qquad C_\Theta = \exp(2\Theta).$$

This illustrates the fact that when $\nu = 0$ the negative solution plays no special role among the solutions with $\Theta \neq +$.

Using the commutativity of the convolution maps when $\phi(x)$ is in $\dot{C}_c^\infty(\mathbb{R}^+)$,

$$K_\Theta(f)(t) = (K_\Theta *_t NE_+)(\phi)(x,t)$$

so

$$E_\Theta(\phi)(x) = E_+(\phi)(x,t) + F(K_\Theta *_t NE_+)(\phi))(x,t).$$

This proves the lemma by the $\mathcal{L}^2(\mathbb{R}^+)$ closure of the initial data. □

Corollary 3.8. *For* $0 < \nu < 1$,

$$E_\Theta(x,x',t) = E_-(x,x',t) - \Theta K_\Theta *_t \{E_-(x,x',t) - E_+(x,x',t)\}$$

PROOF: When $0 < \nu < 1$, the kernel $K_\Theta(t)$ can be written recursively as

$$K_\Theta(t) = \{\delta(t) - \Theta K_\Theta(t)\} *_t K(t)$$

This is just a manipulation of the Fourier transform of $K_\Theta(t)$:

$$\mathcal{F}(K_\Theta)(t) = \frac{\mathcal{F}(K)(\zeta)}{1+\Theta\mathcal{F}(K)(\zeta)} = \{1 - \frac{\Theta\mathcal{F}(K)(\zeta)}{1+\Theta\mathcal{F}(K)(\zeta)}\}\mathcal{F}(K)(\zeta).$$

Then

$$\begin{aligned}
&2\nu N'E_+(x,t) *_t K_\Theta(t) *_t NE_+(x',t)\\
&\qquad = \{\delta(t) - \Theta K_\Theta(t)\} *_t 2\nu(N'E_+(x,t) *_t K(t) *_t NE_+(x',t)\\
&\qquad\qquad = \{\delta(t) - \Theta K_\Theta(t)\} *_t (E_-(x,x',t) - E_+(x,x',t)).
\end{aligned}$$

□

Lemma 3.9. *For* $0 < \nu < 1$ *and for any* N *in* $\mathbb{N}$,

$$\Theta K_\Theta(t) = -\sum_{j=1}^{N-1}(-c_\nu\Theta)^j\frac{\Gamma^j(\nu)}{\Gamma(j\nu)}t_+^{j\nu-1} + O(t^{Nj-1})$$

PROOF: Let $K^j(t)$ be the convolution operator $K(t) = c_\nu t_+^{\nu-1}$ applied to itself j times, $K^j(t) = K *_t K *_t \cdots *_t K(t)$. For ν in this range the Fourier transform gives

$$t_+^{j\nu-1} *_t t_+^{\nu-1} = \Gamma(j\nu)\Gamma(\nu)\mathcal{F}(e^{-i\frac{\pi}{2}(j+1)\nu}(\zeta - i0)^{-(j+1)\nu}) = \frac{\Gamma(j\nu)\Gamma(\nu)}{\Gamma((j+1)\nu)}t_+^{(j+1)\nu-1}$$

and a formula for $K^j(t)$ is given by

$$K^j(t) = K^{j-1} *_t K(t) = c_\nu^j\frac{\Gamma^{j-1}(\nu)}{\Gamma((j-1)\nu)}t_+^{(j-1)\nu-1} *_t t_+^{\nu-1} = c_\nu^j\frac{\Gamma^j(\nu)}{\Gamma(j\nu)}t_+^{j\nu-1}.$$

Applying the recursive formula for $K_\Theta(t)$ gives

$$K_\Theta(t) = \sum_{j=1}^{N-1}(-\Theta)^{j+1}K^j(t) + (-\Theta)^{N+1}K^N(t) *_t K_\Theta(t)$$

for any positive integer N. $\square$

Proof of theorem 3.1:

$E_\Theta(x,x',t)$ is in the domain D_Θ in x for fixed $t>0$ by the Spectral Theorem 1.6. The restriction of the positive heat kernel to the faces $x=0$ or $x'=0$ vanishes to infinite order at $t=0$ away from the corner $x=x'=0$, so the term

$$\beta^*(N'E_+ *_t K_\Theta *_t NE_+)(\tilde{X})$$

is smooth on the interior and and vanishes to infinite order at $\mathfrak{T}$ and $\mathfrak{D}$ and the expansion of $\beta^*(E_\Theta)(\tilde{X})$ at the diagonal face $\mathfrak{D}$ and the temporal face $\mathfrak{T}$ is the same as that of $\beta(E_+)(\tilde{X})$. For $0<\nu<1$,

$$\begin{aligned}
E_\Theta(x,x',t) &= E_-(x,x',t) - K_\Theta(t) *_t \{E_-(x,x',t) - E_+(x,x',t)\}\\
&= E_-(x,x',t) + \sum_{j=1}^{N-1}(-\Theta)^j K^j(t) *_t \{E_-(x,x',t) - E_+(x,x',t)\}\\
&\qquad + (-\Theta)^{N+1}K^N(t) *_t K_\Theta(t) *_t \{E_-(x,x',t) - E_+(x,x',t)\}
\end{aligned}$$

$\beta^*(E_-)(\tilde{X})$ and $\beta^*(E_- - E_+)(\tilde{X})$ are homogeneous in $\rho_{\mathfrak{D}}$ of degree -1, and $K^j(t)$ is homogeneous in t of degree $j\nu-1$, so $\beta^*(E_\Theta)(\tilde{X})$ has an expansion at $\mathfrak{D}$ in powers of $2j\nu-1$ with leading term equal to the leading term of $\beta^*(E_-)(\tilde{X})$.

When $\nu=0$ the operator H_- is not homogeneous under the scaling μ_s, so the negative extension cannot be used to construct the heat kernels for Θ boundary conditions from the difference between the positive and negative heat kernels. Since

$$\ln(C_\Theta(s\zeta - i0)) = lim_{\epsilon\to 0}\ln(C_\Theta(s\zeta - i\epsilon)) = \ln(C_\Theta(\zeta - i0)) + \ln(s)$$

the inverse Fourier transform of its reciprocal satisfies

$$K_\Theta(st) = s^{-1}K_\Theta(t) + O(s^{-1})$$

The difference between the positive and the Θ heat kernels is

$$N'E_+(x,t) *_t K_\Theta *_t NE_+(x',t)$$
$$= \int_0^t N'E_+(x,t-t') \int_0^{t'} K_\Theta(t'-t'')NE_+(x',t'')dt''dt'$$

$$\mu_s^*(N'E_+(x,t) *_t K_\Theta(t) *_t NE_+(x',t))$$
$$s^4\mu_s^*(N'E_+)(x,t) *_t K_\Theta(s^2t) *_t \mu_s^*(NE_+)(x',t))$$
$$= s^{-1}N'E_+(x,t) *_t K_\Theta(t) *_t NE_+(x',t) + O(s^{-1})$$

and $\beta^*(E_\Theta)(\tilde{X})$ is conormal to the front face $\mathfrak{D}$ with a leading term of degree -1 $\square$

4. A formula for the coefficients of the trace at the conic point

Let $e^{-tH}(x,x')$ be the heat kernel for the Laplacian on $\mathbb{R}$. The operator with kernel

$$E_{\Theta}(x,x',t) - e^{-tH}(x,x')$$

restricted to $x, x' \geq 0$ is trace class and $tr(E_{+} - e^{-tH})(t)$ can be written as the sum of two distributions, the first having an expansions at $t = 0$ with coefficients coming from terms supported on the diagonal Δ_0 and the second with coefficients arising from the Taylor series of $k(x)$.

In [1] Callias shows that for $\nu > 1$ and ν not in $\mathbb{Z}$, $tr(E_{+} - e^{-tH})(t)$ has a constant term $\nu = \sqrt{\kappa + 1/4}$. Following Callias, I outline a construction of the positive heat kernel using a Bessel function representation of the resolvant and use this to compute an exact formula for the difference $E_{-}(x,x',t) - E_{+}(x,x',t)$ between the positive and negative heat kernels for the model operator $H_{\kappa} = \partial_x^2 - \kappa x^{-2}$ with $\kappa > -1/4$, as well as the constant c_{ν} in the convolution kernel $K(t) = c_{\nu} t_{+}^{\nu-1}$ that takes positive boundary data to negative boundary data for solutions to H_{κ}. These are used compute the constants in the first two terms of the expansion of $tr(E_{\Theta} - e^{-tH})(t)$, the modified trace for the self-adjoint extension of H_{κ} with Θ boundary data.

Theorem 4.1.

$$tr(E_{\Theta} - e^{-tH})(t) \sim_{t=0} tr(E_{+} - e^{-tH})(t) + \nu + \Theta \frac{\nu 4^{\nu-\frac{1}{2}}\sqrt{\pi}}{\Gamma(3/2-\nu)} t^{\nu} + O(t^{2\nu})$$

Therefore the constant $\kappa = \nu^2 - 1/4$ in the operator H_{κ} and the boundary condition Θ are recoverable from the expansion of $tr(E_{\Theta} - e^{-tH})(t)$ at $t = 0$.

Proof: The positive heat kernel for H_{κ} for $t > 0$ can be written as a contour integral of the Green's function constructed from modified Bessel functions. Because the domains of the self-adjoint extensions with mixed boundary conditions are not $\mathbb{R}^{+}$ invariant, an analogous construction will not work in those cases. The following theorem and its proof directly follow Callias' paper [1].

Theorem 4.2. *For $t > 0$ and $1 > \nu > 0$, the positive heat kernel is given by*

$$E_{+}(x,x',t) = (xx')^{\nu+\frac{1}{2}} \int_{\gamma} \frac{dz}{2\pi i} e^{tz} z^{\nu} g(zx^2) g(z(x')^2)$$

Where Γ is the canonical contour in the complex plane starting at $-\infty$ in the lower half plane, circling the origin and continuing to $-\infty$ in the upper half plane. The *integral converges for all* x, x' *with* $t > 0$, *is a smooth function of* x, $x' > 0$ *and* $g(zx^2)$ *is an analytic function of its argument with*

$$z^{\nu}(xx')^{\nu+\frac{1}{2}}g(zx^2)g(z(x')^2) = \frac{-\pi(xx')^{\frac{1}{2}}}{2sin(\nu\pi)}I_{\nu}(\sqrt{z}x)I_{\nu}(\sqrt{z}x') \tag{4.1}$$

PROOF: For $\nu \notin \mathbb{Z}$ and z a positive real number the equation

$$(\partial_x^2 - \kappa x^{-2} - z)\psi(x,z) = 0$$

has two solutions

$$\psi_1(x,z) = (\sqrt{z}x)^{\frac{1}{2}}I_{\nu}(\sqrt{z}x) \quad \text{and} \quad \psi_2(x,z) = (\sqrt{z}x)^{\frac{1}{2}}K_{\nu}(\sqrt{z}x) \quad \text{where}$$

$$I_{\nu}(y) = (y/2)^{\nu}\sum_{k=0}^{\infty}(y/2)^{2k}(k!\Gamma(\nu+k+1))^{-1}$$

$$K_{\nu}(y) = \frac{\pi}{2sin(\nu\pi)}(I_{-\nu}(y) - I_{\nu}(y))$$

are modified Bessel functions with asymptotic behavior

$$\begin{array}{llll} K_{\nu}(y) \sim y^{-\nu} & I_{\nu}(y) \sim y^{\nu} & \text{at} & y = 0 \\ K_{\nu}(y) \sim y^{-\frac{1}{2}}e^{-y} & I_{\nu}(y) \sim y^{-\frac{1}{2}}e^{y} & \text{as} & y \to \infty. \end{array}$$

The Green's function for the operator $H_+ - z$ defined on D_+ is given by

$$G(x,x',z) = (xx')^{\frac{1}{2}}\{I_{\nu}(\sqrt{z}x)K_{\nu}(\sqrt{z}x')H(x'-x) + I_{\nu}(\sqrt{z}x')K_{\nu}(\sqrt{z}x)H(x-x')\}$$

where $H(x)$ is the Heavyside function, so for $\psi(x)$ and $\phi(x)$ in D_+,

$$\int_0^{\infty}\int_0^{\infty}(H_+ - z)G(x,x',z)\psi(x')\phi(x)dxdx' = \int_0^{\infty}\psi(x)\phi(x)dx.$$

From the asymptotics of the modified Bessel functions,

$$G(x,x',z) \sim \sqrt{z}xx'e^{\sqrt{z}|x-x'|} \quad \text{as} \quad z \to \infty \quad \text{and}$$

$$E_+(x,x',t) = \int_{\gamma}\frac{dz}{2\pi i}e^{tz}G_+(x,x',t)$$

is well-defined for $t > 0$ since the e^{tz} term dominates when $|z| \gg 0$.

This integral can be simplified by writing the product of the two Bessel functions

$$I_{\nu}(\sqrt{z}x')K_{\nu}(\sqrt{z}x) = x^{-\nu}(x')^{\nu}g(z(x')^2)\tilde{g}(zx^2) + z^{\nu}(xx')^{\nu}g(zx^2)g(z(x')^2)$$

where $\tilde{g}(y)$ and $g(y)$ are analytic functions of their arguments (depending on ν) for $\nu > 0$, $\nu \notin \mathbb{Z}$. Then $G(x, x', z)$ is re-written as

$$G(x,x',z) = x^{\nu}(x')^{-\nu}\tilde{g}(zx^2)g(z(x')^2)H(x'-x) \\ + x^{-\nu}(x')^{\nu}\tilde{g}(z(x')^2)g(zx^2)H(x-x') + z^{\nu}(xx')^{\nu}g(zx^2)g(z(x')^2)$$

The first two terms are analytic in z and drop out of the integral for $E_+(x, x', t)$, and this gives a formula for the integrand. □

Proposition 4.3. *For* $1 > \nu > 0$

$$K(t) = -\frac{4^{\nu}t_+^{\nu-1}}{\Gamma(-\nu)} \qquad \textit{and}$$

$$E_-(x,x',t) - E_+(x,x',t) = \frac{sin(\nu\pi)(xx')^{\frac{1}{2}}}{\pi t}\exp\{-\frac{x^2+(x')^2}{4t}\}K_{\nu}[\frac{xx'}{2t}]$$

PROOF: As before, let $G(t)$ be the convolution kernel of the map between negative and positive boundary data functions. By Lemma 3.5,

$$\mathcal{F}(G)(\zeta) = \frac{(\zeta - i0)^{\nu}e^{i\pi\nu/2}}{c_{\nu}\Gamma(\nu)}$$

so by the Fourier inversion formula 3.4

$$G(t) = \frac{t_+^{-\nu-1}}{c_{\nu}\Gamma(\nu)\Gamma(-\nu)}$$

$G(h)(t)$ is the positive data for the solution to the signaling problem for $h(t)$ in $C^{\infty}(\mathbb{R}^+)$ given by

$$F(h)(x,t) = 2\nu h(t) *_t N_+E_+(x,t).$$

A second expression for $G(t)$ can be derived by restricting $F(h)(x,t)$ to $x = 0$ and computing its positive boundary data function. Since $F(h)(x,t)$ must contain both positive and negative boundary data, we have to first make sense of what the restriction means in this case.

$N_+E_+(x,t)$, the restriction of $E_+(x,x',t)$ to the face $x = 0$, is singular at $x = t = 0$ but is smooth in t away from this corner, and has positive boundary data for $t > 0$. Let $\gamma_{\epsilon}(t)$ be a smooth function on $\mathbb{R}^+$ such that $\gamma_{\epsilon}(t) = 1$ for $t > \epsilon$ and $\gamma_{\epsilon}(t)$ vanishes to infinite order at $t = 0$. Then

$$\gamma_{\epsilon} \cdot N'E_+ *_t h(x,t) = \int_0^t h(t-t')\gamma_{\epsilon}(t')N'E_+(x,t')dt'$$

is the convolution of two smooth functions and

$$\lim_{\epsilon \to 0} h(t) *_t \gamma_\epsilon(t) N' E_+(x,t)$$

has positive boundary data, so for $t > 0$

$$G(t) = 2\nu \lim_{\epsilon \to 0} \{(xx')^{-\nu-\frac{1}{2}} \gamma_\epsilon(t) E_+(x,x',t)\}|_{x,x'=0}$$

Using the Bessel function form of the positive heat kernel for $t > 0$

$$G(t) = 2\nu N N' E_+(t) = 2\nu g^2(0) \int_\gamma \frac{dz}{2\pi i} z^\nu e^{tz}.$$

Lemma 4.4. *For k a positive integer,*

$$\int_\gamma \frac{dz}{2\pi i} z^{\nu+k} e^{tz} = (-1)^k \frac{sin(\nu\pi)}{\pi} \Gamma(\nu+k+1) t_+^{-\nu-k-1}$$

PROOF: With the change of variables $w = tz$ for $t > 0$, the integral becomes

$$\begin{aligned}\int_\gamma \frac{dz}{2\pi i} z^{\nu+k} e^{tz} &= t^{-\nu-k-1} \int_\gamma \frac{dw}{2\pi i} w^{\nu+k} e^w \\ &= \frac{e^{i\pi(\nu+k)} - e^{-\pi i(\nu+k)}}{2\pi i} \int_0^\infty w^{\nu+k} e^{-w} dw \\ &= (-1)^k \frac{sin(\pi\nu)}{\pi} \Gamma(\nu+k+1)\end{aligned}$$

□

$$G(t) = 2\nu g^2(0) \frac{sin(\pi\nu)}{\pi} \Gamma(\nu+1) t_+^{-\nu-1}.$$

By equation 4.1 this gives

$$2\nu g^2(0) = \frac{-\nu\pi}{sin(\nu\pi)} \{z^{-\nu} x^{-2\nu} I_\nu^2(\sqrt{z}x)\}|_{x=0} = \frac{-4^{-\nu}\nu\pi}{sin(\nu\pi)\Gamma^2(\nu+1)}$$

c_ν is found by equating the two expressions for $G(t)$

$$\begin{aligned}G(t) &= \frac{t_+^{-\nu-1}}{c_\nu \Gamma(\nu)\Gamma(-\nu)} = \frac{-\nu t_+^{-\nu-1}}{4^\nu \Gamma(\nu+1)} \\ c_\nu &= \frac{4^\nu \Gamma(\nu+1)}{\nu\Gamma(\nu)\Gamma(-\nu)} = \frac{-4^\nu}{\Gamma(-\nu)}.\end{aligned}$$

These same techniques give an explicit formula for the difference between the positive and negative heat kernels. Using the representation of the modified Bessel function

$I_\nu(\sqrt{z}x)$ as an asymptotic sum and 4.1,

$$g(zx^2)g(0) = \frac{-4^{-(\nu+\frac{1}{2})}\pi}{sin(\nu\pi)\Gamma(\nu+1)} \sum_{k=0}^{\infty} (\frac{x^2}{4})^k \frac{z^k}{k!\Gamma(\nu+k+1)}$$

Under the change of variables $w = zt$ this gives

$$N'E_+(x,t) = -\frac{\pi 4^{-(\nu+\frac{1}{2})} x^{\nu+\frac{1}{2}}}{sin(\nu\pi)\Gamma(\nu+1)} \int_\gamma \frac{dz}{2\pi i} z^\nu e^{tz} \sum_{k=0}^{\infty} (\frac{x^2}{4})^k \frac{z^k}{k!\Gamma(\nu+k+1)}$$

$$= \frac{t^{-\nu-1} x^{\nu+\frac{1}{2}}}{4^{\nu+\frac{1}{2}}\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (\frac{x^2}{4t})^k = \frac{t^{-\nu-1} x^{\nu+\frac{1}{2}}}{4^{\nu+\frac{1}{2}}\Gamma(\nu+1)} \exp\{-\frac{x^2}{4t}\}$$

Recall the following Laplace transforms with the dual variable s:

$$\mathcal{L}(t^{\nu-1}e^{-a/t}) = 2(a/s)^{\nu/2} K_\nu[2(as)^{\frac{1}{2}}] \qquad \mathcal{L}(t^{\nu-1}) = s^{-\nu}\Gamma(\nu)$$

$$\mathcal{L}(\frac{1}{2}t^{-1}\exp\{-\frac{a^2+b^2}{2t}\}K_\nu[\frac{ab}{2t}]) = K_\nu[at^{\frac{1}{2}}]K_\nu[bt^{\frac{1}{2}}]$$

Using the Gamma function identities

$$\Gamma(\nu+1) = \nu\Gamma(\nu) \qquad \Gamma(\nu)\Gamma(1-\nu) = \frac{\pi}{\sin(\nu\pi)}$$

these give the Laplace transforms of $N'_+E_+(x,t)$ and $K(t)$;

$$\mathcal{L}(N'E_+)(x,s) = \frac{x^{\frac{1}{2}} s^{\frac{\nu}{2}}}{2^\nu\Gamma(\nu+1)} K_\nu[xs^{\frac{1}{2}}], \qquad \mathcal{L}(K)(s) = -s^{-\nu}4^\nu \frac{\Gamma(\nu)}{\Gamma(-\nu)}$$

To compute the difference between the positive and negative heat kernels use the formula

$$E_-(x,x',t) - E_+(x,x',t) = 2\nu N'E_+(x,t) *_t K(t) *_t NE_+(x',t).$$

Convolution is dual to multiplication under the Laplace transform, and

$$2\nu\mathcal{L}(N'E_+)(x,s)\mathcal{L}(K)(s)\mathcal{L}(NE_+)(x',s)$$

$$= -\frac{2\nu(xx')^{\frac{1}{2}}\Gamma(\nu)}{\Gamma^2(\nu+1)\Gamma(-\nu)} K_\nu[xs^{\frac{1}{2}}]K_\nu[x's^{\frac{1}{2}}] = 2(xx')^{\frac{1}{2}} \frac{sin(\nu\pi)}{\pi} K_\nu[xs^{\frac{1}{2}}]K_\nu[x's^{\frac{1}{2}}]$$

$$= \mathcal{L}(\frac{(xx')^{\frac{1}{2}} sin(\nu\pi)}{\pi t} \exp\{-\frac{x^2+(x')^2}{4t}\}K_\nu[\frac{xx'}{2t}])$$

□

Proof of Theorem 4.1: The difference between the positive and negative kernels has zero boundary data and the $\mathbb{R}_+$ invariance for these extensions means that the $tr(E_- - E_+)(t)$ is given by a convergent integral along the diagonal in $\tilde{X}$ on the front face, which is in turn equivalent to the integral along the diagonal on any parabolic octant Q_p. Under the change of variables $w = x^2/2t$,

$$tr(E_- - E_+)(t) = \int_0^\infty \{E_-(x,x,t) - E_+(x,x,t)\}dx$$
$$= \int_0^\infty \frac{x sin(\nu\pi)}{\pi t} \exp\{-\frac{x^2}{2t}\} K_\nu[\frac{x^2}{2t}]dx$$
$$= \frac{sin(\nu\pi)}{\pi} \int_0^\infty \exp\{-w\} K_\nu[w]dw = \nu$$

This gives the trace at the corner of the heat kernel for negative boundary data in terms of the trace of the positive heat kernel $E_+(x,x',t)$.

$$tr(E_- - e^{-tH})(t) = tr(E_+ - e^{-tH})(t) + tr(E_- - E_+)(t) = tr(E_+ - e^{-tH})(t) + \nu$$

For the $\Theta \neq \pm$ the first two terms of the trace are

$$tr(E_\Theta - e^{-tH})(t) = tr(E_+ - e^{-tH})(t) + \nu - \Theta tr(K(t) *_t \{E_-(x,x',t) - E_+(x,x',t)\}) + O(t^{2\nu})$$

To compute the second term in the trace it is easiest to first restrict to the diagonal and then use Laplace transforms to compute the convolution.

$$\mathcal{L}(K)\mathcal{L}(E_- - E_+)(x,x,s) = \frac{-4^{\nu+\frac{1}{2}} s^{-\nu} x}{\Gamma(1-\nu)\Gamma(-\nu)} K_\nu^2[x s^{\frac{1}{2}}]$$

Taking the inverse Laplace transform

$$K(t) *_t \{E_-(x,x,t) - E_+(x,x,t)\} = \frac{-4^\nu x^{-\nu+1} t^{2\nu-1}}{\Gamma(1-\nu)\Gamma(-\nu)} \exp\{-\frac{x^2}{2t}\} K_0(\frac{x^2}{2t})$$

With the change of variables $w = x^2/2t$,

$$tr(K(t) *_t \{E_-(x,x',t) - E_+(x,x',t)\}) =$$
$$\frac{-2^\nu t_+^\nu}{\Gamma(1-\nu)\Gamma(-\nu)} \int_0^\infty w^{-\nu} \exp\{-w\} K_0(w)dw =$$
$$\frac{-\nu 2^\nu t_+^\nu}{\Gamma^2(1-\nu)} \cdot \frac{2^{\nu-1}\sqrt{\pi}}{\Gamma(3/2-\nu)} \Gamma^2(1-\nu) = \frac{-\nu 4^{\nu-\frac{1}{2}}\sqrt{\pi}}{\Gamma(3/2-\nu)} t_+^\nu$$

□

There is one value of κ for which the kernels $E_+(x,x',t)$ and $E_-(x,x',t)$ can be conveniently expressed in a closed form; that is when $\kappa = 0$ and the operator is no longer singular. In this case $H_0 = \partial_x^2$ and the positive and negative solutions are given by

$$E_+(x,x',t) = \frac{t^{-\frac{1}{2}}}{2\sqrt{\pi}}\{\exp\{-\frac{(x-x')^2}{4t}\} - \exp\{-\frac{(x+x')^2}{4t}\}\}$$
$$E_-(x,x',t) = \frac{t^{-\frac{1}{2}}}{2\sqrt{\pi}}\{\exp\{-\frac{(x-x')^2}{4t}\} + \exp\{-\frac{(x+x')^2}{4t}\}\}$$

When $\nu = 1/2$, $\Gamma(-1/2) = -2\sqrt{\pi}$ and $c_{\frac{1}{2}} = -1/\sqrt{\pi}$

$$N_+E_+(x',t) = \frac{x't^{-3/2}}{2\sqrt{\pi}}\exp\{-\frac{(x')^2}{4t}\}, \qquad N_-E_-(x',t) = \frac{t^{-\frac{1}{2}}}{\sqrt{\pi}}\exp\{-\frac{(x')^2}{4t}\}$$

Under the Laplace transform $\mathcal{L}$ with dual variable s

$$\mathcal{L}(K)(s) = s^{-\frac{1}{2}}, \qquad \mathcal{L}(N_-E_-)(x',s) = s^{-\frac{1}{2}}\exp\{-x's^{\frac{1}{2}}\}$$
$$\mathcal{L}(N_+E_+)(x',s) = \exp\{-x's^{\frac{1}{2}}\}$$

Which shows that the value calculated for $c_{\frac{1}{2}}$ is correct.

$$K_{\frac{1}{2}}[\frac{xx'}{2t}] = \sqrt{\pi}t^{\frac{1}{2}}(xx')^{-\frac{1}{2}}\exp\{-\frac{xx'}{2t}\}$$

and the various methods of computing the difference between the positive and the negative heat kernels agree.

$$E_-(x,x',t) - E_+(x,x',t) =$$
$$2\nu N'E_-(x,t) *_t NE_+(x',t) = \frac{t^{-\frac{1}{2}}}{\sqrt{\pi}}\exp\{-\frac{(x+x')^2}{4t}\}$$
$$= \frac{(xx')^{\frac{1}{2}}}{\pi t}\exp -\frac{x^2+(x')^2}{4t}K_{\frac{1}{2}}[\frac{xx'}{2t}].$$

The modified trace for these operator kernels is then

$$tr(E_+ - e^{-tH})(t) = -\frac{t^{-\frac{1}{2}}}{2\sqrt{\pi}}\int_0^\infty \exp\{-\frac{x^2}{t}\}dx$$

Let $w = x/t^{\frac{1}{2}}$ valid for $t > 0$, then

$$tr(E_+ - e^{-tH})(t) = \frac{-1}{2\sqrt{\pi}}\int_0^\infty \exp\{-w^2\}dw = -\frac{1}{4}$$

And by theorem 4.1,

$$trE_-(t) = \frac{1}{4}, \qquad trE_\Theta(t) = \frac{1}{4} + \Theta\frac{\sqrt{\pi}}{2}t_+^{\frac{1}{2}} + O(t)$$

5. The heat kernel for the variable coefficient operator in one dimension

Let $\kappa(x)$ be a smooth compactly supported function on the half line such that $\kappa(0) = \kappa_0 \geq -1/4$. The positive heat kernel $E_+(x, x', t)$ for the constant coefficient operator

$$H_{\kappa_0} = -\partial_x^2 + \kappa_0 x^{-2}$$

and the positive heat kernel $E_0(x, x', t)$ for the Laplacian on $[0, \infty)$

$$E_0(x, x', t) = \frac{t^{-\frac{1}{2}}}{2\sqrt{\pi}}\{\exp\{-\frac{(x - x')^2}{4t}\} - \exp\{-\frac{(x + x')^2}{4t}\}\}$$

will be used to construct the heat kernel $E_+^\kappa(x, x', t)$ for the operator

$$H_\kappa = -\partial_x^2 + \kappa(x)x^{-2}$$

defined on the domain D_+ defined in 1.2.

Theorem 5.1. *For $0 \leq \nu < 1$, the heat kernel $E_+^\kappa(x, x', t)$ for the positive extension of H_κ with domain D_+ has a boundary expansion on $\tilde{X}$, the blown-up space defined in 2.2*

$$\beta^*(E_+^\kappa)(\tilde{X}) = \rho_{\mathfrak{T}}^\infty \rho_{\mathfrak{D}}^{-1}(\rho_{\mathfrak{R}}\rho_{\mathfrak{L}})^{\nu+\frac{1}{2}}\rho_{\mathfrak{D}}^{-1} \times C^\infty(\tilde{X}) \in \Phi_{phg}^2(X)$$

The coefficients in the expansion at $\mathfrak{L}$ and $\mathfrak{R}$ depend on the Taylor series expansion of $\kappa(x)$ at the boundary, and the leading coefficient at the front face $\mathfrak{D}$ is equal to that of $\beta^*(E_+)(\tilde{X})$, the positive heat kernel for the constant coefficient operator.

Proof: It is initially convenient to do the construction for the operator

$$t(\partial_t - \partial_x^2 + \kappa(x)x^{-2})$$

which, unlike the original operator $\partial_t + H_\kappa$, is not singular at the front face $\mathfrak{D}$ when blown up under β to $\tilde{X}$. The heat kernel constructed for $t(\partial_t - \partial_x^2 + \kappa(x)x^{-2})$ will be exact, so dividing by t gives a solution to the original operator.

Let $\phi(x, x')$ be a compactly supported symmetric function with $\phi(x, x') \equiv 1$ for $(x, x') \in supp(\kappa) \times supp(\kappa)$ and set

$$E_1(x, x', t) = \phi(x, x')E_+(x, x', t) + (1 - \phi(x, x'))E_0(x, x', t)$$

Proposition 5.2.

$$t(\partial_t - H_\kappa)E_1(x,x',t) = R_1(x,x',t) \in \Phi^3_{phg}(X) \qquad E(x,x',0) = \delta(x-x')$$

with $R_1(X)$ compactly supported in x and rapidly decreasing as x' goes to infinity.

PROOF: For large x,

$$t(\partial_t + H_\kappa)E_1(X) = t(\partial_t - \partial_x^2)E_0(X) = 0$$

so $R_1(X)$ is compactly supported in x. Likewise for large x', $E_1(X) = E_0(X)$ is rapidly decreasing as x' goes to infinity for fixed x, therefore so is $R_1(X)$. When x and x' are both in $supp(\kappa)$ then near $x = 0$ $\kappa(x) = \kappa_0 + x\kappa_1(x)$ and

$$t(\partial_t + H_\kappa)E_1(X) = t(\partial_t - \partial_x^2 + \kappa_0 x^{-2} + \kappa_1(x)x^{-1})E_+(X) = tx^{-1}\kappa_1(x)E_+(X)$$

$$\beta^*(tx^{-1}\kappa_1(x)E_+)(\tilde{X}) = \rho_{\mathfrak{D}}\beta^*(\kappa_1)(\rho_{\mathfrak{D}}\rho_{\mathfrak{R}})\beta^*(E_+)(\tilde{X})$$

proving the proposition. □

Lemma 5.3. *There is a distribution $D(X)$ in $\Phi^3_{phg}(X)$ compactly supported in x and x' such that $\beta^*(D)(\tilde{X}) = 0$ along $\mathfrak{D}$ and*

$$t(\partial_t - \partial_x^2 + \kappa(x)x^{-2})D(X) - R_1(X) \in \dot{\Phi}^3_{phg}(X)$$

PROOF: Under the map $\alpha : X \to X$ defined by 2.3, in a coordinate patch bounded away from the left face $x' = 0$

$$\alpha^*(t(\partial_t + H_\kappa)) = \alpha^*(t(\partial_t - \partial_x^2 + \kappa(x)x^{-2})) = s(\partial_s - \partial_y^2 + \kappa(ry)y^{-2}).$$

Let $\psi(y)$ in $\dot{C}^\infty_c(\mathbb{R}^+)$ be a cut-off function supported away from $y = 0$ such that

$$\psi(y) \equiv 1 \ \text{ for } \ y \in (1-\epsilon, 1+\epsilon), \qquad \epsilon > 0$$

and define a family of second order differential operators on $[0,\infty)$ by

$$H^r_\psi = -\partial_y^2 + \kappa(ry)\psi(y)y^{-2}.$$

The Friedrich's heat kernel $E^r_\psi(y,y',s)$ for H^κ_ψ is the family of distributions depending smoothly on the parameter r with

$$E^r_\psi(y,y',0) = \delta(y-y') \qquad \text{and} \qquad H^0_\psi = \partial_y^2 - \kappa_0\psi(y)y^{-2}.$$

On X with coordinates (y, r, s) let

$$F(y, r, s) = r^{-1} E_\psi^r(y, 1, s).$$

Near $y = 1$, so that $\psi(y) = 1$,

$$s(\partial_s - \partial_y^2 + \kappa(yr)y^{-2})F(y, r, s) = 0 \qquad \text{and} \qquad F(y, r, 0) = r^{-1}\delta(y-1).$$

On $\tilde{X}$ near $\mathfrak{D}$ there is a map $\beta_{\mathfrak{D}} : \tilde{X} \rightarrow \alpha^{-1}(X)$ that is an isomorphism on the interiors and giving coordinates on $\tilde{X}$ satisfying

$$\beta_{\mathfrak{D}}^*(r) = \rho_{\mathcal{D}}, \qquad \beta_{\mathfrak{D}}^*(\frac{y-1}{s^{\frac{1}{2}}}) = \tau, \qquad \beta_{\mathfrak{D}}^*(s^{\frac{1}{2}}) = \rho_{\mathfrak{D}} \qquad \text{for} \qquad s > (y-1)^2\epsilon, \;\; \epsilon > 0$$

where τ is not a boundary defining function. Since H_ψ^r is the Laplacain with a smooth perturbation, $F(y, r, s)$ and $\alpha^*(E_1)(y, r, s)$ lift under $\beta_{\mathfrak{D}}$ to have expansions along $\mathfrak{D}$, the lift of the diagonal, with leading term $\rho_{\mathfrak{D}}^{-1}$ [14]

$$\beta_{\mathfrak{D}}^*(F)(\tau, \rho_{\mathcal{D}}, \rho_{\mathfrak{D}}) \sim \sum_{k=0}^{\infty} \rho_{\mathfrak{D}}^{k-1} f_k(\tau, \rho_{\mathcal{D}}), \qquad \beta_{\mathfrak{D}}^*(\alpha^*(E_1))(\tau, \rho_{\mathcal{D}}, \rho_{\mathfrak{D}}) \sim \sum_{k=0}^{\infty} \rho_{\mathfrak{D}}^{k-1} e_k(\tau, \rho_{\mathcal{D}})$$

with $f_k(\tau, \rho_{\mathcal{D}})$ polyhomogeneous conormal to $\mathcal{D}$ with leading term $\rho_{\mathcal{D}}^{-1}$ and $e_k(\tau, \rho_{\mathcal{D}})$ homogeneous in $\rho_{\mathcal{D}}$ of degree -1. Define $D(X)$ by letting $\beta^*(D)(\tilde{X})$ be the Borel sum of $\sum_{k=0}^{\infty} d_k(\tau)\rho_{\mathfrak{D}}^{k-1}$ cut off near $\tau = 0$, where

$$d_0(\tau) = d_1(\tau) = 0 \;\text{ and }\; d_k(\tau) = e_k(\tau, \rho_{\mathcal{D}}) - f_k(\tau, \rho_{\mathcal{D}}) \;\text{ for }\; k > 1.$$

$\alpha^*(E_1)(X)$ and $F(X)$ both vanish to infinite order at $s = 0$ away from the singular submanifold $y = 1$, and since

$$F(y, r, 0) = \alpha^*(E_1)(y, r, 0) = r^{-1}\delta(y-1)$$

the first two coefficients in their expansions are equal at $\mathfrak{D}$, giving the vanishing to first order of $\beta^*(D)(\tilde{X})$ on $\mathfrak{D}$. Since $H_\psi^0 = H_{\kappa_0}$ near $y = 1$, the coefficient of the leading term r^{-1} in the expansion of $F(X)$ at $r = 0$ is the same as that of $\alpha^*(E_+)(X)$. By the smooth dependence of $E_\psi^r(y, 1, s)$ on the parameter r, the expansion of $\beta^*(F)(\tilde{X})$ at $\mathcal{D}$ is smooth, so $\beta^*(D)(\tilde{X})$ is smooth up to the front face $\mathcal{D}$ and vanishes to zeroth order there. $F(X)$ is an exact solution to $\alpha^*(t(\partial_t + H_\kappa))$ near the lifted diagonal $y = 1$, so $\beta^* t(\partial_t + H_\kappa)\{E_1 - D\}(\tilde{X})$ vanishes to infinite order at $\mathfrak{D}$ and $\mathfrak{T}$. □

Lemma 5.4. *There is a distribution $B(X)$ in $\dot{\Phi}^3_{phg}(X)$ compactly supported in x and rapidly decreasing as x' goes to infinity such that*

$$t(\partial_t + H_\kappa)B(X) - R(X) \in \Phi^3_{phg}(X)$$

and $\beta^*(t(\partial_t + H_\kappa)B(X) - R(X))$ vanishes to infinite order at $\mathfrak{R}$.

PROOF: The powers in the Taylor series expansion for $E_+(X)$ and $E_0(X)$ at $x = 0$ are different, therefore the distribution $B(X) = B_+(X) + B_0(X)$ will be constructed in two steps.

First let $b_j(s,r)$ be functions such that

$$s(\partial_s - \partial_y^2 + \kappa(yr)y^{-2})\{\beta^*(\phi)(yr,r)\beta^*(E_+)(\tilde{X}) - \sum_{j=0}^{\infty} b_j(s,r)y^{\nu+j+\frac{1}{2}}\}$$

vanishes to infinite order at $y = 0$. The first term $B_+(X)$ will be given by the Borel sum of these coefficients. It is immediate that the $b_j(s,r)$ vanish to infinite order at $s = 0$ for r large by the vanishing of $E_+(X)$. In order to check that $B_+(X)$ is in $\Phi^k(X)$ it is sufficient to assume that r is small enough so that $\beta^*(\phi)(ry,r) \equiv 1$. The lifted functions $\alpha^*(\kappa_1)(ry)$ and $\alpha^*(E_+)(y,r,s)$ have expansions at $y = 0$

$$\alpha^*(\kappa_1)(ry) \sim_{y=0} \sum_{k=0}^{\infty} r^k y^k \kappa_{k+1} \qquad \text{and} \qquad \alpha^*(E_+)(y,r,s) \sim_{y=0} \sum_{j=0}^{\infty} y^{\nu+\frac{1}{2}+j} e_j(s,r)$$

where the $e_j(s,r)$ vanish to infinite order at $s = 0$ and are homogeneous in r to degree -1. The lifted heat operator applied to $\alpha^*(E_+)(y,r,s)$ gives an error term with expansion at $y = 0$

$$\frac{rs}{y}\alpha^*(\kappa_1)(yr)\alpha^*(E_+)(y,r,s) \sim_{y=0} s\sum_{j=0}^{\infty} y^{\nu-\frac{1}{2}+j} \sum_{k=0}^{j} e_k(s,r)r^{j-k+1}\kappa_{j-k+1}$$

Recalling the formula $\nu = \pm\sqrt{\kappa_0 + 1/4}$,

$$\kappa_0 - (\nu + \frac{1}{2} + j)(\nu - \frac{1}{2} + j) = j^2 + 2\nu j$$

$$s(\partial_s - \partial_y^2 + \kappa(yr)y^{-2}) \sum_{j=0}^{\infty} y^{\nu+\frac{1}{2}+j} b_j(s,r)$$
$$= s\sum_{j=0}^{\infty} y^{\nu+\frac{1}{2}+j}\partial_s b_j(s,r) + s\sum_{j=0}^{\infty}(j^2+2\nu j)y^{\nu-3/2+j}b_j(s,r)$$
$$+ s\sum_{j=0}^{\infty}\sum_{k=1}^{j} \kappa_{j-k+1}b_k(s,r)r^{2\nu+j-k+1}y^{\nu-\frac{1}{2}+j}$$

$b_0(s,r) = 0$ since the leading term in the expansion of $\alpha^*(E_+)(y,r,s)$ is removed by applying $t(\partial_t - H_\kappa)$. Equating coefficients to satisfy

$$\frac{rs}{y}\alpha^*(\kappa_1)(yr)\alpha^*(E_+)(y,r,s) \sim_{y=0} \sum_{j=1}^{\infty} s(\partial_s - \partial_y^2 + \alpha^*(\kappa)(yr)y^{-2})y^{\nu+\frac{1}{2}+j}b_j(s,r)$$

gives $b_1(s,r) = \kappa_1(1+2\nu)^{-1}re_0(s,r)$ and for $j > 1$,

$$b_j(s,r) = (j^2+2\nu j)^{-1}\{\sum_{k=0}^{j-1} k_{j-k}r^{j-k}\{e_k(s,r) - b_k(s,r)\} - \partial_s b_{j-2}(s,r)\}$$

By the above formula the coefficients $b_j(s,r)$ have a smooth expansion in r and vanish to infinite order at $s=0$ uniformly up to $r=0$. Let $B_+(X)$ be the operator such that $\alpha^*(B_+)(X)$ is the Borel sum of the $b_j(s,r)$ cut off near $y=0$. By construction, $B_+(X)$ is in $\dot{\Phi}^3_{phg}(X)$ and has positive boundary data at $x=0$.

Let $\hat{\phi}(x)$ be a cut-off function such that $\hat{\phi}(x) \equiv 1$ near $x=0$ and $\hat{\phi}(x)(1-\phi(x,x'))$ is supported away from the diagonal, and set

$$B_0(X) = \hat{\phi}(x)(1-\phi(x,x'))E_0(X)$$

$B_0(X)$ is rapidly decreasing as x' goes to infinity and $B(X) = B_+(X)+B_0(X)$ satisfies the lemma. □

Definition 5.1. Denote by $\tilde{\Phi}^k(X) \subset \dot{\Phi}^k_{phg}(X)$ the subspaces of distributions that lift under β to vanish to infinite order at the face $\mathfrak{R}$, are compactly supported in x and rapidly vanishing as x' goes to infinity.

Proposition 5.5.

$$\textit{Let} \quad \tilde{E}(X) = E_1(X) - D(X) - B(X) \quad \textit{then}$$
$$\tilde{E}(x,x',0) = \delta(x-x'), \qquad t(\partial_t - H_\kappa)\tilde{E}(X) = R(X) \in \tilde{\Phi}^3(X).$$

PROOF: The initial data for $\tilde{E}(X)$ is the same as that of $E_1(X)$, since $D(X)$ and $B(X)$ both vanish at $t = 0$. Since $R_1(X)$, $D(X)$ and $B(X)$ are compactly supported in x and rapidly decreasing at infinity in x', so is $R(X)$. The Taylor series of the remainder term at $\mathfrak{T}$, $\mathfrak{D}$ and $\mathfrak{R}$ has been removed without changing the initial data. $\square$

There is a composition calculus for operator kernels in $\tilde{\Phi}^*(X)$ defined in 5.1 that allows the error term for the solution $\tilde{E}(X)$ constructed above to be removed by iteration.

Definition 5.2. Let $\Omega(M)$ denote the density bundle on the manifold M.

Definition 5.3. Let $\dot{I}(Y; x = 0)$ for $Y = \mathbb{R}^+ \times \mathbb{R}^+ \ni (x, t)$ denote the space of smooth functions on Y compactly supported in x and vanishing to infinite order at $x = 0$.

The action of $\Phi^*(X)$ on the space of solutions to the heat equation for H_κ on the domain D_+ is defined by $\mathcal{L}^2(\mathbb{R}^+)$ closure on $\dot{I}(Y; x = 0)$. In order to use the properties of b-fibrations to show how $\tilde{\Phi}^*(X)$ acts on $\dot{I}(Y; x = 0)$ the origin $(0,0)$ in Y will be blown up parabolically in the direction of $sp(dt)$ in the co-sphere bundle to get the manifold

$$Y_Q = [Y; (0,0), sp(dt)]$$

with map $\alpha : Y_Q \to Y$ and boundary faces $\mathfrak{R}$, $\mathfrak{D}$, and $\mathfrak{T}$. Boundary defining functions can be chosen satisfying

$$\alpha^*(x) = \rho_{\mathfrak{D}}\rho_{\mathfrak{R}}, \quad \alpha^*(t) = \rho_{\mathfrak{D}}^2\rho_{\mathfrak{T}}$$

The density bundle on Y lifts to Y_Q by

$$\alpha^*(\Omega(Y)) = \alpha^*(dxdt) = \rho_{\mathfrak{D}}^2\Omega(Y_Q).$$

The space X_Q defined in 2.1

$$X_Q = [X \setminus \{0\}] \sqcup \{Q_p\}$$

has boundary faces $\mathfrak{R}$, $\mathfrak{L}$, $\mathfrak{T}$ and $\mathfrak{D}$ and there is a map $\beta_Q : X_Q \to X$ that is an isomorphism on the interiors of the two spaces and boundary defining functions satisfying

$$\beta_Q^*(x) = \rho_{\mathfrak{R}}\rho_{\mathfrak{D}}, \qquad \beta_Q^*(x') = \rho_{\mathfrak{L}}\rho_{\mathfrak{D}}, \qquad \beta_Q^*(t) = \rho_{\mathfrak{T}}\rho_{\mathfrak{D}}^2.$$

By the definition of $\tilde{\Phi}^k(X)$, the lift under β_Q of the remainder term $\beta_Q^*(R)(X_Q)$ is a function polyhomogeneous conormal to the boundary faces of X_Q.

Proposition 5.6. *For all $R(X)$ in $\tilde{\Phi}^k(X)$ the pairing*

$$R(u)(x,t) = \int_0^t \int_0^\infty R(x,x',t-t')u(x',t')dx'dt'$$

defines a map from $\dot{I}(Y; x=0)$ to itself.

PROOF: The integral in x' is well-defined by the compact support of $u(x,t)$ in x and the vanishing at $x = 0$. Let X_2 be the direct product of four half lines $\mathbb{R}^+$ with variables

$$X_2 = \{(x,x',t-t',t');\ x,x' \geq 0,\ t \geq t' \geq 0\}$$

and define projection maps

$$\pi_{2,R}(X_2) = (x,x',t-t'),\quad \pi_{2,L}(X_2) = (x',t'),\quad \pi_{2,C}(X_2) = (x,t)$$

The maps $\pi_{2,R}$ and $\pi_{2,L}$ give fibrations but $\pi_{2,C}$ does not. In order to use the theory of b-fibrations as outlined in [15] the submanifold $\{t = t' = 0\}$ in X_2 will be blown up to make the projection $\pi_{2,C}$ into a b-fibration. This will mess up the other two projections, but they will still be sufficiently well-behaved for the composition of their lifts to make sense on the blown-up space. Since the distributions in $\tilde{\Phi}^k(X)$ and $I(Y; x=0)$ are defined as functions polyhomogeneous conormal to the boundary faces of X_Q and Y_Q, the origin $\{0\}_2$ in X_2 will be blown up parabolically in the direction of $sp(dt, dt')$. and the lifts of these three origins to X_2 will also be blown-up. The subspaces of X_2 defined by the three lifted origins are

$$\pi_{2,R}^{-1}(\{0\}) = (0,0,0,t), \qquad \pi_{2,L}^{-1}((0,0)) = (x,0,t,0), \qquad \pi_{2,C}^{-1}((0,0)) = (0,x',0,0)$$

For the first two blow-ups set

$$X_2' = [X_2; \{0\}_2, sp(dt,dt'); cl(\{t=t'=0\} \setminus \{0\}_2)]$$

where $sp(dt, dt')$ is the span in the co-sphere bundle of X_2 at $\{0\}_2$ of dt and dt' and $cl(\{t = t' = 0\} \setminus \{0\}_2)$ is the closure in $[X_2; \{0\}_2, sp(dt, dt')]$ of the lift of the subspace $\{t = t' = 0\}$ in X_2 with the origin removed. X_2' has six boundary faces with boundary defining functions $(\rho_O, x, x', \tau_R, \tau_L, \tau_C)$ where ρ_O is the boundary defining function for the blow up of the origin in X_2, τ_R of blow up of $\{t = t' = 0\}$ and τ_L and τ_C of the lifts of the temporal faces in Y and X. There is a map $\gamma : X_2' \to X_2$ such that

$$\gamma^*(x) = x\rho_O, \quad \gamma^*(x') = x'\rho_O, \quad \gamma^*(t - t') = \tau_R\tau_C\rho_O \text{ and } \gamma^*(t') = \tau_L\tau_C\rho_O$$

Define the blown-up space $\tilde{X}_2$ over X_2 by blowing up X_2' along the lifts of the three origins in X and Y,

$$\tilde{X}_2 = [X_2'; B_L, sp(d\tau_L); B_R, sp(d\tau_R); B_C, sp(d\tau_C)]$$

where B_L is the closure of the lift under γ^{-1} of the lift of the origin in Y to X_2 minus the origin in X_2,

$$B_L = cl\{\gamma^{-1}(\pi_{2,L}^{-1}((0,0)) \setminus \{0\}_3)\} \qquad \text{and likewise}$$
$$B_C = cl\{\gamma^{-1}(\pi_{2,C}^{-1}((0,0)) \setminus \{0\}_3)\}, \qquad B_R = cl\{\gamma^{-1}(\pi_{2,R}^{-1}(\{0\}) \setminus \{0\}_3)\}$$

where the order of the blow-ups is not important. $\tilde{X}_2$ has boundary defining functions $(\rho_O, \rho_R, \rho_L, \tilde{\rho}_L, \tilde{\rho}_R, \tilde{\rho}_C, \tau_R, \tau_L, \tau_C)$ where τ_R, τ_L, τ_C are the defining functions for the lifts of the three temporal boundary faces, ρ_O for the blow-up O of the origin in X_2, ρ_R, ρ_L for the lifts R and L of the two boundary faces corresponding to x and x' and $\tilde{\rho}_L$, $\tilde{\rho}_R$, $\tilde{\rho}_C$ the boundary defining functions for the blow-ups $\tilde{L}$, $\tilde{R}$ and $\tilde{C}$ of the submanifolds B_L, B_R and B_C. There is a map $\beta_2 : \tilde{X}_2 \to X_2$ such that

$$\beta_2(x) = \rho_O\rho_R\tilde{\rho}_R\tilde{\rho}_C \qquad \beta_2(x') = \rho_O\rho_L\tilde{\rho}_L\tilde{\rho}_R$$
$$\beta_2(t') = \tau_L\tau_C\rho_O^2\tilde{\rho}_L^2\tilde{\rho}_C^2 \qquad \beta_2(t - t') = \tau_R\tau_C\rho_O^2\tilde{\rho}_R^2\tilde{\rho}_C^2 \qquad \beta_2(t) = \tau_C\rho_O^2\tilde{\rho}_C^2$$

and blow down maps

$$\tilde{\pi}_{2,R} : \tilde{X}_2 \to X_Q, \qquad \tilde{\pi}_{2,C} : \tilde{X}_2 \to Y_Q, \qquad \tilde{\pi}_{2,L} : \tilde{X}_2 \to Y_Q$$

defined so that the following diagram commutes: The density bundle $\Omega(X_2)$ lifts under β_2

$$\beta_2^*(\Omega(X_2)) = \beta_2^*(dxdx'dtdt') = \rho_O^5\tilde{\rho}_C^4\tilde{\rho}_L^2\tilde{\rho}_R\tau_C\Omega(\tilde{X}_2)$$

The lifts of the boundary defining functions on X_Q and Y_Q are

$$\tilde{\pi}_{2,R}^*(\rho_{\mathfrak{R}}) = \rho_R\tilde{\rho}_C \;\; \tilde{\pi}_{2,R}^*(\rho_{\mathfrak{L}}) = \rho_L\tilde{\rho}_L \;\; \tilde{\pi}_{2,R}^*(\rho_{\mathfrak{T}}) = \tau_R\tau_C\tilde{\rho}_C^2 \;\; \tilde{\pi}_{2,R}^*(\rho_{\mathfrak{O}}) = \rho_O\tilde{\rho}_R$$
$$\tilde{\pi}_{2,L}^*(\rho_{\mathfrak{R}}) = \rho_L\tilde{\rho}_R \;\; \tilde{\pi}_{2,L}^*(\rho_{\mathfrak{T}}) = \tau_L\tau_C\tilde{\rho}_C^2 \;\; \tilde{\pi}_{2,L}^*(\rho_{\mathfrak{O}}) = \rho_O\tilde{\rho}_L$$
$$\tilde{\pi}_{2,C}^*(\rho_{\mathfrak{R}}) = \rho_R\tilde{\rho}_R \;\; \tilde{\pi}_{2,C}^*(\rho_{\mathfrak{T}}) = \tau_C \;\; \tilde{\pi}_{2,C}^*(\rho_{\mathfrak{O}}) = \rho_O\tilde{\rho}_C$$

Where the asymmetry in the lifts of ρ_{Υ} reflects the asymmetry of the blown-up manifold. Although the projection $\tilde{\pi}_{2,R}$ is not a b-fibration, this set of projection maps is sufficient because the projection $\tilde{\pi}_{2,R}$ will only be used to pull back functions to X_2. In fact, if the operator kernels in $\tilde{\Phi}^k(X)$ were written as Schwartz kernels in t as well as in x, ignoring the translation invariance, then there would be an extra temporal variable in the blown-up space and this would give b-fibrations in all the factors.

Lifting $\alpha^*(u)(Y_Q)$ with $u(x,t)$ in $\dot{I}(Y, x=0)$ to $\tilde{X}_2$ by the left projection map gives

$$\tilde{\pi}_{2,L}^*(\alpha^*(u))(\tilde{X}_2) = (\rho_O \rho_l \tilde{\rho}_L \tilde{\rho}_R)^\infty \tilde{\pi}_{2,L}^*(\tilde{u}(Y_Q))$$

where $\tilde{\pi}_{2,L}^*(\tilde{u}(Y_Q))$ is smooth. For $R(X)$ in $\tilde{\Phi}^k(X)$,

$$\tilde{\pi}_{2,R}^*(\beta_Q^*(R))(\tilde{X}_2) = (\tilde{\rho}_C \rho_R \tau_C \tau_R)^\infty (\rho_O \tilde{\rho}_R)^{3k} \tilde{\pi}_{2,R}^*(\alpha(\tilde{X}_2))$$

with $\tilde{\pi}_{2,R}^*(\alpha(\tilde{X}_2))$ polyhomogeneous conormal to L and $\tilde{L}$. The lifts of these two objects pair on $\tilde{X}_2$ to give

$$\beta_2^*(\Omega(X_2))\tilde{\pi}_{2,R}^*(R)\tilde{\pi}_{2,L}^*(u) = (\rho_O \rho_R \tilde{\rho}_R \tilde{\rho}_L \tilde{\rho}_C \tau_C \tau_R)^\infty \alpha(\tilde{X}_2)$$

With $\alpha(\tilde{X}_2)$ polyhomogeneous conormal to $\tilde{R}$ and L.

An element $w(x,t)$ of $\dot{I}(Y; x=0)$ lifts to $\tilde{X}_2$ as

$$\tilde{\pi}_{2,C}^*(\alpha^*(w))(\tilde{X}_2) = (\rho_O \tilde{\rho}_C \rho_R \tilde{\rho}_R)^\infty \tilde{\pi}_{2,C}^*(\tilde{w}(Y_Q))$$

with $\tilde{\pi}_{2,C}^*(\tilde{w}(Y_Q))$ smooth. therefore by the push-forward theorem for polyhomogeneous conormal functions under b-fibrations given in [13], the density

$$\beta_2^*(\Omega(X_2))\tilde{\pi}_{2,R}^*(R)\tilde{\pi}_{2,L}^*(u)$$

pushes forward under $\tilde{\pi}_{2,C}$ to a density

$$\alpha^*(\Omega(Y))\alpha^*(u)(Y_Q)$$

with $u(x,t)$ in $\dot{I}(Y; x=0)$. □

Proposition 5.7. *The composition of two operators with kernels $R_i(X)$ in $\tilde{\Phi}^{k_i}(X)$, for $i = 1,2$ given by*

$$R_1 \circ R_2(x, x', t) = \int_0^\infty \int_0^t R_1(x, z, t-t') R_2(z, x', t') dt' dz$$

is an element of $\tilde{\Phi}^{k_1+k_2}(X)$.

PROOF: Again $R_1(X)$ and $R_2(X)$ are compactly supported in x so the integral in z is over a finite interval and well-defined by the vanishing to infinite order at $x = 0$. Let X_3 be the direct product of five copies of the half-line $\mathbb{R}^+$,

$$X_3 = \{(x, x', z, t - t', t'); x, x', z \geq 0, t \geq t' \geq 0\}$$

with three projection maps give by

$$\pi_{3,R}(X_3) = (x, z, t'), \qquad \pi_{3,L}(X_3) = (x', z, t - t'), \qquad \pi_{3,C}(X_3) = (x, x', t)$$

The construction of the composition space $\tilde{X}_3$ for operators in $\tilde{\Phi}^*(X)$ is almost identical to the construction of $\tilde{X}_2$, the main difference being that there is one extra boundary face which makes it a little more complicated to write down.

X_3 is a five dimensional manifold with five boundary faces. The heat space $\tilde{X}_3$ will be a five dimensional manifold with nine boundary faces. In order for $\tilde{X}_3$ to give a b-fibration under the lift of $\pi_{3,C}$ the origin $\{0\}_3$ in $\tilde{X}_3$ will be blown up parabolically, as well as the submanifold $t = t' = 0$ and the lifts of the origin $\{0\}$ in X under the three projection maps:

$$\pi_{3,R}^{-1}(\{0\}) = (0, x', 0, t, 0), \qquad \pi_{3,L}^{-1}(\{0\}) = (x, 0, 0, 0, t), \qquad \pi_{3,C}^{-1}(\{0\}) = (0, 0, z, 0, 0)$$

The blow up of $\{0\}_3$ is done first, and then the blow-up of the sub-manifold $\{t = t' = 0\}$, so that the final result will be a b-fibration in the center projection. Define

$$X_3' = [X; \{0\}_3, sp(dt, dt'); cl(\{t = t' = 0\} \setminus \{0\}_3)]$$

with boundary defining functions $(x, x', z, \rho_O, \tau_L, \tau_R, \tau_C)$. The three lifts of $\{0\} \subset X$ to X_3' are disjoint and can be blown up in any order, and the heat space is

$$\tilde{X}_3 = [X_3'; B_R, sp(\tau_R); B_L, sp(\tau_L); B_C, sp(\tau_C)]$$

where the three submanifolds B_L, B_R and B_C are defined in a manner analogous to the previous definition. There is a map $\beta_3 : \tilde{X}_3 \to X_3$ and boundary defining functions

$$(\rho_O, \tilde{\rho}_R, \tilde{\rho}_L, \tilde{\rho}_C, \rho_R, \rho_L, \rho_C, \tau_R, \tau_L, \tau_C) \text{ on } \tilde{X}_3$$

such that the boundary defining functions on X_3 lift by

$$\beta_3^*(z) = \rho_O \rho_C \tilde{\rho}_R \tilde{\rho}_L, \qquad \beta_3^*(x) = \rho_O \rho_R \tilde{\rho}_R \tilde{\rho}_C, \qquad \beta_3^*(x') = \rho_O \rho_L \tilde{\rho}_L \tilde{\rho}_C,$$
$$\beta_3^*(t') = \tau_L \tau_C \rho_O^2 \tilde{\rho}_R^2 \tilde{\rho}_C^2, \qquad \beta_3^*(t - t') = \tau_R \tau_C \rho_O^2 \tilde{\rho}_L^2 \tilde{\rho}_C^2, \qquad \beta_3^*(t) = \tau_C \rho_O^2 \tilde{\rho}_C^2.$$

Finally there are maps $\tilde{\pi}_{3,\cdot} : \tilde{X}_3 \to X_Q$ defined so that the following diagram is commutative: The density $\Omega(X_3)$ lifts under β_3 by

$$\beta_3^*(\Omega(X_3)) = \beta_3(dxdx'dzdtdt') = \rho_O^6 \tilde{\rho}_C^5 \tilde{\rho}_R^3 \tilde{\rho}_L \tau_C \Omega(\tilde{X}_3)$$

With coordinates on X_Q given as before, the lift of boundary defining functions are

$$\tilde{\pi}_{3,R}^*(\rho_{\mathfrak{R}}) = \rho_R \tilde{\rho}_C, \quad \tilde{\pi}_{3,R}^*(\rho_{\mathfrak{L}}) = \rho_C \tilde{\rho}_L, \quad \tilde{\pi}_{3,R}^*(\rho_{\mathfrak{D}}) = \rho_O \tilde{\rho}_R, \quad \tilde{\pi}_{3,R}^*(\rho_{\mathfrak{T}}) = \tau_R \tau_C \tilde{\rho}_C^2$$

$$\tilde{\pi}_{3,L}^*(\rho_{\mathfrak{R}}) = \rho_C \tilde{\rho}_R, \quad \tilde{\pi}_{3,L}^*(\rho_{\mathfrak{L}}) = \rho_L \tilde{\rho}_C, \quad \tilde{\pi}_{3,L}^*(\rho_{\mathfrak{D}}) = \rho_O \tilde{\rho}_L, \quad \tilde{\pi}_{3,L}^*(\rho_{\mathfrak{T}}) = \tau_L \tau_C \tilde{\rho}_C^2$$

$$\tilde{\pi}_{3,C}^*(\rho_{\mathfrak{R}}) = \rho_R \tilde{\rho}_R, \quad \tilde{\pi}_{3,C}^*(\rho_{\mathfrak{L}}) = \rho_L \tilde{\rho}_L, \quad \tilde{\pi}_{3,C}^*(\rho_{\mathfrak{D}}) = \rho_O \tilde{\rho}_C, \quad \tilde{\pi}_{3,C}^*(\rho_{\mathfrak{T}}) = \tau_C$$

If $R_1(X)$ is in $\tilde{\Phi}^{k_1}(X)$ and $R_2(X)$ is in $\tilde{\Phi}^{k_2}(X)$ then

$$\tilde{\pi}_{3,R}^*(\beta_Q^*(R_1))(X_3) = (\tilde{\rho}_C \rho_R \tau_R \tau_C)^\infty (\rho_O \tilde{\rho}_R)^{3k_1} \tilde{\pi}_{3,R}^*(\alpha_R(X_Q))$$

$$\tilde{\pi}_{3,L}^*(\beta_Q^*(R_2))(\Omega^{\frac{1}{2}}) = (\tilde{\rho}_C \rho_C \tilde{\rho}_R \tau_L \tau_C)^\infty (\rho_O \tilde{\rho}_L)^{3k_2} \tilde{\pi}_{3,L}^*(\alpha_L(X_Q))$$

where $\tilde{\pi}_{3,R}^*(\alpha_R(X_Q))$ is polyhomogeneous conormal to $\tilde{L}$ and $\tilde{\pi}_{3,L}^*(\alpha_L(X_Q))$ polyhomogeneous conormal to L. The composition of these two operators on $\tilde{X}_3$ with the lifted density $\beta_3^*(\Omega(X_3))$ is

$$\beta_3^*(\Omega(X_3))\tilde{\pi}_{3,R}^*(\beta_Q^*(R_1))\tilde{\pi}_{3,L}^*(\beta_Q^*(R_2))(\tilde{X}_3) = (\tilde{\rho}_C \rho_C \tilde{\rho}_R \rho_R \tau_R \tau_L \tau_C)^\infty \rho_O^{6+3(k_1+k_2)} \gamma(\tilde{X}_3; \Omega)$$

Where $\gamma(\tilde{X}_3, \Omega)$ is a integrable density valued function polyhomogeneous conormal to L and $\tilde{L}$. $R_3(X)$ in $\tilde{\Phi}^{k_3}(X)$ lifts under $\tilde{\pi}_{3,C}$ to a distribution on $\tilde{X}_3$

$$\tilde{\pi}_{3,C}^*(\beta_Q^*(R_3))(\tilde{X}_3) = (\rho_R \tilde{\rho}_R \tau_C)^\infty (\rho_O \tilde{\rho}_C)^{3k_3} \tilde{\pi}_{3,C}^*(\alpha_C(X_Q))$$

where $\tilde{\pi}_{3,C}^*(\alpha_C(X_Q))$ is a function polyhomogeneous conormal to L and $\tilde{L}$. The lift of the density bundle on X to X_Q is

$$\beta_Q^*(dxdx'dt) = \rho_{\mathfrak{D}}^3 \Omega(X_Q)$$

therefore the push-forward to the composition under $\tilde{\pi}_{3,C}$

$$\tilde{\pi}_{3,C,*}\{\beta_3^*(\Omega(X_3))\tilde{\pi}_{3,R}^*(\beta_Q^*(R_1))\tilde{\pi}_{3,L}^*(\beta_Q^*(R_2))(\Omega^{\frac{1}{2}})\} = (\rho_{\mathfrak{R}} \rho_{\mathfrak{T}})^\infty \rho_{\mathfrak{D}}^{6+3(k_1+k_2)} \alpha(\tilde{X}_3)$$

with $\alpha(\tilde{X}_3)$ a function polyhomogeneous conormal to L and the push forward is an element of $\tilde{\Phi}^{k_1+k_2}(X)$. This completes the proof that composition in the calculus $\tilde{\Phi}^*(X)$ is well defined. □

Proposition 5.8. *There exists an operator $S(X)$ in $\tilde{\Phi}^3(X;\Omega^{\frac{1}{2}})$ and an operator $T(X)$ in $\tilde{\Phi}^\infty(X)$ such that*

$$(\partial_t + H_\kappa)\tilde{E} \circ (Id + S) \circ (Id + T)^{-1} = 0$$

PROOF: $t(\partial_t + H_\kappa)\tilde{E}(X) = R(X)$ with $R(X)$ in $\tilde{\Phi}^3(X, \Omega^{\frac{1}{2}})$. The asymptotic expansion of $R(X)$ at $\mathfrak{D}$ can be iterated away, since it is in the calculus $\tilde{\Phi}^*(X)$.

$$R^j(X) = R \circ \cdots \circ R(X), \qquad R^j(X) \in \tilde{\Phi}^{3k}(X)$$

Let $Id(X)$ in $\Phi^0(X)$ be the identity operator for the calculus $\tilde{\Phi}^*(X)$. For $\tilde{E}(X)$ acting on $I(Y; x = 0)$,

$$t(\partial_t + H_\kappa)\{\tilde{E} \circ u\}(Y) = u(Y) + R \circ u(Y), \qquad u \in I(Y; x = 0)$$

$$\begin{aligned} t(\partial_t + H_k)(\tilde{E} \circ \{Id + \sum_{j=1}^{N}(-1)^j R^j\}) &= (Id + R) \circ (Id + \sum_{j=0}^{N}(-1)^j R^j) \\ &= Id + R + \sum_{j=0}^{N}(-1)^j R^j + \sum_{j=0}^{N}(-1)^j R^{j+1} = Id + (-1)^N R^{N+1} \end{aligned}$$

where $R^{N+1}(X) \in \tilde{\Phi}^{(N+1)3}(X)$. Since $\beta_Q^*(R^j)(\tilde{X}) \sim_{\mathfrak{D}} O(\rho_{\mathfrak{D}}^{3j})$ the series $\sum_{j=1}^\infty \beta_Q^*(R^j)(X_Q)$ can be summed near $\mathfrak{D}$. Let $S(X)$ be the push forward under β_Q of this sum. Then

$$t(\partial_t + \tilde{H}_k)(\tilde{E} \circ (Id - S))(X) = (Id - T)(X)$$

with $T(X)$ in $\tilde{\Phi}^\infty(X)$.

Since $T(X)$ vanishes to infinite order at $\mathfrak{D}$, it can be considered as a function defined on X. $T(X)$ vanishes to infinite order at $t = 0$ and $x = 0$ and is polyhomogeneous conormal at $x' = 0$ and compactly supported in x, therefore for any $\tau > 0$ there exists a constant C_τ and integers p and q such that $p + q > 0$ and

$$T(x, x', t) \leq C_\tau x^p (x')^q (1 + x)^{-1}(1 + x')^{-1} \text{ for } t \leq \tau.$$

Let $K_\tau = \int_0^\infty z^{p+q}(1 + z)^{-2} dz$ and $T^j(X) = T \circ \cdots \circ T$ j times and assume

$$T^j(x, x', t) \leq C_\tau^j x^p (x')^q (1 + x)^{-1}(1 + x')^{-1} \frac{(K_\tau t)^{j-1}}{(j-1)!} \text{ for } t \leq \tau$$

$$T^{j+1}(x,x',t) = T \circ T^j(x,x',t) = \int_0^t \int_0^\infty T(x,z,t-t')T^j(z,x',t')d'dz$$
$$\leq \int_0^t \int_0^\infty C_\tau^{j+1} x^p (x')^q (1+x)^{-1}(1+x')^{-1} z^{p+q}(1+z)^{-2}\frac{(K_\tau t)^{j-1}}{(j-1)!}dt'dz$$
$$\leq C_\tau^{j+1} x^p (x')^q (1+x)^{-1}(1+x')^{-1}\frac{(K_\tau t)^j}{j!}$$

for $t < \tau$ therefore the Neumann series $\sum_{j=1}^{\infty}(-1)^j T^j(X)$ converges to an operator in $\tilde{\Phi}^\infty(X)$, so for $T(X)$ in $\tilde{\Phi}^\infty(X)$ the inverse of the remainder $(Id-T)(X)$ is well-defined and the parametrix

$$E_+^\kappa = (E_+ - D - B) \circ (Id - S) \circ (Id - T)^{-1}(x,x',t)$$

solves the forcing problem

$$t(\partial_t - \partial_x^2 + \kappa(x)x^{-2})(E_+^\kappa \circ u)(x,t) = u(x,t),$$

where $u(x,t)$ is a function that is in the positive domain D_+ for fixed t. $E_+^\kappa(x,x',t)$ is unique and is the Schwartz kernel for the initial value problem, so it is the heat kernel for the positive extension of the variable coefficient operator H_κ. □

Since $B(X)$ and $D(X)$ are in $\Phi^3_{phg}(X)$, the leading coefficient of $\tilde{E}(X)$ at $\mathfrak{D}$ is given by the restriction to the parabolic octant Q_p of the kernel $E_+(X)$ for the constant coefficient problem,

$$\{\rho_{\mathfrak{D}}\beta^*(\tilde{E})(\tilde{X}) - \rho_{\mathfrak{D}}\beta^*(E_+)(\tilde{X})\}|_{\rho_{\mathfrak{D}}=0} = 0$$

$T(X)$ vanishes to infinite order at $\mathfrak{D}$, so it does not effect the expansion there. Consider the composition of $S(X)$ with $\tilde{E}(X)$. For any N in $\mathbb{N}$ with $R^0(X) = Id(X)$

$$S(X) \sim_{\rho_{\mathfrak{D}}=0} \sum_{j=0}^{N-1}(-1)^j R^j(X) + S^N(X), \qquad S^N(X) \in \tilde{\Phi}^{3N}(X)$$

$R^j(X)$ is in $\tilde{\Phi}^{3j}(X)$ so

$$\mu_s^*(R^j)(X) \sim_{s=0} O(s^{3(j-1)})$$
$$\tilde{E} \circ R^j(X) = \int_0^\infty \int_0^t \tilde{E}(x,z,t-t')R^j(z,x',t')dzdt'$$
$$\mu_s^*(\tilde{E} \circ R^j)(X) = s^3 \mu_s^*(\tilde{E}) \circ \mu_s^*(R^j)(X) \sim_{s=0} O(s^{3j-1}).$$

Since μ_s lifts under β_Q as a scaling in $\rho_{\mathfrak{D}}$,

$$\beta^*(\tilde{E} \circ S)(\tilde{X}) = \rho_{\mathfrak{D}}^{-1}\alpha(\tilde{X})$$

with $\alpha(\tilde{X})$ a function polyhomogeneous conormal to the boundary faces of $\tilde{X}$ and smooth in $\rho_{\mathfrak{D}}$. This gives the expansion of $\beta^*(E_+^\kappa)(X)$ at $\mathfrak{D}$. The expansions at $\mathfrak{R}$ and $\mathfrak{L}$ are given by inclusion in the domain D_+. This completes the proof of theorem 5.1. $\square$

6. The heat kernel for the variable coefficient operator with general self-adjoint boundary conditions

Theorem 6.1. *Denote by $E^{\kappa}_{\Theta}(x,x',t)$ the heat kernel for the self-adjoint extension of the operator*

$$H_{\kappa} = -\partial_x^2 + \kappa(x)x^{-2}$$

with domain D_{Θ} for $\Theta \neq +$. $E^{\kappa}_{\Theta}(x,x',t)$ is in D_{Θ} in x and x' and for any N in $\mathbb{N}$ when $0 < \nu < 1$,

$$\beta^*(E^{\kappa}_{-})(X) = \rho_{\mathfrak{T}}^{\infty}\rho_{\mathfrak{D}}^{-1}(\rho_{\mathfrak{R}}\rho_{\mathfrak{L}})^{-\nu+\frac{1}{2}}\rho_{\mathfrak{D}}^{-1} \times C^{\infty}(\tilde{X}) \in \Phi^2_{phg}(X)$$

$$E^{\kappa}_{\Theta}(X) = E^{\kappa}_{-}(X) + \sum_{j=1}^{N-1} E^{\kappa,j}_{\Theta}(X) + R^{\kappa,N}_{\Theta}(X) \in \Phi^2_{phg}(X)$$

$E^{\kappa,j}_{\Theta}(X)$ in $\dot{\Phi}^{2\nu j+2}_{phg}(X)$ with $\rho_{\mathfrak{D}}^{1-2\nu j}\beta^*(E^{\kappa,j}_{\Theta})(\tilde{X})$ smooth in $\rho_{\mathfrak{D}}$ and $R^{\kappa,N}_{\Theta}(X)$ in $\dot{\Phi}^{2\nu N+2}_{phg}(X)$. For $\nu = 0$,

$$E^{\kappa}_{\Theta}(X) = E^{\kappa}_{+}(X) + \sum_{j=1}^{N-1} E^{\kappa,j}_{\Theta}(X) + R^{\kappa,N}_{\Theta}(X) \in \Phi^2(X)$$

$E^{\kappa,j}_{\Theta}(X) = \mathcal{F}^{-1}(\ln^{-j}(C_{\Theta}(\zeta - i0)) *_t R^j(X)$ with $R^j(X)$ in $\dot{\Phi}^{2+j}_{phg}(X)$ and $\beta^*(R^j)(X)$ smooth in $\rho_{\mathfrak{D}}$ and $R^{\kappa,N}_{\Theta}(X)$ in $\dot{\Phi}^{2+N}(X)$ defined in 2.2.

Proof: The construction of the heat kernel $E^{\kappa}_{\Theta}(x,x',t)$ for $\Theta \neq +$ boundary conditions for the variable coefficient operator H_{κ} is by modifying the positive heat kernel $E^{\kappa}_{+}(x,x',t)$.

Proposition 6.2. *Let F(h)(x,t) be the solution to the signaling problem for H_{κ} with $h(t)$ smooth and supported on the positive half line and*

$$F(h)(x,t) \sim_{x=0} h(t)\phi_{-}(x) + O(x^{\nu+\frac{1}{2}})$$

Then

$$F(h)(x,t) = \begin{cases} 2\nu h(t) *_t N'E^{\kappa}_{+}(x,t) & 0<\nu<1 \\ h(t) *_t N'E^{\kappa}_{+}(x,t) & \nu = 0 \end{cases}$$

where $N'E^{\kappa}_{+}(x,t) = \{(x')^{-\nu-\frac{1}{2}}E^{\kappa}_{+}(x,x',t)\}|_{x=0}$

PROOF: The proof is analogous to the proof in the constant coefficient case with $\psi_-(x)$ replaced by the function

$$\psi_-(x) \in C_0^\infty(\mathbb{R}\setminus 0) \qquad \psi_-(x) = \begin{cases} x^{-\nu+\frac{1}{2}}\omega'(x) & 0<\nu<1 \\ x^{\frac{1}{2}}\ln(x)\omega'(x) & \nu = 0 \end{cases}$$

and using the heat kernel $E_+^\kappa(x,x',t)$ for the variable coefficient problem. The cross terms in the construction of $F(h)(x,t)$ give only the leading term of $E_+^\kappa(x,x',t)$ since all other terms vanish at the boundary. □

Proposition 6.3. *There is a pseudo-differential operator with convolution kernel $K_\Theta^\kappa(t)$ that takes boundary data functions for positive solutions of the variable coefficient operator H_k to boundary data functions for $\Theta \neq +$ solutions and supported on $t \geq 0$. For $0<\nu<1$ let $K^\kappa(t)$ be the convolution kernel taking positive boundary data functions to negative ones, then*

$$K^\kappa(t) \sim_{t=0} \sum_{j=0}^{\infty} t_+^{j/2+\nu-1} c_\nu^j \qquad c_\nu^j \in \mathbb{R}$$

$$K_\Theta^\kappa(t) = \sum_{j=0}^{N-1}(-\Theta)^j *_t K^{\kappa,j}(t) + (-\Theta)^N K^{\kappa,N} *_t K_\Theta^\kappa(t)$$

for any positive integer N with $K^{\kappa,j}(t) = K^\kappa *_t \cdots *_t K^\kappa(t)$ convolution of $K^\kappa(t)$ with itself j times. For $\nu = 0$,

$$K_\Theta^\kappa(t) \sim_{t=0} K_\Theta(t) + \sum_{j=1}^{\infty} L_\Theta^j(t)$$

$$\sigma(L_\Theta^j)(\zeta) \sim \ln^{-j-1}(C_\Theta(\zeta - i0)) \sum_{k=0}^{\infty} l^{j,k}(\zeta)$$

$$l^{j,k}(s\zeta) = s^{-j/2-k/2} l^{j,k}(\zeta) \qquad \textit{for } \; s>0$$

PROOF: A map can be found from the negative data $h(t)$ of $F(h)(x,t)$ to the corresponding positive data by making sense of the restriction of $F(h)(x,t)$ by N_+ defined in 3.2. For $\epsilon > 0$ let $\gamma_\epsilon(t)$ be a cut-off function with $\gamma_\epsilon(t) = 1$ for $t > \epsilon$ and vanishing to infinite order at $t = 0$. Let $h(t)$ be a smooth function that vanishes to infinite order at $t = 0$. The restriction of the positive heat kernel $N'E_+^\kappa(x,t)$ for the variable coefficient operator is smooth in t away from the corner $x = t = 0$, so

$$\gamma_\epsilon \cdot N'E_+^\kappa *_t h(x,t) = \int_0^t h(t-t')\gamma_\epsilon(t')N'E_+^\kappa(x,t')dt'$$

is the convolution of two smooth functions and

$$\lim_{\epsilon \to 0} \gamma_\epsilon \cdot N_+ E_+^\kappa *_t h(x,t)$$

has positive boundary data. For $t > 0$ the kernel mapping negative boundary data functions to positive ones is

$$G^\kappa(t) = 2\nu \lim_{\epsilon \to 0} \{(xx')^{-\nu-\frac{1}{2}} \gamma_\epsilon(t) E_+^\kappa(x,x',t)\}|_{x,x'=0}$$

By Theorem 5.1, $\rho_{\mathfrak{D}} \beta^*(E_+^\kappa)(\tilde{X})$ is smooth in $\rho_{\mathfrak{D}}$ and

$$G^\kappa(s^2 t) = s^{-2\nu-1} 2\nu \lim_{\epsilon \to 0} \{(xx')^{-\nu-\frac{1}{2}} \gamma_\epsilon(s^2 t) \mu_s^*(E_+^\kappa)(x,x',t)\}|_{x,x'=0}.$$

therefore $t^{\nu+1} G^\kappa(t)$ is smooth in $t^{\frac{1}{2}}$ and supported on $[0,\infty)$. The leading term of $\beta^*(E_+^\kappa)(\tilde{X})$ at $\mathfrak{D}$ is the same as that $\beta^*(E_+)(\tilde{X})$, so

$$\{t^{\nu-1}(G^\kappa(t) - G(t))\}|_{t=0} = 0$$

and $G^\kappa(t)$ is the kernel of an polyhomogeneous elliptic pseudo-differential operator supported on the positive real line with symbol denoted by $\sigma(G^\kappa)(\zeta)$

$$\sigma(G^\kappa)(\zeta) \sim \sum_{j=0}^{\infty} p_j^\kappa(\zeta) \qquad \text{with} \qquad p_j^\kappa(s\zeta) = s^{\nu-1-j/2} p_j^\kappa(\zeta), \qquad j > 0$$

and principle symbol given by 3.5,

$$p_0^\kappa(\zeta) = p(\zeta) = \mathcal{F}(G)(\zeta) = \begin{cases} (\zeta - i0)^\nu c_\nu^{-1} & \text{for } 0 < \nu < 1 \\ ln(\zeta - i0) c_\nu^{-1} & \text{for } \nu = 0 \end{cases}$$

For $0 < \nu < 1$,

$$\sigma(G^\kappa)(\zeta)/p(\zeta) = 1 + R(\zeta)$$

$$R(\zeta) \sim \sum_{j=1}^{\infty} r^j(\zeta), \qquad r^j(s\zeta) = s^{-j/2} r^j(\zeta) \qquad \text{for } s > 0$$

the Neumann series for $R(\zeta)$ converges and

$$\mathcal{F}(K^\kappa)(\zeta) \sim (\zeta - i0)^{-\nu}\{1 + \sum_{j=0}^{\infty} (-1)^j R^j(\zeta)\}$$

is the symbol of the pseudo-differential operator that maps positive boundary data functions to negative boundary data functions.

If $w(x,t)$ is a solution to the self-adjoint extension of H_κ on the domain D_Θ for $\Theta \neq +$ boundary conditions there is a boundary defining function $h(t)$ for $w(x,t)$ given by

$$NE^\kappa_\Theta(x',t) = \begin{cases} \{x^{\nu-\frac{1}{2}}E^\kappa_\Theta(x,x',t)\}|_{x=0} & \text{for } 0<\nu<1 \\ \{x^{-\frac{1}{2}}/\ln(x)E^\kappa_\Theta(x,x',t)\}|_{x=0} & \text{for } \nu=0 \end{cases}$$

$$h(t) = N^\kappa_\Theta(\phi)(t) = \int_0^\infty NE^\kappa_\Theta(x',t)\phi(x')dx'.$$

Solving to correct the boundary data leads to the same set of equations as in the constant coefficient case. The map from positive boundary data to Θ boundary data with convolution kernel $K^\kappa_\Theta(t)$ is the inverse of the elliptic pseudo-differential operator with kernel $G^\kappa(t)+\Theta\delta(t)$ supported on $t \geq 0$. For $0<\nu<1$

$$\sigma(K^\kappa_\Theta)(\zeta) = (\sigma(G^\kappa)(\zeta)-\Theta)^{-1}$$

satisfies the same recursive relations as when $\kappa(x)\equiv\kappa$ is a constant. For $\nu=0$

$$(\sigma(G^\kappa)(\zeta)+\Theta)/\ln(C_\Theta(\zeta-i0)) = 1 + R(\zeta)/\ln(C_\Theta(\zeta-i0))$$

$$R(\zeta) \sim \sum_{j=1}^\infty r^j(\zeta), \qquad r^j(s\zeta) = s^{-j/2}r^j(\zeta) \qquad s>0$$

$$\sigma(K^\kappa_\Theta)(\zeta) \sim \ln^{-1}(C_\Theta(\zeta-i0)) + \sum_{k=1}^\infty \ln^{-k-1}(C_\Theta(\zeta-i0))R^k(\zeta)$$

□

For $0<\nu<1$, the difference between the positive and negative heat kernels for the variable coefficient operator H_κ is

$$E^\kappa_-(x,x',t) - E^\kappa_+(x,x',t) = 2\nu N'E^\kappa_+(x,t) *_t K^\kappa(t) *_t NE^\kappa_+(x',t)$$

$$\mu^*_s(N'E^\kappa_+(x,t) *_t K^\kappa(t) *_t NE^\kappa_+(x',t)) =$$
$$s^{-1}N'E^\kappa_+(x,t) *_t K^\kappa(t) *_t NE^\kappa_+(x',t) + O(1)$$

so $\beta^*(E^\kappa_-)(\tilde{X})$ has an integer expansion at $\mathfrak{D}$ with leading term of order -1. Since $\beta^*(NE^\kappa_+)(\rho_{\mathfrak{R}},\rho_{\mathfrak{D}},\rho_{\mathfrak{T}})$ vanishes to infinite order at $\mathfrak{T}$, $\beta^*(E^\kappa_-)(\tilde{X})$ has an expansion at $\mathfrak{D}$ like that of $\beta^*(E^\kappa_+)(\tilde{X})$ and vanishes to infinite order at $\mathfrak{T}$. The expansion at $\mathfrak{L}$ and $\mathfrak{R}$ is given by inclusion in the negative domain. For $\Theta \neq \pm$,

$$E^\kappa_\Theta(x,x',t) = E^\kappa_-(x,x',t) + \Theta K^\kappa_\Theta(t) *_t \{E^\kappa_-(x,x',t) - E^\kappa_+(x,x',t)\}$$

$$\mu_s^*(K_\Theta^\kappa(t) *_t \{E_-^\kappa(x,x',t) - E_+^\kappa(x,x',t)\})$$
$$= K_\Theta^\kappa(t) *_t \{E_-^\kappa(x,x',t) - E_+^\kappa(x,x',t)\} + O(s)$$

$\beta^*(E_-)(\tilde{X}) - \beta^*(E_+)(\tilde{X})$ vanishes to infinite order at $\mathfrak{T}$ and $\mathfrak{D}$, so $E_\Theta^\kappa(x,x',t)$ has leading term at the front face $\mathfrak{O}$ equal to that of $E_-^\kappa(x,x',t)$ and the same expansion as $E_+^\kappa(x,x',t)$ at the faces $\mathfrak{T}$ and $\mathfrak{D}$ and is in D_Θ in x and x', the domain of the self-adjoint extension with Θ boundary conditions. The terms in the decomposition of $E_\Theta^\kappa(\tilde{X})$ are given by

$$(-\Theta)^{j+1} K^{\kappa,j} *_t \{E_-(x,x',t) - E_+(x,x',t)\}$$

When $\nu = 0$,

$$E_\Theta(x,x',t) - E_+(x,x',t) = N' E_+^\kappa(x,t) *_t K_\Theta^\kappa(t) *_t N E_+^\kappa(x',t)$$

$\beta^*(N' E_+^\kappa *_t K_\Theta^\kappa *_t N E_+^\kappa)(\tilde{X})$ vanishes to infinite order at $\mathfrak{T}$ and $\mathfrak{D}$ so $\beta^*(E_\Theta^\kappa)(\tilde{X})$ vanishes to infinite order at $\mathfrak{T}$, has an expansion like $\beta^*(E_+)(\tilde{X})$ at $\mathfrak{D}$ and boundary expansion at $\mathfrak{L}$ and $\mathfrak{R}$ given by the lift of the Θ boundary conditions. The terms in the decomposition of $E_\Theta^\kappa(\tilde{X})$ are given by

$$N' E_+^\kappa *_t L_\Theta^j *_t N E_+^\kappa(x,x',t)$$

$\square$

7. Self-adjoint extensions of the Laplacian on metric cones with constant cross-sectional metric

I denote by $C(N)$ the $m = n+1$ dimensional manifold with boundary $N \times \mathbb{R}^+ \ni (x,r)$, for N an n dimensional compact manifold without boundary, with metric on $C(N)$ given by

$$dr^2 + r^{-2} g(r)$$

for $g(r)$ a smooth family of metrics on N. In this section I will take $g(r) = g$ constant in r. The hodge star on the k-forms $\omega_k(r)$ and $dr \wedge \omega_{k-1}(r)$ for this metric is

$$*\omega_k(r) = r^{n-2k} dr \wedge \tilde{*}\omega_k(r), \qquad *(dr \wedge \omega_{k-1}(r)) = r^{n-2k+2} \tilde{*}\omega_{k-1}(r).$$

A tilde will denote the operator restricted to the cross sectional manifold N. In [6] the Laplacian Δ^k on k-forms on the metric cone $C(N)$ with $\dim N = n$ is shown to be

$$\begin{aligned}
&\Delta(h(r)\phi + f(r) dr \wedge \psi) = \\
&\qquad (-h''(r) - (n-2k) r^{-1} h'(r))\phi - r^{-2} h(r) \tilde{\Delta}\phi - 2r^{-3} h(r) dr \wedge \tilde{\delta}\phi \\
&\qquad + (-f''(r) - (n-2k+2) r^{-1} f'(r) + (n-2k+2) r^{-2} f(r)) dr \wedge \psi \\
&\qquad\qquad + r^{-2} f(r) dr \wedge \tilde{\Delta}\psi - 2r^{-1} f(r) d\psi
\end{aligned}$$

Following [8], define the map

$$\begin{aligned}
\psi_k : C_0^\infty((0,\epsilon), \Lambda^k(N) \times \Lambda^{k-1}(N)) &\rightarrow \Lambda^k(C_\epsilon(N)) \\
(\omega_k, \omega_{k-1}) &\rightarrow r^{k-\frac{n}{2}} \pi^* \omega_k(r) + r^{k-\frac{n}{2}-1} \pi^* \omega_{k-1}(r) \wedge dr
\end{aligned}$$

with $\pi : C(N) \rightarrow N$ the projection map. ψ_k is a unitary transformation and the $\mathcal{L}^2$ measure of the rescaled k-forms on $C(N)$ is

$$\begin{aligned}
\|r^{k-\frac{n}{2}} \omega_k\|^2 &= \int_0^\infty \int_N \omega_k(r) \wedge \tilde{*}\omega_k(r) dr \\
\|r^{k-\frac{n}{2}-1} \omega_{k-1}\|^2 &= \int_0^\infty \int_N \omega_{k-1}(r) \wedge \tilde{*}\omega_{k-1}(r) dr
\end{aligned}$$

Since N is compact, the integriblity of the k-forms depends on that of $\omega_k(r)$ and $\omega_{k-1}(r)$ in the radial variable r. The Lapalcain on $C(N)$ with constant cross sectional metric g can be written

$$H_A = -\partial_r^2 + \frac{A}{r^2}$$

$$A = \begin{pmatrix} \tilde{\Delta} + (k - \frac{m}{2} - 1)^2 - \frac{1}{4} & -2d \\ -2\tilde{\delta} & \tilde{\Delta} + (k - \frac{m}{2} + 1)^2 - \frac{1}{4} \end{pmatrix}$$

with A acting on $C^\infty(N, \Lambda^k \oplus \Lambda^{k-1})$. There is an orthogonal basis of $C^\infty(N, \Lambda^k \oplus \Lambda^{k-1})$ given by eigen forms $\{\tilde{\psi}_i^k\}$ for A with eigen values $\{\kappa_i^k\}$.

Proposition 7.1. *H_A is symmetric on $\dot{C}_0^\infty(C(N), \Lambda^k)$.*

PROOF: If $u(r,x)$ is in $\dot{C}_0^\infty(C(N), \Lambda^k)$ then

$$H_A(u)(r,x) = \partial_r^2(u)(r,x) - r^2 A(u)(r,x)$$

is in $\dot{C}_0^\infty(C(N), \Lambda^k)$ and integration by parts is valid. For $v(r,x)$ in $\dot{C}_0^\infty(C(N), \Lambda^k)$

$$\int_0^\infty \int_N H_A(u)(r,x) \wedge \tilde{*}v(r,x) dr =$$

$$\lim_{r \to 0} \int_N \{\partial_r(u)(r,x) \wedge \tilde{*}v(r,x) - u(r,x) \wedge \tilde{*}\partial_r v(r,x)\}$$

$$+ \int_0^\infty \int_N u(r,x) \wedge \tilde{*}H_A(v)(r,x) dr$$

and the cross term vanishes in the limit since $u(r,x)$ and $v(r,x)$ and vanish at the boundary to infinite order. □

The various self-adjoint extensions of H_A on differential k-forms on the cone $C(N)$ are extensions of the operator defined on $\dot{C}_0^\infty(C(N), \Lambda^k)$. Self-adjoint extensions are fixed by the choice of harmonic forms included in the domain.

Proposition 7.2. *Let $\{\tilde{\psi}_i(x)\}$ in $\mathcal{L}^2(N, \Lambda^k \oplus \Lambda^{k-1})$ be a basis of eigen forms for A on N with values in the form bundle $\Lambda^k \oplus \Lambda^{k-1}$ and eigenvalues $A\tilde{\psi}_i(x) = \kappa_i \tilde{\psi}_i(x)$. Each $\tilde{\psi}_i(x)$ corresponds to a generalized harmonic form $r^{\frac{1}{2} + \nu(\kappa_i)} \tilde{\psi}_i(x)$ for H_A in $\mathcal{L}^2(C(N), \Lambda^k)$ with $\nu(\kappa_i) \geq 0$ a real valued function of κ_i.*

If $|k - \frac{m}{2}| < 2$ then for A with small eigen values there is a finite set $\{\tilde{\psi}_i\}_{i=1}^{p_k}$ of eigen k-forms for A with $0 \leq \nu(\kappa_i) < 1$ that correspond to pairs of positive and negative generalized harmonic k-forms for H_A.

PROOF: Using the matrix form of the operator A, computing the harmonic k-forms is messy but straightforward. Since the cross-sectional manifold N is compact, the k-forms on N with the metric g can be divided into spaces of closed, co-closed and harmonic forms which form an orthonormal basis for $\mathcal{L}^2(N, \Lambda^k \oplus \Lambda^{k-1})$. A generalized harmonic k-forms on $C(N)$ can be written

$$u = r^l(\phi + dr \wedge \psi)$$

where (ϕ, ψ) in $(\Lambda^k(N), \Lambda^{k-1}(N))$ are eigen forms for $\tilde{\Delta}$ on N with eigen values $\mu \geq 0$. For $\mu > 0$ the harmonic forms can be classified into four types:

$$\begin{array}{llll}
\pm 1) & u_\pm = r^{\frac{1}{2} \pm \nu_1(\mu)} \phi & \delta\phi = 0 & \nu_1(\mu) = \sqrt{\mu + (k - \frac{m}{2} - 1)^2} \\
\pm 2) & u_\pm = r^{\frac{1}{2} \pm \nu_2(\mu)} dr \wedge \psi & d\psi = 0 & \nu_2(\mu) = \sqrt{\mu + (k - \frac{m}{2} + 1)^2} \\
\pm 3) & u_\pm = r^{\frac{1}{2} \pm \nu_3(\mu)} (d\psi + c_+(\mu) dr \wedge \psi) & \delta\psi = 0 & \nu_3(\mu) = \sqrt{\mu + (k - \frac{m}{2})^2} + 1 \\
\pm 4) & u_\pm = r^{\frac{1}{2} \pm \nu_4(\mu)} (d\psi + c_-(\mu) dr \wedge \psi) & \delta\psi = 0 & \nu_4(\mu) = |\sqrt{\mu + (k - \frac{m}{2})^2} - 1|
\end{array}$$

$$c_\pm(\mu) = (k - \frac{m}{2}) \pm \sqrt{\mu + (k - \frac{m}{2})^2}$$

If $\nu_i(\mu) = 0$ there is a negative solution with leading term equal to $r^{\frac{1}{2}} \ln(r)$. When $\tilde{\Delta}\phi = \tilde{\Delta}\psi = 0$ the matrix A is diagonal,

$$A = ((k - \frac{m}{2} - 1)^2 - \frac{1}{4}, (k - \frac{m}{2} + 1)^2 - \frac{1}{4})$$

and there are only solutions of types $\pm 1)$ and $\pm 2)$ with

$$\nu_1(0) = \|k - \frac{m}{2} - 1\|, \qquad \nu_2(0) = \|k - \frac{m}{2} + 1\|.$$

The harmonic k forms fall into three ranges:

+1) du and δu are in $\mathcal{L}^2(C(N), \Lambda^k)$

−1) u is in $\mathcal{L}^2(C(N), \Lambda^k)$ but du and δu are not if

$$0 \leq \mu + (k - \frac{m}{2} - 1)^2 < 1$$

+2) du and δu are in $\mathcal{L}^2(C(N), \Lambda^k)$

−2) u is in $\mathcal{L}^2(C(N), \Lambda^k$, but du and δu are not if

$$0 \leq \mu + (k - \frac{m}{2} + 1)^2 < 1$$

+3) du and δu are in $\mathcal{L}^2(C(N), \Lambda^k)$

−3) u is never in $\mathcal{L}^2(C(N), \Lambda^k)$

+4) du and δu are in $\mathcal{L}^2(C(N), \Lambda^k)$

−4) u is in $\mathcal{L}^2(C(N), \Lambda^k)$ but du and δu are not if

$$0 \leq \mu + (k - \frac{m}{2})^2 < 4$$

The following is a list of possible generalized negative harmonic eigen forms for the operator H_A on $C^\infty(C(N),\Lambda^k)$ by form dimension k with respect to the dimension m of the cone $C(N)$.

$k=\frac{m}{2}$	$r^{\frac{1}{2}-\nu_4(\mu)}(d\psi + c_-(\mu)dr\wedge\psi)$	$0<\mu<1$
	$r^{\frac{1}{2}}\ln(r)(d\psi + c_-(\mu)dr\wedge\psi)$	$\mu=1$
$k=\frac{m}{2}+\frac{1}{2}$	$r^{\frac{1}{2}-\nu_1(\mu)}\phi$	$0\le\mu<\frac{3}{4}$
	$r^{\frac{1}{2}-\nu_4(\mu)}(d\psi + c_-(\mu)dr\wedge\psi)$	$0\le\mu<\frac{15}{4}$
$k=\frac{m}{2}-\frac{1}{2}$	$r^{\frac{1}{2}-\nu_2(\mu)}dr\wedge\psi$	$0\le\mu<\frac{3}{4}$
	$r^{\frac{1}{2}+\nu_4(\mu)}(d\psi + c_-(\mu)dr\wedge\psi)$	$0\le\mu<\frac{15}{4}$
$k=\frac{m}{2}+1$	$r^{\frac{1}{2}}\ln(r)\phi$	$\mu=0$
	$r^{\frac{1}{2}-\nu_1(\mu)}\phi$	$0<\mu<1$
	$r^{\frac{1}{2}-\nu_4(\mu)}(d\psi + c_-(\mu)dr\wedge\psi)$	$0<\mu<1$
$k=\frac{m}{2}-1$	$r^{\frac{1}{2}}\ln(r)dr\wedge\psi$	$\mu=0$
	$r^{\frac{1}{2}-\nu_2(\mu)}dr\wedge\psi$	$0<\mu<1$
	$r^{\frac{1}{2}-\nu_4(\mu)}(d\psi + c_-(\mu)dr\wedge\psi)$	$0<\mu<1$
$k=\frac{m}{2}+\frac{3}{2}$	ϕ	$\mu=0$
	$r^{\frac{1}{2}-\nu_1(\mu)}\phi$	$0<\mu<\frac{3}{4}$
	$r^{\frac{1}{2}-\nu_4(\mu)}(d\psi + c_-(\mu)dr\wedge\psi)$	$0<\mu<\frac{7}{4}$
$k=\frac{m}{2}-\frac{3}{2}$	$dr\wedge\psi$	$\mu=0$
	$r^{\frac{1}{2}-\nu_2(\mu)}dr\wedge\psi$	$0<\mu<\frac{3}{4}$
	$r^{\frac{1}{2}-\nu_4(\mu)}(d\psi + c_-(\mu)dr\wedge\psi)$	$0<\mu<\frac{7}{4}$

□

Let $\{\psi_i^\pm\}_{i=1}^{p_k}$ be the pairs of orthonormal eigen k-form with eigenvalues $\{\kappa_i\}_{i=1}^{p_k}$ for A acting on $\mathcal{L}^2(N,\Lambda^k\oplus\Lambda_{k-1})$ with $0\le\nu(\kappa_i)<1$. For $\gamma(r)$ a cut off function with $\gamma(r)=1$ near $r=0$ define the set of compactly supported functions for $i\in(1,\cdots,p)$

$$\psi_i^+(r,x) = r^{\nu_i+\frac{1}{2}}\tilde{\psi}_i(x)\gamma(r) \qquad \psi_i^-(r,x) = \begin{cases} r^{\frac{1}{2}-\nu_i}\tilde{\psi}_i(x)\gamma(r) & \text{for } 0<\nu_i<1 \\ r^{\frac{1}{2}}\ln(r)\tilde{\psi}(x)\gamma(r) & \text{for } \nu_i=0 \end{cases}$$

Definition 7.1.

$$D^m_{min} = \{u(r,x)\in\mathcal{L}^2(C(N),\Lambda^k) : \exists\{u_n\}\in\dot{C}_0^\infty(C(N),\Lambda^k) \text{ such that } u_n \xrightarrow{\kappa} u\}$$

$$D^m_{max} = \{u(r,x)\in\mathcal{L}^2(C(N),\Lambda^k) :< u, H_A v > \;\le\; C\|v\|_{\mathcal{L}^2}$$
$$\text{for all } v(x)\in C_0^\infty(C(N),\Lambda^k)\}$$

Proposition 7.3. *If $u(r,x)$ is in D^m_{max} then*

$$u(r,x) = u'(r,x) + u''(r,x)$$

with $u''(r,x)$ in D^m_{min} and

$$u'(r,x) = \sum_{k=1}^{p}\{u_i^+\psi_i^+(r,x) + u_i^-\psi_i^-(r,x)\}, \qquad u_i^\pm \in \mathbb{R}$$

PROOF: Let $\pi_i(u)(r,x)$ be projection onto the *ith* eigenspace for A, then

$$(\partial_r^2 - r^{-2}A)\pi_i(u)(r,x) = (\partial_r^2 - r^{-2}\kappa_i)\pi_i(u)(r,x)$$

$\pi_i(u)(r,x)$ is in D_Θ in r and the proof reduces to the 1 dimensional case treated in Theorem 1.2. □

Proposition 7.4. *If $u(r,x)$ and $v(r,x)$ are in D^m_{max} then*

$$< H_A u, v > - < u, H_A v > \;=\; \sum_{i=1}^{p} c_i(u_i^+ v_i^- - u_i^- v_i^+) \qquad c_i = \begin{cases} 2\nu_i & \text{if } \nu_i \neq 0 \\ 1 & \text{if } \nu_i = 0 \end{cases}$$

PROOF: Since $u(r,x)$ and $v(r,x)$ are in D^m_{max},

$$u(r,x) = u'(r,x) + u''(r,x), \qquad v(r,x) = v'(r,x) + v''(r,x)$$

with $u''(r,x)$ and $v''(r,x)$ in D^m_{min} and

$$u'(r,x) = \sum_{i=1}^{p} u_i^+\psi_i^+(r,x) + u_i^-\psi_i^-(r,x) \qquad v'(r,x) = \sum_{j=1}^{p} v_j^+\psi_j^+(r,x) + v_j^-\psi_j^-(r,x)$$

$$\begin{aligned}
&< H_A u, v > - < u, H_A v > \\
&\quad = \lim_{r\to 0}\int_N \{\partial_r(u)(r,x)\wedge \tilde{*}v(r,x) - u(r,x)\wedge\tilde{*}\partial_r(v)(r,x)\} \\
&\quad = \lim_{r\to 0}\int_N \{\partial_r(u')(r,x)\wedge \tilde{*}v'(r,x) - u'(r,x)\wedge\tilde{*}\partial_r(v')(r,x)\} \\
&= \lim_{r\to 0}\int_N \sum_{i=1}^{p}\{\partial_r(u_i^+\psi_i^+(r,x) + u_i^-\psi_i^-(r,x))\wedge\tilde{*}(v_i^+\psi_i^+(r,x) + v_i^-\psi_i^-(r,x)) \\
&\quad - (u_i^+\psi_i^+(r,x) + u_i^-\psi_i^-(r,x))\wedge\tilde{*}\partial_r(v_i^+\psi_i^+(r,x) + v_i^-\psi_i^-(r,x))\} \\
&\quad = \sum_{i=1}^{p} c_i\{u_i^+ v_i^- - u_i^- v_i^+\}
\end{aligned}$$

by restriction to the ith eigen space of A and Theorem 1.3. □

Definition 7.2. Let S_k be the $2p_k$ dimensional vector space spanned by the set of vectors $\{\psi_i^{\pm}\}_{i=1}^{p_k}$ with $\psi_i^-(r,x)$ in $\mathcal{L}^2(C(N),\Lambda^k)$, with bilinear form $\omega_k(\cdot,\cdot)$ on S_k given by

$$\omega_k(\psi_i^+,\psi_i^-) = -\omega_k(\psi_i^-,\psi_i^+) = c_i \qquad \omega_k(\psi_i^+,\psi_i^+) = \omega_k(\psi_i^-,\psi_i^-) = \omega_k(\psi_i^{\pm},\psi_j^{\pm}) = 0$$

Let G_k be the Grassmannian of Lagrangian p_k-planes in S_k, the manifold of all p_k-planes on which the form $\omega_k(\cdot,\cdot)$ vanishes. A point Γ in G_κ is a set of p_k linearly independent vectors in S_k

$$\Gamma = \{\sum_{i=1}^{p}\{a_{i1}^+\psi_i^+(r,x) + a_{i1}^-\psi_i^-(r,x)\},\cdots,\sum_{i=1}^{p}\{a_{ip}^+\psi_i^+(r,x) + a_{ip}^-\psi_i^-(r,x)\}\} \qquad a_{ij}^{\pm}\in\mathbb{R}$$

Proposition 7.5. *A point Γ in G_k can be represented as a matrix with diagonal entries $b_{ii}\psi_i^-(r,x)+\Theta_{ii}\psi_i^+(r,x)$ and off diagonal entries $\Theta_{ij}\psi_i^+(r,x)$ with $b_{ij},\Theta_{ij}\in\mathbb{R}$,*

$$\Gamma = \begin{pmatrix} b_{11}\psi_1^-(r,x)+\Theta_{11}\psi_1^+(r,x) & \cdots & \Theta_{1p}\psi_1^+(r,x) \\ \Theta_{21}\psi_2^+(r,x) & \cdots & \Theta_{2p}\psi_2^+(r,x) \\ \vdots & \vdots & \vdots \\ \Theta_{p1}\psi_p^+(r,x) & \cdots & b_{pp}\psi_p^-(r,x)+\Theta_{pp}\psi_p^+(r,x) \end{pmatrix}$$

such that either $b_{ii}=1$ or $b_{ii}=0$, in which case $\Theta_{ii}=1$ and $\Theta_{ij}=0$ for $i\neq j$. The rows of Γ are vectors in S_k pairwise orthogonal with respect to $\omega_k(\cdot,\cdot)$.

PROOF: Γ can be represented as a matrix with each column a basis vector for Γ. The p_k-plane Γ is preserved under column reduction of this matrix, bringing it into the above form by reduction on the negative elements $\psi_i^-(r,x)$. By scaling of the basis vectors, $b_{ii}=1$ or 0.

The vanishing of the bilinear form $\omega_k(\cdot,\cdot)$ on Γ means that for $(i,j)\in(1,\cdots,p)$

$$\omega(b_{ii}\psi_i^- + \sum_{k=1}^{p}\Theta_{ki}\psi_k^-, b_{jj}\psi_j^- + \sum_{l=1}^{p}\Theta_{lj}\psi_l^-) = c_ib_{ii}\Theta_{i,j} + c_jb_{jj}\Theta_{ji}, \qquad c_i,c_j\neq 0$$

If $b_{ii}=0$ for some i, assume that $b_{ii}=0$ for $i\in(1,\cdots,m)$, $1\leq m<p$ and $b_{ii}=1$ for $i\in(m+1,\cdots p)$. Then the vanishing of ω shows that for $j\in(m+1,\cdots,p)$, $\Theta_{ji}=0$

and Γ has the form

$$\Gamma = \begin{pmatrix} \Theta & A \\ O & B \end{pmatrix} \qquad \Theta = \begin{pmatrix} \Theta_{11}\psi_1^+(r,x) & \cdots & \Theta_{1m}\psi_m^+(r,x) \\ \vdots & & \vdots \\ \Theta_{m1}\psi_1^+(r,x) & \cdots & \Theta_{mm}\psi_m^+(r,x) \end{pmatrix}$$

O a $(p-m)\times m$ matrix of zeros, so by column reduction preserving the p-plane, $\Theta_{ii}=1$ and $\Theta_{ij}=0$ for $i\neq j$. □

Definition 7.3. For $\Gamma = (\Gamma_1,\cdots,\Gamma_p)\in G_k$, Γ_i vectors in S^k, let D_Γ^m be the space of functions $\phi(r,x)$ in $D_{\max}^m$ with

$$\phi(r,x) = \sum_{k=1}^{p}\{\phi_i^+\psi_i^+(r,x) + \phi_i^-\psi_i^-(r,x)\} + v(r,x), \qquad v(r,x)\in D_{min}$$

near $r=0$ such that

$$\omega_k(\sum_{k=1}^{p}\{\phi_i^+\psi_i^+ + \phi_i^-\psi_i^-\},\Gamma_j) = 0, \qquad \text{all } j\in(1,\cdots,p)$$

Theorem 7.6. *The set of self-adjoint boundary conditions for H_A is isomorphic to the Grassmannian G_k of Lagrangian p_k-planes in S_k.*

PROOF: Each point Γ in G_k determines a unique domain D_Γ^m. If $u(r,x)$ and $v(r,x)$ are in D_{max} then by Prop. 7.3

$$u(x) = u'(x)+u''(x) \qquad v(x) = v'(x)+v''(x)$$

$$u''(x),\ \ v''(x)\in D_{min}, \qquad u'(x),\ \ v'(x)\in S_k$$

$$\omega(u',v') \ = \ <H_Au,v> - <u,H_Av>$$

therefore each D_Γ^m represents the domain of a self-adjoint extension of the operator H_A. □

The diagonal matrices

$$\Gamma_+ = \{\psi_1^+,\cdots\psi_p^+\} \qquad \text{and} \qquad \Gamma_- = \{\psi_1^-,\cdots\psi_p^-\}$$

fix the domains $D_\pm^m$ of the positive and negative extensions of H_A, respectively.

Theorem 7.7. *Let $E_\Gamma(r,r',x,x',t)$ be the heat kernel for H_A with domain D_Γ^m.*

$$(\partial_t - \partial_r^2 + r^{-2}A)E_\Gamma(r,r',x,x',t) = 0 \qquad \textit{and} \qquad E_\Gamma(r,r',x,x',0) = \delta_{(r,x)}(r',x')$$

$E_\Gamma(r,r',x,x',t)$ is smooth in the interior of X^m and symmetric in (r,x) and (r',x') and for $t>0$, $E_\Gamma(r,r',x,x',t)$ is in D^m_Γ in (r,x) and (r',x') up to $r=r'=0$.

PROOF: The proof follows from the Spectral Theorem 1.6. □

Definition 7.4. Let μ_s for $s\in(0,\infty)$ be the 1-parameter family of scalings on X

$$\mu_s(r,r',x,x',t)=(sr,sr',x,x',s^2t)$$

Lemma 7.8. *The positive heat kernel $E_+(r,r',x,x',t)$ is invariant under μ_s,*

$$\mu_s^*(E_+)(r,r',x,x',t)=s^{-1}E_+(r,r',x,x',t)$$

PROOF: For A constant in r, H_A is invariant under μ_s. If $\phi(r,x)$ is in D^m_+, the domain of the positive extension, then $\phi(r,x)=\phi'(r,x)+\phi''(r,x)$ with $\phi''(r,x)$ in D^m_{min} invariant under μ_s and

$$\phi'(r,x)=\sum_{j=1}^{p}\phi_i r^{\nu_i+\frac{1}{2}}\tilde{\psi}_i(x)\gamma(r),\qquad \phi_i\in\mathbb{R}$$

with no relations on the ϕ_i's. $\gamma(sr)=\gamma(r)=1$ near $r=0$ so

$$\mu_s^*(\phi')(r,x)\sim_{r=0}\sum_{j=1}^{p}\phi_i s^{\nu_i+\frac{1}{2}}r^{\nu_i+\frac{1}{2}}\tilde{\psi}_i(x)\gamma(sr)$$

satisfies the g_+ boundary conditions, and $\mu_s^*(\phi)(r,x)$ is in D^m_+ and the degree of the scaling is by reduction to the one dimensional case treated in Prop. 1.9. □

8. Heat kernels on the metric cones with general self-adjoint boundary conditions

In the following discussion the subscripts referring to the form dimension k will be suppressed. The heat space for H_A is a $2m+1$ dimensional manifold $N \times N \times [0,\infty)^3 = X^m$. The heat kernel for self-adjoint extensions of H_A will be described on the blown up space $\tilde{X}^m$ constructed as before by first blowing up parabolically in the t variable along the diagonal at time zero,

$$\Delta_0^m = \{(r, r, x, x, 0) \in X^m : x \in N, r \in \mathbb{R}^+\}$$

then blowing up parabolically in the t variable the sub-manifold corresponding to the boundary of $C(N)$

$$N \times N = \{(0, 0, x, x', 0) \in X^m : x, x' \in N\}.$$

Define the parabolic quadrant and the parabolic half-sphere defined in 2.1

$$Q_p^m = \{(\phi, x, x') \in \mathbb{R}^{2m+1} : \phi \in Q_p, x \in N\}$$
$$S_p^m = \{\omega = (\omega_0, \omega') \in \mathbb{R}^{m+1} : \omega_0 \in \mathbb{R}^+, \omega' \in \mathbb{R}^m\}$$

and denote by X_Q^m the parabolic blow up of the heat space X^m for H_A at the boundary face $N \times N$. Define

$$X_Q^m = [X^m \setminus N \times N] \sqcup [Q_p^m].$$

This blow-up is independent of the cross-sectional variables x and x'. The final blown-up space $\tilde{X}^m$ is constructed by blowing up X_Q^m along the lift $\tilde{\Delta}_0^m$ of the diagonal in X^m to X_Q^m

$$\tilde{X}^m = [X_Q^m \setminus \tilde{\Delta}_0^m] \sqcup [S_p^m \times \mathbb{R}^m].$$

$\tilde{X}^m$ has five boundary faces, the left and right faces $\mathfrak{L}$ and $\mathfrak{R}$ corresponding under this blow-up to the faces $x = 0$ and $x' = 0$, a front face $\mathfrak{O}$ and a diagonal face $\mathfrak{D}$ corresponding to the blow-ups of the boundary $N \times N$ and the diagonal face Δ_0^m and a temporal face $\mathfrak{T}$ corresponding to $t = 0$ away from the blown-up faces.

There is a blow-down map $\beta : \tilde{X}^m \to X$ that is an isomorphism on the interiors of these spaces. Let $\rho_{\mathfrak{F}}$ be a boundary defining function on $\tilde{X}^m$ for the face $\mathfrak{F}$. Coordinate functions can be chosen to satisfy

$$\beta^*(x_i) = x_i, \quad \beta^*(x_i') = x_i' \quad \beta^*(r) = \rho_{\mathfrak{R}}\rho_{\mathfrak{O}}, \quad \beta^*(r') = \rho_{\mathfrak{L}}\rho_{\mathfrak{O}}, \quad \beta^*(t) = \rho_{\mathfrak{T}}\rho_{\mathfrak{D}}^2\rho_{\mathfrak{O}}^2.$$

Definition 8.1. For k in $\mathbb{N}$ let $\Phi^k(X^m)$ be the space of distributions $R(X^m)$ on X^m such that $\beta^*(R)(\tilde{X}^m)$ is a function conormal to the boundary faces of $\tilde{X}^m$ with an integer expansion at $\mathfrak{D}$ with leading power $-m$, vanishing to infinite order at $\mathfrak{T}$, and with leading term of order $k-3$ at $\mathfrak{O}$.

Let $\dot{\Phi}^k(X^m)$ be the space of distributions $R(Base^m)$ in $\Phi^k(X^m)$ such that $\beta^*(R)(\tilde{X}^m)$ vanishes to infinite order at $\mathfrak{T}$ and $\mathfrak{D}$. Let $\dot{\Phi}^k_{phg}(X^m) \subset \dot{\Phi}^k(X^m)$ and $\Phi^k_{phg}(X^m) \subset \Phi^k(X^m)$ be distributions that lift to $\tilde{X}^m$ to have classical polyhomogeneous conormal expansions at the boundary faces of $\tilde{X}^m$.

Theorem 8.1. *$E_+(X)$ is in $\Phi^2_{phg}(X^m)$ and is in D^m_+ defined in 7.3 in (r,x) and (r',x') and $\beta^*(E_+)(\tilde{X}^m)$ is homogeneous in $\rho_{\mathfrak{O}}$ of degree -1.*

PROOF: the method of proof is identical to the one dimensional case for Theorem 2.2 so I will omit the details here. The parabolic scaling μ_s on X^m lifts under β to $\tilde{X}^m$ as a scaling in $\rho_{\mathfrak{O}}$, therefore the homogeneity of $E_+(X^m)$ under μ_s shows that

$$\beta^*(E_+)(\tilde{X}^m) = \rho_{\mathfrak{O}}^{-1}\beta^*(E_+)(Q^m_p).$$

For $t>0$, by the Spectral Theorem 1.6, $E_+(X^m)$ is in D^m_+ in (r,x) and (r',x') up to the corner $r=r'=0$, and by scaling in $\rho_{\mathfrak{O}}$ this is valid up to the front face $\mathfrak{O}$ away from the lift of $t=0$. There is a map $\alpha : X^m \rightarrow X^m|_{r'>\epsilon}$ with coordinates (y,x,x',ρ,s) that is an isomorphism for $\epsilon>0$ and satisfying

$$\beta^*_\alpha(\frac{r}{r'}) = y, \quad \beta^*_\alpha(\frac{t}{(r')^2}) = s, \quad \beta^*_\alpha(r') = \rho, \quad \beta^*_\alpha(x_i) = x_i, \quad \beta^*_\alpha(x'_i) = x'_i$$

With these coordinates the operator $\partial_t + H_A$ lifts to

$$\beta^*_\alpha(\partial_t + H_A) = \rho^{-2}\{\partial_s - \partial_y^2 + y^{-2}A\}$$

and the initial condition for $E_+(X^m)$ lifts to

$$\beta^*_\alpha(E_+(r,r',x,x',0)) = \rho^{-1}\delta_{y,x}(y',x')$$

a construction analogous to the one used in the one dimensional case shows that the expansion of $\beta^*(E_+)(\tilde{X}^m)$ at the diagonal face $\mathfrak{D}$ is in the same powers as that of the lift to $\tilde{X}^m$ of the kernel of an operator with a smooth perturbation, and that $E_+(X^m)$ vanishes to infinite order at the temporal face $t=0$ away from the diagonal. By homogeneity, this expansion is valid up to the front face $\mathfrak{O}$. □

Theorem 8.2. *The heat kernel $E_\Gamma(X^m)$ for the extension of H_A with domain D^m_Γ defined in 7.3 is in $\Phi^2(X^m)$ and in D^m_Γ in (r,x) and (r',x'). Let $\{\nu_i\}_{i=1}^p$ be the set of values given by 7.2 for which negative solutions are in $\mathcal{L}^2(C(N),\Lambda^k)$. If $b_{ii}=0$ for $i\in(1,\cdots,q)$ then*

$$E_\Gamma(X^m) = E_+(X^m) + \sum_{j,i=q+1}^{p} E_\Gamma^{ij}(X^m) \in \Phi^2(X^m)$$

with $\beta^*(E_\Gamma^{ij})(\tilde{X}^m)$ vanishes to infinite order at $\mathfrak{T}$ and $\mathfrak{D}$ and satisfying

$$\begin{array}{ll} \nu_i,\nu_j \neq 0 & E_\Gamma^{ij}(X^m)\in \dot{\Phi}_{phg}^{\nu_i+\nu_j+2}(X^m) \\ \nu_i=0,\nu_j\neq 0 & E_\Gamma^{ij}(X^m)\in \dot{\Phi}^{\nu_j+2}(X^m) \\ \nu_i=\nu_j=0 & E_\Gamma^{ij}(X^m)\in \dot{\Phi}^{2}(X^m) \end{array}$$

PROOF: The theorem is proved by modifying the boundary behavior of the heat kernel $E_+(X^m)$ for the domain D^m_+. The construction of the heat kernel $E_-(X^m)$ for the self-adjoint extension of H_A defined on D^m_- defined in 7.3 is analogous to the one dimensional case, via the decomposition of the heat kernels into eigen spaces for A acting on $C^\infty(N,\Lambda^k\oplus\Lambda^{k-1})$. For each $i\in(1,\cdots,p)$ there is a distribution $NE^i_+(r',x',t)$ on $N\times[0,\infty)^2$ defined by

$$NE^i_+(r',x',t) = \{r^{-\nu_i-\frac{1}{2}}\int_N E_+(r,r',x,x',t)\tilde{\psi}_i(x)dx\}|_{r=0}$$

and similarly for $N'E^i_+(r,x,t)$. Let $u(r,x,t)$ be a solution to the heat equation ∂_t+H_A with D^m_+ boundary data and initial data

$$u(r,x,0)=\phi(r,x)\in\dot{C}_0^\infty(C(N),\Lambda^k).$$

Boundary data functions $f_i(t)$ for $u(r,x,t)$ can be defined by setting

$$f_i(t) = \int_0^\infty\int_N NE^i_+(r',x',t)\phi(r',x')dx'dr'.$$

Let $\{h_i(t)\}_{i=1}^p$ be a set of smooth functions on $\mathbb{R}^+$ vanishing to infinite order at $t=0$. We construct a solution to $\partial_t - H_A$ with boundary data $\sum_{i=1}^p h_i(t)\psi_i^-(r,x)$ Applying the heat operator gives the error term

$$(\partial_t-\partial_r^2+r^{-2}A)\sum_{i=1}^p h_i(t)\psi_i^-(r,x) = \sum_{i=1}^p\{\partial_t h_i(t)\psi_i^-(r,x)+h_i(t)H_A\psi_i^-(r,x)\}.$$

This is removed by subtracting off the term

$$\sum_{i=1}^p\int_0^t\int_0^\infty\int_N E_+(r,r',x,x',t-t')\{\partial_t h_i(t')\psi_i^-(r',x')+h_i(t')H_A\psi_i^-(r',x')\}dx'dr'dt'.$$

For each i, integration by parts in the t variable gives

$$h_i(t)\psi_i^-(r,x) + h_i(t) *_t \{H'_A(E_+)(\psi_i^-)(r,x,t) + E_+(H'_A(\psi_i^-))(r,x,t)\}$$

where as before the prime on the operator H_A means that it acts on the primed variables. For fixed $i \in (1,\cdots,p)$,

$$\int_0^\infty \int_N \{H'_A(E_+)(r,r',x,x',t)\psi_i^-(r',x') + E_+(r,r',x,x',t)H'_A(\psi_i^-)(r',x')\}dr'dx'$$
$$= lim_{\epsilon\to 0} \int_N \{\partial_{r'}E_+(r,r',x,x',t)\psi_i^-(r',x') + E_+(r,r',x,x',t)\partial_{r'}\psi_i^-(r',x')\}dx'|_{r=\epsilon}$$
$$= c_i N' E_+^i(r,x,t)$$

with c_i defined by 7.2. This gives a set of p independent convolution equations that are equivalent to those from the one dimensional case, and there is a map $K_i(f_i)(t) \to h_i(t)$ for each i that is a convolution operator in t given by the kernels

$$K_i(t) = \begin{cases} c_i t_+^{\nu_i - 1} & \text{if } 0 < \nu_i < 1 \\ \mathcal{F}^{-1}((ln(\zeta - i0))^{-1})(t) & \text{if } \nu_i = 0 \end{cases}$$

therefore the heat kernel for the extension of H_A with domain D_- is

$$E_-(r,r',x,x',t) = E_+(r,r',x,x',t) + \sum_{i=1}^{p} c_i N' E_+^i(r,x,t) *_t K_i(t) *_t NE_+^i(r',x',t).$$

Similar maps are used to construct the heat kernel for general self-adjoint extensions of H_A fixed by the boundary condition Γ. Let $w(r,x,t)$ be a solution to the heat equation for the self-adjoint extension of H_A on the domain D_Γ^m with initial data $\phi(r,x)$ in $\dot{C}_c^\infty(C(N),\Lambda^k)$. Boundary data functions for $w(r,x,t)$ are defined by setting

$$NE_\Gamma^i(r',x',t) = \{r^{\nu_i - \frac{1}{2}} \int_N E_\Gamma(r,r',x,x',t)\tilde{\psi}(x')dx'\}|_{r=0}$$
$$h_i(t) = \int_0^\infty \int_N NE_\Gamma^i(r',x',t)\phi(r',x')dr'dx'.$$

The solution $w(r,x,t)$ will be constructed by adding the purely negative initial data of $w(r,x,t)$,

$$\sum_{i=1}^{p} b_{ii} h_i(t)\psi_i^-(r,x)$$

for $h_i(t)$ smooth functions on $\mathbb{R}^+$ vanishing to infinite order at the boundary $t = 0$, to the solution $u(r,x,t)$ for the positive extension with the same initial data and adjusting the positive boundary data to equal that of $w(r,x,t)$,

$$\sum_{i=1}^{p} h_i(t) \sum_{j=1}^{p} \Theta_{ij}\psi_j^+(r,x).$$

Theorem 8.3. *There is a matrix $\{K_\Gamma^{ij}(t)\}$ of convolution kernels in t that maps the boundary data functions $f_i(t)$ for $u(r,x,t)$ to the boundary data functions $h_i(t)$ of $w(r,x,t)$. For $b_{ii}=0$,*

$$K_\Gamma^{ij}(t) = \begin{cases} \delta(t) & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

For $b_{ii}=1$, if $\nu_i=0$ for all $i\in(1,\cdots,q)$ and $\nu_i\neq 0$ all $i\in(q+1,\cdots,p)$ for $1\le q\le p$ then

$$K_\Gamma^{ii}(t) = \begin{cases} t_+^{\nu_i-1} + o(t^{\nu_i-1}) & \text{if } i\notin(1,\cdots,q) \\ \tau_{\Theta_{ii}}(t) + o(t^{-1}) & \text{if } i\in(1,\cdots,q) \end{cases}$$

$$K_\Gamma^{ij}(t) = \begin{cases} t_+^{\nu_i+\nu_j-1} + o(t^{\nu_i+\nu_j-1}) & \text{if } i,j\notin(1,\cdots,q) \\ t_+^{\nu_i-1} *_t \tau_{\Theta_{jj}}(t) + o(t^{\nu_i-1}) & \text{if } i\notin(1,\cdots,q),\ j\in(1,\cdots,q) \\ \tau_{\Theta_{ii}}(t) *_t \tau_{\Theta_{jj}}(t) + o(t^{-1}) & \text{if } i,j\in(1,\cdots,q) \end{cases}$$

with $\tau_{\Theta_{ii}}(t) = \mathcal{F}^{-1}(1/ln(C_\Theta(\zeta - i0)))$.

Proof: If $b_{ii}=0$ for some i then $f_i(t)=h_i(t)$, by the diagonalization of Γ. Assume that $b_{ii}=1$ for all i. Solving the signaling problem for each i gives the term

$$\int_0^t\int_0^\infty\int_N E_+(r,r',x,x',t-t')\{\partial_t h_i(t')\psi_i^-(r',x') + h_i(t)H_A'(\psi_i^-)(r',x')\}dx'dr'dt'$$
$$= h_i(t)\psi_i^-(r,x) + h_i(t) *_t 2\nu_i N' E_+^i(r,x,t)$$

The positive data for the solution to the signaling problem with negative boundary data $\{h_1(t),\cdots,h_p(t)\}$ is

$$\{G_1(t) *_t h_1(t),\cdots,G_p(t) *_t h_p(t)\}.$$

Where by Prop. 3.5

$$\mathcal{F}(G_j)(\zeta) = \begin{cases} c_j\Gamma(\nu_j)e^{i\pi\nu_j/2}(\zeta+i0)^{\nu_j} & 0<\nu<1 \\ \frac{1}{2}\ln(\zeta - i0) & \nu=0 \end{cases}$$

Adding this to $u(r,x,t)$ and correcting the positive boundary data to match that of Γ gives a set of convolution equations,

$$f_i(t) = G_i(t) *_t h_i(t) + \sum_{j=1}^{p} \Theta_{ij} h_i(t).$$

Let $\hat{f}_i(t)$ denote the Fourier transform of $f_i(t)$. Taking the Fourier transforms of the above equations leads to the matrix equation

$$\begin{pmatrix} \hat{f}_1(\zeta) \\ \hat{f}_2(\zeta) \\ \vdots \\ \hat{f}_p(\zeta) \end{pmatrix} = \begin{pmatrix} \mathcal{F}(G_1)(\zeta) + \Theta_{11} & \Theta_{12} & \cdots & \Theta_{1p} \\ \Theta_{21} & \mathcal{F}(G_2)(\zeta) + \Theta_{22} & \cdots & \Theta_{2p} \\ \vdots & \vdots & & \vdots \\ \Theta_{p1} & \Theta_{p2} & \cdots & \mathcal{F}(G_p)(\zeta) + \Theta_{pp} \end{pmatrix} \begin{pmatrix} \hat{h}_1(\zeta) \\ \hat{h}_2(\zeta) \\ \vdots \\ \hat{h}_p(\zeta) \end{pmatrix}$$

Denote this matrix by $F(\zeta)$ and let $F_{ij}(\zeta)$ be the ijth minor of $F(\zeta)$, the matrix obtained by deleting the ith row and the jth column of $F(\zeta)$.

Lemma 8.4. *Let $B(l_1, \cdots, l_q)$ for $(l_1, \cdots, l_q) \in (1, \cdots, p)$ be the matrix obtained by replacing the l_jth column of the identity matrix with the column $(\Theta_{1l_j}, \cdots, \Theta_{pl_j})$ for $1 \le j \le q$. Then*

$$|B(l_1, \cdots, l_q)_{ij}| = 0 \qquad \text{for} \qquad i \notin (j, l_1, \cdots, l_q)$$
$$B(i) = B(i)_{ij} = B(i)_{jj} = \Theta_{ii} \qquad \text{for} \qquad i,j \in (1, \cdots, p)$$

PROOF: If i is not in the set $(j, l_1, \cdots, l_q)$ then the column of $B(l_1, \cdots, l_q)_{i,j}$ corresponding to the *ith* column of the identity matrix is all zeros, so the determinate is zero. □

Let P_q be the set of sets of q elements from $(1, \cdots, p)$. With the simplification $b_{ii} \neq 0$ for all $i \in (1, \cdots, p)$, the determinate $|F|(\zeta)$ of the matrix $F(\zeta)$ can be written

$$|F|(\zeta) = \prod_{k=1}^{p} \mathcal{F}(G_k)(\zeta) + \sum_{l \in P_1} \Theta_{ll} \prod_{k=1, k \neq l}^{p} \mathcal{F}(G_k)(\zeta) + \cdots +$$
$$\sum_{(l_1, \cdots, l_q) \in P_q} |B(l_1, \cdots, l_q)| \prod_{k=1, k \notin (l_1, \cdots, l_q)}^{p} \mathcal{F}(G_k)(\zeta) + \cdots + |\{\Theta_{ij}\}| \quad (8.1)$$

Let P_q^j be the set of sets of q elements from $(1, \cdots, \hat{j}, \cdots, p)$. Then for all i, j,

$$|F_{ii}|(\zeta) = \prod_{k=1,k\neq j}^{p} \mathcal{F}(G_k)(\zeta) + \sum_{l\in P_1^i} \Theta_{ll} \prod_{k=1,k\notin(i,l)}^{p} \mathcal{F}(G_k)(\zeta) + \cdots +$$

$$\sum_{(l_1,\cdots,l_q)\in P_q^i} |B(l_1,\cdots,l_q)_{ii}| \prod_{k=1,k\notin(i,l_1,\cdots,l_q)}^{p} \mathcal{F}(G_k)(\zeta) + \cdots + |\{\Theta_{ij}\}_{ii}| \quad (8.2)$$

For $i \neq j$ with P_q^{ij} the set of sets of q elements in $(1, \cdots, \hat{j}, \cdots, \hat{i}, \cdots, p)$

$$|F_{ij}|(\zeta) = \Theta_{ii} \prod_{k=1,k\notin(i,j)} F(G_k)(\zeta) + \sum_{l\in P_1^{i,j}} |B(i,l)_{ij}| \prod_{k=1,k\notin(i,j,l)} F(G_k)(\zeta) + \cdots +$$

$$\sum_{(l_1,\cdots,l_q)\in P_q^{i,j}} |B(i,l_1,\cdots,l_q)_{ij}| \prod_{k=1,k\notin(i,j,l_1,\cdots,l_q)}^{p} \mathcal{F}(G_k)(\zeta) + \cdots + |\{\Theta_{ij}\}_{ij}| \quad (8.3)$$

Proposition 8.5. *$|F|(\zeta)$ is the symbol of an elliptic pseudo-differential operator with principal symbol*

$$\sigma_p(|F|)(\zeta) = C(\nu)(\zeta - i0)^{\sum_{k=1}^{p}\nu_k} \qquad \nu_i \neq 0 \qquad 1 \leq i \leq p$$

$C(\nu)$ a constant depending on $\nu = \{\nu_1, \cdots, \nu_p\}$. If $\nu_i = 0$ for some i then assume that $\nu_i = 0$ for $i \in (1, \cdots, q)$ with $1 \leq q \leq p$. Then

$$\sigma_p(|F|)(\zeta) = C(\nu)|A|(\zeta)(\zeta - i0)^{\sum_{q+1}^{p}\nu_k}$$

where $A(\zeta)$ is the $q \times q$ matrix

$$A(\zeta) = \begin{pmatrix} ln(C_{\Theta_{11}}(\zeta - i0)) & \cdots & \Theta_{1,q} \\ \vdots & & \vdots \\ \Theta_{q1} & \cdots & ln(C_{\Theta_{qq}}(\zeta - i0)) \end{pmatrix}$$

with diagonal entries $ln(C_{\Theta_{ii}}(\zeta - i0))$ and off diagonal entries Θ_{ij}.

Proof: If $i \neq 0$ for $1 \leq i \leq p$ then by 8.1 $|F|(\zeta)$ is a polynomial in ζ of degree $\sum_{i=1}^{p}\nu_i > 0$, so there are constants C and C' such that

$$||F|(\zeta)| > C'(\zeta)^{\sum_{i=1}^{p}\nu_i} \qquad \text{for} \qquad |\zeta| > C.$$

If $\nu_i = 0$ for $1 \leq i \leq q$ then

$$\mathcal{F}(G_i)(\zeta) = ln(C_{\Theta_{ii}}(\zeta - i0))$$

and $|A|(\zeta)$ is a polynomial in $ln(\zeta - i0)$, so there exist constants C and C' such that

$$||A|(\zeta)| > C' \qquad \text{for} \qquad |\zeta| > C.$$

□

Lemma 8.6. *$K_\Gamma^{ij}(t)$ is the kernel of a pseudo-differential operator with principal symbol $\sigma_p(|F_{ij}|)/\sigma_p(|F|)(\zeta)$. Assume $\nu_i = 0$ for $i \in (1, \cdots, q)$ and $\nu_i \neq 0$ for $i \in (q+1, \cdots, p)$ with $1 \le q \le p$, then*

$$\frac{\sigma_p(|F_{ii}|)}{\sigma_p(|F|)}(\zeta) = \begin{cases} (\zeta - i0)^{-\nu_i} & q < i \le p \\ \{ln(C_{\Theta_{ii}}(\zeta - i0))\}^{-1} & 1 \le i \le q \end{cases}$$

For $i \neq j$

$$\frac{\sigma_p(|F_{ij}|)}{\sigma_p(|F|)}(\zeta) = \begin{cases} \Theta_{ii}(\zeta - i0)^{-\nu_i - \nu_j} & q < j < i \le p \\ \Theta_{ii}(\zeta - i0)^{-\nu_i}\{ln(C_{\Theta_{jj}}(\zeta - i0))\}^{-1} & 1 \le j \le q < i \le p \\ \Theta_{ii}\{ln(C_{\Theta_{ii}}(\zeta - i0))ln(C_{\Theta_{jj}}(\zeta - i0))\}^{-1} & 1 \le j < i \le q \end{cases}$$

PROOF: For $\zeta \in \mathbb{R}$ with $|F|(\zeta) \neq 0$ the matrix $F(\zeta)$ has an inverse with ijth entry $|F_{ij}|(\zeta)/|F|(\zeta)$. Let $|F|(D)$ be the operator with symbol $|F|(\zeta)$. Since $|F|(D)$ is elliptic it has a well-defined inverse $|F^{-1}|(D)$ with principal symbol

$$\sigma_p(|F^{-1}|)(\zeta) = \sigma_p(|A^{-1}|)(\zeta)(\zeta - i0)^{-\sum_{q+1}^{p} \nu_k}$$

The form of the principal symbol of the operators $K_\Gamma^{ij}(D) = |F^{ij}|/|F|(D)$ follows from the expansions 8.2 and 8.3 for $|F^{ij}|(\zeta)$. □

This completes the proof of theorem 8.3 □

This construction gives the boundary data functions for the Γ boundary condition

$$h_i(t) = \sum_{j=1}^{p} K_\Gamma^{ij} *_t (f_j)(t)$$

The solution to the heat equation for H_A with $\phi(r,x,t)$ initial data and Γ boundary data can be written

$$w(r,x,t) = u(r,x,t) + w'(r,x,t)$$

where if $b_{ii} = 0$ for $i \in (1, \cdots, q)$ with $c_j = 2\nu_j$ for $\nu_j \neq 0$ and $c_j = 1$ for $\nu_j = 0$,

$$w'(r,x,t) = \sum_{i=q+1}^{p} c_j h_i(t) *_t N'E_+^i(r,x,t) = \sum_{i,j=q+1}^{p} c_j K_\Gamma^{ji}(f_i)(t) *_t N'E_+^j(r,x,t)$$

For fixed i, j, since $\phi(r,x,t)$ is compactly supported,

$$K_\Gamma^{ji}(f_i)(t) *_t N'E_+^j(r,x,t) = \{NE_+^i *_t K_\Gamma^{ji} *_t N'E_+^j\}(\phi)(r,x,t)$$

so the heat kernel $E_\Gamma(r,r',x,x',t)$ for the self-adjoint extension fixed by including Γ in the domain is

$$E_\Gamma(r,r',x,x',t) = E_+(r,r',x,x',t) + E'_\Gamma(r,r',x,x',t)$$
$$E'_\Gamma(r,r',x,x',t) = \sum_{i,j=q+1}^{p} NE^i_+(r,x,t) *_t K^{ji}_\Gamma(t) *_t N'E^j_+(r',x',t)$$

Proposition 8.7. *$\beta^*(NE^i_+(r',x',t) *_t N'E^j_+(r,x,t))$ vanishes to infinite order at $\mathfrak{T}$ and $\mathfrak{D}$ and is homogeneous in $\rho_{\mathfrak{D}}$ of degree $-\nu_i - \nu_j - 1$.*

PROOF: From the scaling of the positive heat kernel

$$\mu_s^*(N'E_+)(r,x,t) = s^{-\nu_i - \frac{3}{2}} N'E_+(r,x,t)$$

for fixed i, j the homogeneity at the front face is show by the scaling under μ_s,

$$\int_0^{s^2 t} N'E^i_+(sr,x,s^2t-t')NE^j_+(sr',x',t')dt'dt'$$
$$= s^2 \int_0^t \mu_s^*(N'E^i_+)(r,x,t-t')\mu_s^*(NE^j_+)(r',x',t')dt'dt'$$
$$= s^{-\nu_i-\nu_j-1} N'E(r,x,t) *_t NE(r',x',t)$$

The vanish at $\mathfrak{T}$ and $\mathfrak{D}$ is from the vanishing of the positive heat kernel restricted away from the diagonal at the face $t=0$. □

$$\mu_s^*(NE^i_+(r,x,t) *_t K^{ji}_\Gamma(t) *_t N'E^j_+(r',x',t)) \sim_{s=0} \begin{cases} s^{-1} + o(s^{-1}) & i = j \\ s^{\nu_i+\nu_j-1} + o(s^{\nu_i+\nu_j-1}) & i \neq j \end{cases}$$

This, along with the homogeneity of $E_+(r,r',x,x',t)$, gives the expansion of $\beta^*(E_\Gamma)(\tilde{X})$ at $\mathfrak{D}$. the terms in $\beta^*(E'_\Gamma)(\tilde{X})$ all vanish to infinite order at $\mathfrak{T}$ and $\mathfrak{D}$, so the expansion at $\mathfrak{T}$ and $\mathfrak{D}$ is the same as that of $\beta^*(E_+)(\tilde{X})$, and the expansions at $\mathfrak{L}$ and $\mathfrak{R}$ are given by inclusion in the domain of the self-adjoint extension. □

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