Multiplicity Formulas for Orbifolds

by

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ABSTRACT

Given a symplectic space, equipped with a line bundle and a Hamiltonian group action satisfying certain compatibility conditions, it is a basic question to understand the decomposition of the quantization space in irreducible representations of the group. We derive weight multiplicity formulas for the quantization space in terms of data at the fixed points on the symplectic space, which apply to general situations when the underlying symplectic space is allowed to be an orbifold, the group acting is a compact connected semi-simple Lie group, and the fixed points of that action are not necessarily isolated. Our formulas extend the celebrated Kostant multiplicity formulas. Moreover, we show that in the semi-classical limit our formulas converge to the Duistermaat-Heckman measure, that is the push-forward of Lebesgue measure by the moment map.

Thesis Supervisor: Victor W. Guillemin
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CONTENTS

INTRODUCTION

Abstractly, *quantization* is a framework for associating to a manifold M a Hilbert space H, and to each function on M a self-adjoint operator on H, subject to certain conditions. Although it is known that such a general scheme does not exist, it is still very interesting to find which subclasses of manifolds and functions can be quantized in a consistent fashion. Quantization is important both for mathematical physics and for representation theory of Lie groups.

Generally speaking, a quantization functor should also allow for quantum predictions to be formulated in terms of classical concepts. In particular, when a classical system has some group action lifting to a linear representation on the quantized system, it is natural to ask what this representation is in terms of the classical one. If the (compact) group is abelian, this question is translated into knowing all weight multiplicities.

The main goal of this thesis is to actually *compute* quantizations, specifically by extending the domain of application of one of the main techniques for finding multiplicities: Kostant-type multiplicity formulas.

The original Kostant multiplicity formula was discovered by Kostant [Ko1] in the fifties for the setting of coadjoint orbits. It expressed a weight multiplicity in terms of data at the fixed points of the action, and involved partition functions. In the late eighties, Guillemin, Lerman and Sternberg [GLS] extended Kostant's formula to the symplectic manifold setting, for the case of torus actions with only isolated fixed points. Soon afterwards, Guillemin and Prato [GP] generalized it to non-abelian group actions. Our formulas here extend those of Guillemin-Lerman-Sternberg to the cases of arbitrary compact connected semi-simple Lie group actions with fixed points which are not necessarily isolated, and where the original space is not necessarily smooth but is allowed to have *orbifold* singularities.

An orbifold is a topological space that is locally homeomorphic to an open euclidean subset modulo a linear finite group action, with a fixed set of codimension at least 2. We can generalize most notions of differential geometry from manifolds to orbifolds.

Orbifolds arise naturally by symplectic reduction at regular levels of the moment map, where the group action is only locally free. Not only are the generic reduced spaces of symplectic manifolds orbifolds, but also the generic reduced spaces of symplectic *orbifolds* are still orbifolds. Hence, one is lead to studying the category of symplectic orbifolds. Furthermore, even in the simplest cases, this study has remarkable connections to number theory.

Overview. We begin by discussing in section 1 the index of a twisted Dolbeault-Dirac operator on a symplectic orbifold M with a Hamiltonian action of a compact connected Lie group G. We explain how this index becomes a representation of G, and introduce the Kostant-type multiplicity formulas. These are formulas expressing the multiplicities of weights in the index representation in terms of data at the fixed points of the G action on M.

After describing the setting and notation, we state our three main results in section 2. Theorem 1 extends the Kostant multiplicity formula to Hamiltonian actions of compact Lie groups on orbifolds, when the fixed points are not necessarily isolated. Theorem 2 gives the semi-classical limit of the formula in Theorem 1, leading to the relation with the Duistermaat-Heckman measure reported in Theorem 3.

In section 3, we show how our formulas also cover compact *non-abelian* Lie group actions, since by Weyl's integral formula we can indeed restrict the action to a Cartan subgroup. Sections 4, 5 and 6 contain some integral estimates, results on finite abelian groups and Fourier transforms necessary to our calculations.

The idea behind our proof of Theorem 1 in section 7 is Cartier's observation that the Kostant multiplicity formula can be derived from the Weyl character formula by expanding the Weyl denominator into a trigonometric series and computing the coefficient of $e^{i\alpha}$ [Ca]. Guillemin, Lerman and Sternberg also applied a Fourier transform to the Atiyah-Bott-Lefschetz fixed point formula in [GLS]. The non-isolated fixed point analogue of the [GLS] formula was similarly deduced in [CG] from the Atiyah-Segal-Singer equivariant index theorem. Following this spirit, we will show that the orbifold formula can be derived, by essentially the same argument, from the orbifold version of the equivariant index theorem, which was proved by Vergne [V2].[1;2]

The main ingredient for the proof of Theorem 2 in section 8 is an asymptotic result in invariant theory for finite groups. Let ρ be an effective representation of a finite group Γ, and let $\mathcal{S}^n\rho$ be its n-th symmetric power. Then the expectation value of the ratio between the dimensions of the space of invariants in $\mathcal{S}^n\rho$ and that of $\mathcal{S}^n\rho$, tends to the inverse of the order of G, as n tends to infinity.

The proof of Theorem 3 in section 9 is essentially an exercise in Fourier calculus.

In section 10 we give an account of our formulas for the particular case of isolated fixed points. In section 11 we illustrate the main formula of section 10 in the case of a twisted projective space.

The appendix contains the basic facts on orbifolds which we need throughout.

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[1] Duistermaat [D] has also proved a result of which this is a consequence: that the "local" version of the Atiyah-Segal-Singer theorem [ASIII] is true for spin^c-Dirac operator on orbifolds. In [Ch] Sheldon Chang proves the fixed point formula for orbifolds that we use, but unfortunately his preprint was only available to us after the completion of this work.

[2] For "good" orbifolds, *i.e.* for orbifolds which admit a finite presentation of the form $M = X/\Gamma$, where X is a manifold and Γ is a finite group, our results can be obtained from those in [CG] by averaging.

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1. The index on orbifolds

Let M be a compact, connected, $2d$-dimensional orbifold equipped with a symplectic form, ω, an orbifold Hermitian line bundle, $\mathbf{L}$, and an almost complex structure, J. We assume that $\mathbf{L}$ is a "prequantum" line bundle, *i.e.* that $-2\pi i\omega$ is the curvature of $\mathbf{L}$ with respect to some Hermitian orbifold connection. In this case

$$c(\mathbf{L}) = [\omega] \tag{1.1}$$

where $c(\mathbf{L})$ denotes the orbifold Chern class of $\mathbf{L}$. We also assume that J and ω are *compatible* in the sense that for all $p \in M$ the bilinear form

$$g_p(v,w) = \omega_p(J_p v, w) \ , \qquad v, w \in T_pM \ , \tag{1.2}$$

is symmetric and positive definite. For the orbifold definitions, please see the appendix. We assume that the multiplicity of M is 1 (see A.2), which means that the orbifold structure groups act effectively.

Warning: From here on, all notions should be interpreted as the orbifold counterparts of the usual manifold concepts. We will sometimes omit the word *orbifold* to keep the exposition shorter. Please see the appendix for more details.

From J one gets a Dolbeault structure on the exterior algebra of the cotangent bundle of M. That is, the almost complex structure gives a splitting of the complexified tangent bundle of M into the $+i$ and $-i$ eigenspaces of J:[3]

$$TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M \ .$$

The exterior powers of the complexified cotangent bundle also split

$$\wedge^k T^*M \otimes \mathbb{C} = \bigoplus_{i+j=k} \wedge^{i,j}$$

and the differential forms $\Omega^k := C^\infty(M, \wedge^k T^*M \otimes \mathbb{C})$ break up into a direct sum according to form-type

$$\Omega^k = \bigoplus_{i+j=k} \Omega^{i,j} \ .$$

For $\xi \in T_p^*M$, let $\xi_{0,1}$ be the $\wedge^{0,1}$-component of ξ, and let

$$\zeta_p(\xi) : \wedge_p^{0,j} \otimes \mathbf{L}_p \longrightarrow \wedge_p^{0,j+1} \otimes \mathbf{L}_p$$

be the map,

$$\zeta_p(\xi)w = \xi_{0,1} \wedge w \ .$$

The compatibility condition (1.2) implies that g_p and ω_p are the real and imaginary parts of a Hermitian inner product which extends to a Hermitian inner product on

[3]This splitting is given on the tangent bundles $T\widetilde{\mathcal{U}}$ over each orbifold chart $(\widetilde{\mathcal{U}}, \Gamma, \phi)$. The local eigenspaces piece together to orbifold complex vector bundles.

each of the $\wedge_p^{0,j}$; and from this inner product and the Hermitian inner product on $\mathbf{L}_p$ one gets inner products on the domain and range of $\zeta_p(\xi)$. Let $\zeta_p(\xi)^*$ be the adjoint of $\zeta_p(\xi)$, and let

$$\sigma_p(\xi) : \wedge_p^{0,even} \otimes \mathbf{L}_p \longrightarrow \wedge_p^{0,odd} \otimes \mathbf{L}_p \tag{1.3}$$

be the sum of $\zeta_p(\xi)$ and $\zeta_p(\xi)^*$. For $\xi \neq 0$ this map is bijective; so there exists a first order elliptic differential operator, $D_{\mathbf{L}}$, whose symbol is (1.3).[4] We will denote by $\mathrm{Ind}(D_{\mathbf{L}})$ the virtual vector space

$$\mathrm{kernel}(D_{\mathbf{L}}) - \mathrm{cokernel}(D_{\mathbf{L}}) \ . \tag{1.4}$$

The dimension of this virtual vector space is a *symplectic* invariant of M in the sense that it is independent of the choice of compatible $\mathbf{L}$ and J, and also independent of the choice of $D_{\mathbf{L}}$ with the given symbol. We call $\mathrm{Ind}(D_{\mathbf{L}})$ the **index** of (M, ω).

Remark 1.5. The assumption that M is symplectic is not really necessary. We could start from a compact, connected, $2d$-dimensional orbifold equipped with a a Hermitian line bundle, $\mathbf{L}$, an almost complex structure, J, and a Hermitian inner product. ω would then simply be a closed two-form satisfying $c(\mathbf{L}) = [\omega]$. In this case the index would *not* be an invariant of (M, ω), but would also depend on the auxiliary choice of J.

Remark 1.6. Let d be the DeRham exterior derivative, let $\pi^{i,j} : \Omega^* \longrightarrow \Omega^{i,j}$ denote the projection into (i,j)-type forms, and let

$$\overline{\partial} = \pi^{0,j+1} \circ d : \Omega^{0,j} \longrightarrow \Omega^{0,j+1} \ .$$

The symbol of $\overline{\partial}$ is given by the map

$$\begin{array}{rccl} \zeta_p(\xi) : & \wedge_p^{0,j} & \longrightarrow & \wedge_p^{0,j+1} \\ & w & \mapsto & \xi_{0,1} \wedge w \ . \end{array}$$

When M is a *complex analytic* orbifold, the operator $\overline{\partial}$ defines the *orbifold Dolbeault complex*

$$0 \longrightarrow \Omega^{0,0} \xrightarrow{\overline{\partial}} \Omega^{0,1} \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} \Omega^{0,n} \longrightarrow 0 \ .$$

Let $\mathbf{L}$ be a *holomorphic* line bundle over M, *i.e.* a complex line bundle for which the trivializations are given by non-zero complex numbers depending holomorphically on

[4] $D_{\mathbf{L}}$ is assembled from Γ-invariant elliptic differential operators over each orbifold chart $(\widetilde{\mathcal{U}}, \Gamma, \phi)$. These local $D_{\mathbf{L}}^{\widetilde{\mathcal{U}}}$ take Γ-invariant sections to Γ-invariant sections. The compactness of M, together with the ellipticity of each $D_{\mathbf{L}}^{\widetilde{\mathcal{U}}}$, yield that the kernel and cokernel of $D_{\mathbf{L}}$ are finite dimensional complex vector spaces.

the base point.[5] Let ∇ be a Hermitian connection on $\mathbf{L}$, and let $\overline{\partial}_L = \overline{\partial} \otimes 1 + 1 \otimes \nabla$. $\overline{\partial}_L$ defines a "twisted Dolbeault complex" for the $\mathbf{L}$-valued forms

$$0 \longrightarrow C^\infty(\wedge^{0,0} \otimes \mathbf{L}) \xrightarrow{\overline{\partial}_L} C^\infty(\wedge^{0,1} \otimes \mathbf{L}) \xrightarrow{\overline{\partial}_L} \dots \xrightarrow{\overline{\partial}_L} C^\infty(\wedge^{0,2d} \otimes \mathbf{L}) \longrightarrow 0 .$$

The cohomology of this complex is the cohomology of the sheaf of holomorphic sections of $\mathbf{L}$ (see [D]):

$$\frac{\ker \overline{\partial}_L^{(0,j)}}{\operatorname{im} \overline{\partial}_L^{(0,j-1)}} \cong H^j(M, \mathcal{O}(\mathbf{L})) .$$

Let $\overline{\partial}_L^*$ be the adjoint of $\overline{\partial}_L$ with respect to the given Hermitian inner product. The operator

$$\not{D}_{\mathbf{L}} = \overline{\partial}_L + \overline{\partial}_L^* : C^\infty(M, \wedge^{0,even} T^* M \otimes \mathbf{L}) \longrightarrow C^\infty(M, \wedge^{0,odd} T^* M \otimes \mathbf{L})$$

has symbol equal to (1.3). Hence,

$$\operatorname{Ind}(D_{\mathbf{L}}) = \operatorname{Ind}(\not{D}_{\mathbf{L}}) = \sum (-1)^j H^j(M, \mathcal{O}(\mathbf{L})) .$$

When $\mathbf{L}$ is sufficiently positive, Kodaira's vanishing theorem says that $H^j(M, \mathcal{O}(\mathbf{L})) = 0$ for $j > 0$. Therefore, $\operatorname{Ind}(D_{\mathbf{L}})$ is just the space of holomorphic sections of $\mathbf{L}$ over M.

Let G be a compact connected Lie group with $\gamma = \dim G$, and let $\tau : G \times M \longrightarrow M$ be an effective action of G on M which preserves ω and J. We will assume that there is an action, ρ, of G on $\mathbf{L}$ which is compatible with τ and hence, in particular that τ is a *Hamiltonian* action with a moment map, $\Psi : M \longrightarrow \mathfrak{g}^*$ (see §A.8 and §A.9). Below we will consider $\Phi = -2\pi i \Psi : M \longrightarrow \mathbb{R}^\gamma$.[6]

From ρ one gets an induced action of G on the sections of $\wedge^{0,*} \otimes \mathbf{L}$ and, by averaging, one can make $D_{\mathbf{L}}$ commute with this action, thus getting a virtual representation of G on $\operatorname{Ind}(D_{\mathbf{L}})$. This representation is, up to isomorphism, a *Hamiltonian* invariant of M, *i.e.*, depends on (ω, τ, Φ) but not on J nor $D_{\mathbf{L}}$. The character of this virtual representation is called the **index-character** or the **equivariant index** of (M, ω, τ, Φ).

To compute the equivariant index, one can, without loss of generality, assume that G is abelian (see §3) in which case the representation of G on $\operatorname{Ind}(D_{\mathbf{L}})$ is completely determined by its weight multiplicities.

[5] $\mathbf{L}$ is called a *prequantum* line bundle if $\frac{i}{2\pi}\mathrm{curv}(\mathbf{L})$ is a *Kähler* form on M.

[6] Φ is a moment map for the curvature $-2\pi i\omega$. At a fixed point p, Φ_p gives precisely the rational weight of the G-action on $\mathbf{L}_p$.

If M is a manifold and M^G is finite, these weight multiplicities are given by the "Kostant multiplicity formula". This formula expresses the multiplicity $\mathcal{M}(\alpha)$ of the weight α on $\mathrm{Ind}(D_{\mathbf{L}})$ as an alternating sum over the fixed points of the form:

$$\mathcal{M}(\alpha) = \sum_{p \in M^G} (-1)^{\sigma_p} \mathcal{N}_p(\alpha) \tag{1.7}$$

$\mathcal{N}_p$ being the "Kostant partition function" associated with the isotropy representation of G on the tangent space at the fixed point p.[7]

We will show that a formula of this type is true when M is only an orbifold and the fixed points aren't isolated.

[7] For the definition of $\mathcal{N}_p$ and σ_p, please see §2.

2. Statement of the theorems

Denote a connected component of M^G by F. Its normal bundle NF splits into a direct sum of vector subbundles

$$\mathbf{E}_{F,1} \oplus \dots \oplus \mathbf{E}_{F,m}$$

(m depending on F) [D], such that the isotropy representation of G on $\mathbf{E}_{F,j}$ has a fixed weight, $\alpha_{F,j} \in \mathbb{Q}^\gamma$ (where $\alpha_{F,j} \neq \alpha_{F,k}$ for $j \neq k$). Since G is compact, each F is non-degenerate, in the sense that all $\alpha_{F,j} \neq 0$.[8] Hence, we can polarize these weights as in [GLS] by choosing an element, v, of $\mathbb{R}^\gamma$ such that $\alpha_{F,j}(v) \neq 0$ for all i, j, and setting

$$\alpha^+_{F,j} = \epsilon_{F,j}\alpha_{F,j}$$

where

$$\epsilon_{F,j} = \text{sign } \alpha_{F,j}(v) \ .$$

(These polarized weights have the property that they all lie in the half-space $(\xi, v) > 0$.) Let $n_{F,j}$ be the rank of the vector bundle, $\mathbf{E}_{F,j}$, and let

$$\alpha^\#_F = \sum_{\epsilon_{F,j}=-1} n_{F,j}\alpha^+_{F,j} \quad \text{and} \quad \sigma_F = \sum_{\epsilon_{F,j}=-1} n_{F,j} \ .$$

For every m-tuple of non-negative integers, $k = (k_1, \dots, k_m)$, let $\mathbf{E}_F(k)$ be the tensor product

$$\left(\bigotimes_{j=1}^{m} \mathcal{S}^{k_j}(\mathbf{E}^+_{F,j})\right)^* \otimes \left(\bigotimes_{\epsilon_{F,j}=-1} \wedge^{n_{F,j}}(\mathbf{E}^+_{F,j})\right)^*$$

where $\mathbf{E}^+_{F,j} = \mathbf{E}_{F,j}$ or $\mathbf{E}^*_{F,j}$ depending on whether $\epsilon_{F,j}$ is 1 or -1, and $\mathcal{S}^{k_j}(\mathbf{E}^+_{F,j})$ is the k_j-th symmetric product of $\mathbf{E}^+_{F,j}$.

Finally, if α is a weight, let $\Delta_F(\alpha)$ be the convex polytope in $\mathbb{R}^m$ consisting of all m-tuples, $(s_1, \dots, s_m)$, $s_i \geq 0$, for which

$$\Phi_F - \sum_j s_j\alpha^+_{F,j} = \alpha$$

where Φ_F is the value of Φ on F. (Φ is constant on F, since F is a connected component of M^G.) Notice that in this expression $\alpha^+_{F,j}$ and Φ_F are rational, whereas α is integral. The fact that the $\alpha^+_{F,j}$'s are polarized implies that $\Delta_F(\alpha)$ is compact. (The quantity

$$\Phi_F(v) - s_1\alpha^+_{F,1}(v) - \dots - s_m\alpha^+_{F,m}(v)$$

tends to $+\infty$ as $s_1 + \dots + s_m$ tends to $+\infty$.)

[8] As pointed out by Allen Knutson and in [GLS], the connected components of the fixed point set of a compact group action, are always non-degenerate.

From here on, to simplify notation we omit the subscript F's from the double indices and write $\mathbf{E}_{F,j} = \mathbf{E}_j$, $\alpha_{F,j} = \alpha_j$, $\epsilon_{F,j} = \epsilon_j$, etc.

Our generalization of the Kostant formula is the following:

Theorem 1. *The multiplicity with which α occurs as a weight of the representation of G on* $\mathrm{Ind}(D_{\mathbf{L}})$ *is equal to the sum*

$$\sum_{F \subseteq M^G} (-1)^{\sigma_F} \mathcal{N}_F(\alpha)$$

where

$$\mathcal{N}_F(\alpha) = \sum_{k \in \Delta_F(\alpha + \alpha_F^{\#})} \mathrm{KRR}(F, \mathbf{E}_F(k) \otimes \mathbf{L}) \tag{2.1}$$

$\mathrm{KRR}(F, \mathbf{E}_F(k) \otimes \mathbf{L})$ *being the "Kawasaki-Riemann-Roch number" of the orbifold vector bundle* $\mathbf{E}_F(k) \otimes \mathbf{L}$ *over F, with respect to the almost complex structure induced on F by J (see* [Ka] *and* §7*).*

Remark 2.2. If M is a manifold, this agrees with the result in [CG]. In that case, the Kawasaki-Riemann-Roch number is just the usual Riemann-Roch number:

$$\mathrm{RR}(F, \mathbf{E}_F(k) \otimes \mathbf{L}) = \int_F \mathrm{Ch}(\mathbf{E}_F(k) \otimes \mathbf{L})\, \mathrm{Td}(F)$$

where $\mathrm{Ch}(\mathbf{E}(k) \otimes \mathbf{L})$ is the Chern character of $\mathbf{E}_F(k) \otimes \mathbf{L}$, and $\mathrm{Td}(F)$ is the Todd class of F.

The formula (2.1) has an interesting "semi-classical" limit. Let n be a positive integer. Replacing the line bundle, $\mathbf{L}$, by its n-th tensor power, one gets, in analogy with (1.3), an elliptic symbol

$$\sigma_p^{(n)}(\xi) : \wedge_p^{0,even} \otimes \mathbf{L}_p^n \longrightarrow \wedge_p^{0,odd} \otimes \mathbf{L}_p^n \,.$$

Let $D_{\mathbf{L}}^{(n)}$ be a G-invariant elliptic operator with this as its symbol and let $\gamma = \dim G$. Denote by $\mathcal{M}^{(n)}(\alpha)$ the multiplicity with which α occurs as a weight of the representation of G on $\mathrm{Ind}(D_{\mathbf{L}}^{(n)})$.

Theorem 2. *As n tends to infinity, the quantity $n^{-(d-\gamma)} \mathcal{M}^{(n)}(n\alpha)$ tends to*

$$\sum_{F \subseteq M^G} (-1)^{\sigma_F} \int_{\Delta_F(\alpha)} \mathrm{Res}_F(s) ds \tag{2.3}$$

where $\mathrm{Res}_F(s)$ *is the sum of residues of*

$$e^{\Sigma s_j z_j}\frac{1}{|\Gamma_F|}\int_F \frac{\exp \omega_F}{c_{F,1}(z_1)\dots c_{F,m}(z_m)}\ , \tag{2.4}$$

Γ_F *is the structure group of* F *and* $c_{F,j}(z)$ *is the Chern polynomial of* $\mathbf{E}^+_{F,j}$.

In [CG] it was proved that in the case of manifolds the function of α defined by (2.3) is the Radon-Nikodym derivative

$$\frac{d\mu_{DH}}{d\mu_{Leb}} \tag{2.5}$$

where μ_{DH} is the Duistermaat-Heckman measure and μ_{Leb} is the standard Lebesgue measure on $\mathfrak{g}^*$ (suitably normalized). It turns out that the same is true for orbifolds:

Theorem 3. *The piece-wise polynomial function of* α *defined by* (2.3) *is the Radon-Nikodym derivative,* (2.5).

3. Non-abelian groups

Let G be a compact semi-simple Lie group. By the "shifting trick"[9] it suffices to compute the multiplicity with which the trivial representation occurs in the representation of G on the space (1.4) and (as was pointed out to us by Michèle Vergne) this can easily be computed from the weight multiplicities of the representation of the Cartan subgroup, T, of G on the space (1.4). More explicitly the following result is true: Let $\rho : G \longrightarrow U(Q)$ be a representation of G on a finite dimensional Hilbert space, Q. Restricting ρ to T, Q breaks up into weight spaces

$$Q_\beta \quad , \quad \beta \in \mathbb{Z}^T$$

($\mathbb{Z}^T$ being the weight lattice of T). Then

$$\dim Q^G = \frac{1}{|W|} \sum_{\beta \in \mathbb{Z}^T} C_\beta \dim Q_\beta \ , \tag{3.1}$$

the C_β's being the Fourier coefficients of the function, $\theta(\exp \xi) := \prod_{\alpha \in \Delta}(1 - e^{\alpha(\xi)})$. In other words,

$$\prod_{\alpha \in \Delta} (1 - e^{\alpha(\xi)}) = \sum_{\beta \in \mathbb{Z}^T} C_\beta e^{\beta(\xi)} \quad , \quad \xi \in \mathfrak{t} \,. \tag{3.2}$$

(Here Δ is the set of roots of G.)

Proof. (3.1) can be extracted from Weyl's integral formula (see for instance [He], page 194, corollary 5.16).

Theorem. *Let $\chi \in C^\infty(G)$ be a class function, i.e. $\chi(aga^{-1}) = \chi(g)$ for all $a, g \in G$.) Then*

$$\int_G \chi(g) dg = \frac{1}{|W|} \int_T \theta(t)\chi(t) dt \tag{3.3}$$

dg and dt being Haar measures on G and T, $\theta(t)$ being the function (3.2) *and $|W|$ the cardinality of the Weyl group.*

---

[9] Each coadjoint orbit of G comes equipped with a canonical Lie-Kostant-Kirillov Kähler structure. We take the standard Hamiltonian action of G on $\mathcal{O}$ with moment map given by inclusion. Let ρ be an irreducible representation of G. By Kostant's version of the Borel-Weil theorem, there is a unique integral coadjoint orbit, $\mathcal{O}$, such that ρ is the canonical representation of G on the space of holomorphic sections of the prequantum line bundle. Let $\mathcal{O}^-$ denote $\mathcal{O}$ with the opposite Kähler structure. It was shown in [GS1], §6, that the multiplicity of ρ in $\mathrm{Ind}(D_{\mathbf{L}})$ equals the multiplicity of the trivial representation in $\mathrm{Ind}(D_{\mathbf{L}}^{\mathcal{O}})$ where $D_{\mathbf{L}}^{\mathcal{O}}$ is the elliptic operator corresponding to $M \times \mathcal{O}^-$.

Comments

1. $\theta(t)$ is real and non-negative, as one can see by writing it as the product of the function

$$\prod_{\alpha\in\Delta_+}(1-e^{\alpha(\xi)}) \tag{3.4}$$

and its conjugate. In particular, $\theta=\overline{\theta}$, *i.e.*, $C_\beta=C_{-\beta}$.

2. Let δ be half the sum of the positive roots. It is clear from (3.4) that $C_\beta\neq 0 \Rightarrow \beta/2$ lies in the convex hull of $\{w\delta, w\in W\}$.

Apply (3.3) to the character, χ, of representation ρ. Noting that for $\xi\in\mathfrak{t}$:

$$\chi(\exp\xi)=\sum_{\beta\in\mathbb{Z}^T} e^{\beta(\xi)}\dim Q_\beta$$

one gets, by Schur's lemma

$$\dim Q^G=\int_G \chi(g)dg=\langle\chi,1\rangle_{L^2}$$

(1 being the character of the trivial representation), and hence, by (3.3) and (3.4)

$$\begin{aligned}\dim Q^G &= \frac{1}{|W|}\int_T \chi(t)\theta(t)dt\\ &= \frac{1}{|W|}\int_T\left(\sum_\beta e^{\beta(\xi)}\dim Q_\beta\right)\left(\sum_\beta C_\beta e^{-\beta(\xi)}\right)dt\\ &= \frac{1}{|W|}\sum_{\beta\in\mathbb{Z}^T} C_\beta\dim Q_\beta\ .\end{aligned}$$

□

4. Integral estimates

Let V be a d-dimensional complex vector space and $\mathcal{S}^k$ be the standard representation of $GL(V)$ on the k-th symmetric product, $\mathcal{S}^k(V)$.

Theorem 4.1. *For $z \in \mathbb{C}$, z large, and $A \in GL(V)$,*

$$\det(zI - A)^{-1} = z^{-d} \sum_{k=0}^{\infty} z^{-k} \operatorname{trace} \mathcal{S}^k(A) \ . \tag{4.2}$$

Proof. Without loss of generality we can assume that A is diagonalizable with eigenvalues, $\lambda_1, \ldots, \lambda_d$; in which case the left hand side of (4.2) becomes

$$z^{-d} \prod_{j=1}^{d} (1 - \lambda_j z^{-1})^{-1} \ . \tag{4.3}$$

Expanding each of the factors $(1 - \lambda_j z^{-1})^{-1}$ into a geometric series one can rewrite (4.3) in the form

$$z^{-d} \sum_{k=0}^{\infty} z^{-k} t_k$$

where

$$t_k = \sum_{|I|=k} \lambda_1^{i_1} \ldots \lambda_d^{i_d} \ ,$$

and the right hand side of this expression is $\operatorname{trace} \mathcal{S}^k(A)$. □

Corollary 4.4. *Let Γ be a contour about the origin containing the zeroes of $\det(zI - A)$. Then*

$$\frac{1}{2\pi i} \int_{\Gamma} z^{d+k-1} \det(zI - A)^{-1} dz = \operatorname{trace} \mathcal{S}^k(A) \ . \tag{4.5}$$

Remark 4.6. By analyticity, (4.2) and (4.5) are valid for any endomorphism $A : V \longrightarrow V$; *i.e.*, A doesn't necessarily have to be in $GL(V)$.

From (4.5) we will deduce the following two useful estimates:

Theorem 4.7. *Let A, B be commuting elements of $GL(V)$, with A diagonalizable. Then*

(4.8)

$$n^{-(d-1)} \operatorname{trace} \mathcal{S}^k(A \exp B/n) = \begin{cases} \lambda^k n^{-(d-1)} \operatorname{trace} \mathcal{S}^k(\exp B/n) & \textit{if } A = \lambda I \\ O(\frac{1}{n}) & \textit{otherwise.} \end{cases}$$

Proof. Without loss of generality, we can assume that A and B are simultaneously diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_d$ (of A) and $\mu_1, \dots, \mu_d$ (of B) and that $e^{\mu_1}, \dots, e^{\mu_d}$ are distinct. By (4.5), the left hand side of (4.8) is equal to the contour integral

$$n^{-(d-1)} \frac{1}{2\pi i} \int_\Gamma \frac{z^{d+k-1}}{\prod_j \left(z - \lambda_j e^{\mu_j/n}\right)} dz \, .$$

For n large enough, the pole at $\lambda_j e^{\mu_j/n}$ has residue

$$n^{-(d-1)} \frac{\left(\lambda_j e^{\mu_j/n}\right)^{d+k-1}}{\prod_{i \neq j} \left(\lambda_j e^{\mu_j/n} - \lambda_i e^{\mu_i/n}\right)} \tag{4.9}$$

where

$$\lambda_j e^{\mu_j/n} - \lambda_i e^{\mu_i/n} = \begin{cases} \lambda_j \frac{\mu_j - \mu_i}{n} \left(1 + O(\frac{1}{n})\right) & \text{if } \lambda_j = \lambda_i \\ (\lambda_j - \lambda_i)\left(1 + O(\frac{1}{n})\right) & \text{if } \lambda_j \neq \lambda_i \, . \end{cases}$$

There are exactly $(d-1)$ factors in the denominator of (4.9). Therefore, if A has at least two different eigenvalues, all residues (4.9) are of order $O(\frac{1}{n})$. □

Remark 4.10. By analyticity, (4.8) remains true for an arbitrary endomorphism $B : V \longrightarrow V$.

Remark 4.11. When k is of order $O(n)$ (i.e. $k \to \infty$ with n, but $\frac{k}{n} = O(1)$), (4.8) remains true if all eigenvalues of A have absolute value at most 1.

Theorem 4.12. *Let B be an endomorphism of V, and let Γ be a contour about the origin containing the zeroes of* $\det(zI - B)$. *If k is of order $O(n)$, then*

(4.13)
$$n^{-(d-1)} \operatorname{trace} \mathcal{S}^k(\exp B/n) = \frac{1}{2\pi i} \left(\int_\Gamma e^{\left(\frac{d+k-1}{n}\right)z} \det(zI - B)^{-1} dz \right) \left(1 + O\left(\frac{1}{n}\right)\right)$$

the $O(\frac{1}{n})$ being uniform in k.

Proof. Without loss of generality we can assume that B is diagonalizable with eigenvalues $\mu_1, \dots, \mu_d$ and that $e^{\mu_1}, \dots, e^{\mu_d}$ are distinct. Then by (4.5) $\operatorname{trace} \mathcal{S}^k(\exp B/n)$ is equal to

$$\frac{1}{2\pi i} \int_\Gamma z^{d+k-1} (z - e^{\mu_1/n})^{-1} \dots (z - e^{\mu_d/n})^{-1} dz$$

which, by the residue formula, is equal to

$$\sum_{i=1}^{d} e^{(d+k-1)\mu_i/n} \prod_{j \neq i} \left(e^{\mu_i/n} - e^{\mu_j/n}\right)^{-1}$$

or

$$n^{d-1}\left(\sum_{i=1}^{d} e^{(d+k-1)\mu_i/n}\prod_{j\neq i}(\mu_i-\mu_j)^{-1}\right)\left(1+O\left(\frac{1}{n}\right)\right)$$

and, again by the residue formula, this is equal to:

$$n^{d-1}\left(\frac{1}{2\pi i}\int_\Gamma e^{\frac{d+k-1}{n}z}\prod_{i=1}^{d}(z-\mu_i)^{-1}\right)\left(1+O\left(\frac{1}{n}\right)\right)\ .$$

Dividing by n^{d-1} and replacing $\prod(z-\mu_i)$ by $\det(zI-B)$ we obtain (4.13). □

5. Finite abelian groups

Let $\rho : \Gamma \longrightarrow GL(V)$ be a linear representation of a finite abelian group Γ on a d-dimensional complex vector space V. ρ breaks down into d irreducible representations, ρ_j, on 1-dimensional complex vector spaces V_j. Let $\chi_j : \Gamma \longrightarrow S^1$ be the character of ρ_j.

Theorem 5.1. *If ρ is an effective action, then the characters χ_j generate the character group Γ^*.*

Proof. Let χ be a non-trivial character of Γ and choose $g \in \Gamma$ such that $\chi(g) \neq 1$. Let m be the order of the cyclic group generated by g, $\langle g \rangle$. Identifying $\langle g \rangle$ with a subgroup of S^1, we can write

$$\chi(g) = g^r \ , \qquad \chi_j(g) = g^{p_j}$$

for some r, $p_j \in [0, m)$.

Suppose χ is not generated by the χ_j's, meaning that $r \notin \langle p_1, \dots, p_d, m \rangle$, and hence $\langle p_1, \dots, p_d, m \rangle = \langle p \rangle$ with $p \neq \pm 1$. Consider the element $g^q \neq \text{id}$ where $m \doteq q \cdot p$. g^q acts trivially since $\chi_j(g^q) = 1$ for all j. □

Consider the following sublattice of $\mathbb{Z}^d$:

$$\Lambda = \{(k_1, \dots, k_d) \in \mathbb{Z}^d | \prod_j \chi_j^{k_j} = \text{id}\} \ .$$

Theorem 5.2. *If ρ is an effective action, then $\mathbb{Z}^d/\Lambda \cong \Gamma$.*

Proof. By the classification of finite abelian groups, and the fact that for finite abelian groups $\Gamma^* \cong \Gamma$, and $(\Gamma_1 \times \Gamma_2)^* \cong \Gamma_1^* \times \Gamma_2^*$, we can assume Γ to be a cyclic group $\langle g \rangle$ of order m, and identify an irreducible character χ_j with an element g^{p_j}, $p_j \in [0, m)$.

By Theorem 5.1, the χ_j's generate Γ^*, which translates into

$$\langle p_1, \dots, p_d, m \rangle = \mathbb{Z} \ . \tag{5.3}$$

Consider the homomorphism

$$\begin{array}{rccc} \pi : & \mathbb{Z}^d & \longrightarrow & \Gamma \\ & (k_1, \dots, k_d) & \mapsto & \prod_j g^{k_j p_j} \end{array}$$

The kernel of π is precisely Λ. The surjectivity of π follows from (5.3). □

6. Fourier transforms

Let $\alpha_1, \ldots, \alpha_d$ be vectors in $\mathbb{R}^n$ which are "polarized" in the sense that, for some $v \in \mathbb{R}^n$, the inner products (α_i, v) are all positive. Given $\Phi \in \mathbb{R}^n$ consider the function

$$e^{i(\Phi,x)} \prod_{j=1}^{d} (\alpha_j, x)^{-1} . \tag{6.1}$$

Since this function isn't well-defined on all of $\mathbb{R}^n$, its Fourier transform is also not well-defined. However, there is a unique measure, μ, on $\mathbb{R}^n$, with the following two properties:

1. The inverse Fourier transform of μ is equal to (6.1) on the set

$$(\alpha_j, x) \neq 0 \quad , \quad j = 1, \ldots, d .$$

2. μ is supported in the half space

$$(y, v) \geq (\Phi, v) .$$

Proof. *Existence:* One can take for μ the measure

$$\delta_\Phi * H_{\alpha_1} * \ldots * H_{\alpha_d} \tag{6.2}$$

where δ_Φ is the delta-measure at Φ and

$$H_\alpha(f) = \int_0^\infty f(t\alpha)dt$$

for continuous functions of compact support, f.[10]

Uniqueness: Let μ_0 be another measure satisfying (1) and (2), and let $\varepsilon = \mu - \mu_0$. First note that the inverse Fourier transform of ε is a tempered distribution on $\mathbb{R}^n$ supported on the union of the hyperplanes $(\alpha_j, y) = 0$. Therefore, we have

$$\left(\prod_j (\alpha_j, y)\right)^q (\mathcal{F}^{-1}\varepsilon)(y) = 0$$

for sufficiently large q, which implies that

$$\left(\prod_j D_{\alpha_j}\right)^q \varepsilon = 0,$$

[10] Keeping track of constants, we should in fact take

$$\mu = \left(\frac{(2\pi)^{n(d+1)}}{i^d}\right) \delta_\Phi * H_{\alpha_1} * \ldots * H_{\alpha_d} .$$

where D_{α_j} is the derivative in the direction of α_j. Finally, since ε is supported on the half space $(y, v) \geq (\Phi, v)$, an inductive argument shows that $\varepsilon = 0$ (see [GP]). □

Another description of this measure is the following: Let

$$\mathbb{R}^d_+ = \{(s_1, \ldots, s_d), s_j \geq 0\}$$

be the positive orthant in $\mathbb{R}^d$ and let $L : \mathbb{R}^d_+ \longrightarrow \mathbb{R}^n$ be the map

$$L(s_1, \ldots, s_d) = \Phi + \sum_{j=1}^{d} s_j \alpha_j \ .$$

The assumption that the α_j's are polarized implies that this is a proper mapping, so the measure

$$L_* ds_1 \ldots ds_d \tag{6.3}$$

is well-defined.

Theorem 6.4. *The measures* (6.2) *and* (6.3) *are equal.*

Proof. Both measures evaluated at a continuous function of compact support, f, give

$$\int_{\mathbb{R}^d_+} f(\Phi + \Sigma\, s_j \alpha_j) ds_1 \ldots ds_d \ .$$

□

Corollary 6.5. *If the vectors, α_1, …, α_d, span $\mathbb{R}^n$, the measure* (6.2) *is absolutely continuous with respect to Lebesgue measure.*

Proof. If suffices to prove that the set of critical points of the map L is of measure zero, which will be the case if and only if α_1, …, α_d span $\mathbb{R}^n$. □

Thus, if the vectors α_1, …, α_d span $\mathbb{R}^n$, the Radon-Nikodym theorem allows us to write the measure (6.2) in the form

$$\nu(y) dy_1 \ldots dy_n$$

the function ν being in L^1_{loc}. In fact it is easy to see that, up to a scalar multiple,[11]

$$\nu(y) = \text{volume}\, \Delta(y) \ , \tag{6.6}$$

$\Delta(y)$ being the convex polytope:

$$\{s \in \mathbb{R}^d_+, \Phi + \Sigma\, s_j \alpha_j = y\} \ .$$

[11] By an appropriate normalization of Lebesgue measure in the space, $\Sigma\, s_j \alpha_j = 0$, one can make this scalar equal to one.

By abuse of notation we will refer to (6.6) as the *Fourier transform* of the function (6.1). .Let us compute, in the same spirit, the Fourier transform, $\tilde{\nu}$, of the function

$$e^{i(\Phi,x)}\prod_{j=1}^{d}(\alpha_j,x)^{-N_j} .$$

Letting $N = N_1 + \ldots + N_d$, it follows from what we've just proved that $\tilde{\nu}(y)$ is the volume of the polytope consisting of all N-tuples

$$t = (t_{1,1},\ldots,t_{1,N_1},\ldots,t_{d,1},\ldots,t_{d,N_d})$$

in $\mathbb{R}_+^N$ satisfying

$$\Phi + \sum_{j=1}^{d}\left(\sum_{i=1}^{N_j} t_{j,i}\right)\alpha_j = y .$$

Let's denote this polytope by $\widetilde{\Delta}(y)$. From the mapping

$$\mathbb{R}_+^N \longrightarrow \mathbb{R}_+^d \quad , \quad s_j = \sum_{i=1}^{N_j} t_{j,i} ,$$

one gets a fibration of $\widetilde{\Delta}(y)$ over $\Delta(y)$, the volume of the fiber over s being

$$\frac{s_1^{N_1-1}}{(N_1-1)!}\cdots\frac{s_d^{N_d-1}}{(N_d-1)!} .$$

Hence

$$\tilde{\nu}(y) = \text{volume}\,\widetilde{\Delta}(y) = \int_{\Delta(y)}\frac{s_1^{N_1-1}}{(N_1-1)!}\cdots\frac{s_d^{N_d-1}}{(N_d-1)!}ds . \tag{6.7}$$

7. Proof of Theorem 1

The equivariant index fixed point formula for orbifolds [V2, Ch, M2, D] says that for $ix \in \mathfrak{g}$, ix *generic* in the sense that $\exp ix$ generates G, the trace of $\exp ix$ on the virtual vector space (1.4) is equal to the sum over the fixed point components, F, of local traces, $\chi_F(x)$. In order to define these local traces, we will need some notation: Let $\widehat{F}$ be the orbifold associated to F, with natural mapping $\mu : \widehat{F} \longrightarrow F$, as explained in A.10. ($\widehat{F}$ will in general have various connected components of possibly different dimensions.) $\widehat{NF}$ and $\widehat{\mathbf{E}}_j$, $j = 1, \dots, m$, denote the pull-backs of NF and $\mathbf{E}_j$ to $\widehat{F}$ by μ, so that $\widehat{NF}$ splits into $\widehat{\mathbf{E}}_1 \oplus \dots \oplus \widehat{\mathbf{E}}_m$. $\widehat{\mathbf{L}}_{\widehat{F}}$ denotes the pull-back via μ of $\mathbf{L}$ to $\widehat{F}$. Finally, we fix that:

$$A_0 = A(\widehat{\mathbf{L}}_{\widehat{F}}) \;, \qquad A_j = A(\widehat{\mathbf{E}}_j) \;, \quad j = 1, \dots, m \;,$$

where A is the canonical automorphism defined in A.10. With these definitions we have:

$$\chi_F(x) = e^{i\Phi_F(x)} \int_{\widehat{F}} \frac{1}{m_{\widehat{F}}} \cdot \frac{A_0^{-1} \exp[\omega_{\widehat{F}}] \operatorname{Td}(\widehat{F})}{D_{\widehat{F}} \cdot \prod_j \det\left(I - A_j \exp(-i\alpha_j(x) I - \Omega(\widehat{\mathbf{E}}_j))\right)} \tag{7.1}$$

$\omega_{\widehat{F}}$ being the pull-back of ω to $\widehat{F}$, and $-2\pi i \Omega(\widehat{\mathbf{E}}_j)$ being the curvature form associated with a Hermitian connection on $\widehat{\mathbf{E}}_j$. For the definitions of $m_{\widehat{F}}$ and $D_{\widehat{F}}$, please see §A.2 and §A.10.

If $\epsilon_j = -1$, the j-th term in the denominator can be rewritten:

$$(-1)^{n_j} \det A_j \; e^{-in_j\alpha_j(x)} \det\exp\left(-\Omega(\widehat{\mathbf{E}}_j)\right) \det\left(I - A_j^{-1} e^{i\alpha_j(x)} \exp\Omega(\widehat{\mathbf{E}}_j)\right) . \tag{7.2}$$

Let $\widehat{\mathbf{E}}^{\#}$ be the line bundle

$$\bigotimes_{\epsilon_j=-1} \wedge^{n_j}(\widehat{\mathbf{E}}_j^+)$$

and let $A^{\#}$ be the canonical automorphism of $\widehat{\mathbf{E}}^{\#}$

$$\prod_{\epsilon_j=-1} \det A_j^+ \; .$$

Recall that $\exp(\operatorname{trace} B) = \det(\exp B)$ for an endomorphism B, and $\operatorname{trace}\Omega(E) = \Omega(\wedge^n E)$, where n is the rank of the vector bundle E. Hence, if we substitute (7.2) into (7.1) and use the above definitions, we can rewrite (7.1) in "polarized" form

$$(-1)^{\sigma_F} e^{i(\Phi_F - \alpha_F^{\#})(x)} \int_{\widehat{F}} \frac{1}{m_{\widehat{F}}} \cdot \frac{A_0^{-1} A^{\#} \exp\left(\omega_{\widehat{F}} - \Omega(\widehat{\mathbf{E}}^{\#})\right) \operatorname{Td}(\widehat{F})}{D_{\widehat{F}} \cdot \prod_j \det\left(I - A_j^+ e^{-i\alpha_j^+(x)} \exp(-\Omega(\widehat{\mathbf{E}}_j^+))\right)} \; .$$

By Theorem 4.1 this can be expanded into an infinite trigonometric series:

$$(-1)^{\sigma_F} \sum_k c_k e^{i(\Phi_F - \alpha_F^\# - k_1\alpha_1^+ - \ldots - k_m\alpha_m^+)} \tag{7.3}$$

summed over all non-negative integer m-tuples, k, where c_k is equal to

$$\int_{\widehat{F}} \frac{1}{m_{\widehat{F}}} \cdot \frac{\prod_j \left(\operatorname{trace} \mathcal{S}^{k_j}(A_j^+ \exp(-\Omega(\mathbf{E}_j^+)))\right) A_0^{-1} A^\# \exp(\omega_{\widehat{F}} - \Omega(\widehat{\mathbf{E}}^\#)) \operatorname{Td}(\widehat{F})}{D_{\widehat{F}}} \tag{7.4}$$

or simply

$$\int_{\widehat{F}} \frac{1}{m_{\widehat{F}}} \cdot \frac{\operatorname{Ch}(\widehat{\mathbf{E}}(k) \otimes \widehat{\mathbf{L}}) \operatorname{Td}(\widehat{F})}{D_{\widehat{F}}} \tag{7.5}$$

which is the *Kawasaki-Riemann-Roch number* of the orbifold line bundle $\mathbf{E}(k) \otimes \mathbf{L}$ over F [Ka]

$$\operatorname{KRR}(F, \mathbf{E}(k) \otimes \mathbf{L}) \,. \tag{7.6}$$

On the other hand, for $ix \in \mathfrak{g}$ the trace of $\exp ix$ on the vector space (1.4) is equal to

$$\sum_{\alpha \in \mathbb{Z}^G} \mathcal{M}(\alpha) e^{i\alpha(x)} \tag{7.7}$$

and by comparing (7.3) with (7.7) one gets the identity (2.1).

8. Proof of Theorem 2

By Theorem 1, $\mathcal{M}^{(n)}(n\alpha)$ is equal to the sum

$$\sum(-1)^{\sigma_F}\mathcal{N}_F^{(n)}(n\alpha)$$

where

$$\mathcal{N}_F^{(n)}(n\alpha) = \sum_k \int_{\widehat{F}} \frac{1}{m_{\widehat{F}}} \cdot \frac{\operatorname{Ch}(\widehat{\mathbf{E}}_F(k)\otimes\widehat{\mathbf{L}}^n)\operatorname{Td}(\widehat{F})}{D_{\widehat{F}}} \tag{8.1}$$

summed over all non-negative integral solutions, k, of the equation

$$n\Phi_F - k_1\alpha_{F,1}^+ - \ldots - k_m\alpha_{F,m}^+ - \alpha_F^\# = n\alpha\ .$$

(Notice that if we replace $\mathbf{L}$ by $\mathbf{L}^n$ we must replace ω by $n\omega$, Φ by $n\Phi$ and A_0 by A_0^n.) As in §7 we will omit the subscript F's in the double indices, and let $\alpha_{F,j}=\alpha_j$, etc. Let $2p=\dim F$, and $q=\dim\Delta(\alpha)$. By (8.1)

$$n^{-(d-\gamma)}\mathcal{N}^{(n)}(n\alpha) = n^{-(d-\gamma-p)}\sum_k\int_{\widehat{F}} n^{-p}\frac{1}{m_{\widehat{F}}}\cdot\frac{\operatorname{Ch}(\widehat{\mathbf{E}}_F(k)\otimes\widehat{\mathbf{L}}^n)\operatorname{Td}(\widehat{F})}{D_{\widehat{F}}}\ . \tag{8.2}$$

Lemma 8.3. *Up to an error of order $O\left(\frac{1}{n}\right)$, (8.2) is equal to*

(8.4)

$$n^{-(d-\gamma-p)}\sum_k\frac{1}{|\Gamma_F|}\sum_{g\in\Gamma_F}\rho_{F,0}^{-n}(g)\rho_F^\#(g)\int_F\exp\omega_F\prod_j\operatorname{trace}\mathcal{S}^{k_j}\left(\rho_{F,j}^+(g)\exp(-\Omega(\mathbf{E}_j^+)/n)\right)$$

where $\rho_{F,0}$, $\rho_F^\#$, $\rho_{F,j}^+$ are the representations of the structure group Γ_F of F on the orbifold charts of $\mathbf{L}$, $\mathbf{E}^\#$ and $\mathbf{E}_j^+$ over F.

Proof. Denote by $\widehat{F}_l$ the connected components of $\widehat{F}$ and let $2p_l=\dim\widehat{F}_l$. With ω and A_0 replaced by $n\omega$ and A_0^n in (7.4), the integrand in this expression can be expanded into a sum of terms of the form

$$\pm\frac{1}{m_{\widehat{F}_l}}\cdot\frac{A_0^{-n}A^\#\mathcal{A}\ n^{-p}\ (n\omega_{\widehat{F}})^r\wedge\Omega_{i_1}\wedge\ldots\wedge\Omega_{i_s}\wedge\Omega(\widehat{\mathbf{E}}^\#)^\nu\wedge T_\mu}{D_{\widehat{F}_l}}$$

where $\mathcal{A}$ is a factor involving the A_j^+, Ω_{i_a} is a coefficient of the curvature form $\Omega(\widehat{\mathbf{E}}_{i_a}^+)$, and T_μ is the component of degree 2μ of $\operatorname{Td}(\widehat{F}_l)$. However, this term can only contribute to the integral if $r+s+\nu+\mu=p_l$, in which case it can be rewritten as

$$\pm\frac{1}{m_{\widehat{F}_l}}\cdot\frac{A_0^{-n}A^\#\mathcal{A}\ n^{-p+p_l}\ \omega_{\widehat{F}}^r\wedge(\Omega_{i_1}/n)\wedge\ldots\wedge(\Omega_{i_s}/n)\wedge(\Omega(\widehat{\mathbf{E}}^\#)/n)^\nu\wedge T_\mu/n^\mu}{D_{\widehat{F}_l}}\ .$$

The terms in this sum for which ν or μ is positive or $p > p_l$ can be discarded since they contribute errors of order $O(\frac{1}{n})$. We are left with the components of $\widehat{F}$ whose dimension is $2p$. Such components are indexed by the conjugacy classes of Γ_F (see A.10). Hence (8.2) is equal to

$$n^{-(d-\gamma-p)} \sum_k \sum_{\underline{g}\in \mathrm{Conj}(\Gamma_F)} \frac{1}{m_{\widehat{F}_{\underline{g}}}} \int_{\widehat{F}_{\underline{g}}} \frac{A_0^{-n} A^{\#} \exp \omega_{F_{\underline{g}}} \prod_j \operatorname{trace} \mathcal{S}^{k_j} \left(A_j^+(g)\exp(-\Omega(\mathbf{E}_j^+)/n)\right)}{D_{\widehat{F}_{\underline{g}}}} .$$

On the component $\widehat{F}_{\underline{g}}$, corresponding to $\underline{g} \in \mathrm{Conj}(\Gamma_F)$, we have

$$A_0 = \rho_{F,0}(g) , \qquad A^{\#} = \rho_F^{\#}(g) , \qquad A_j^+ \sim \rho_j^+(g)$$

where "$\sim$" means "conjugate to". We also have $D_{\widehat{F}_{\underline{g}}} = 1$ since $\dim \widehat{F}_{\underline{g}} = \dim F$, and $m_{\widehat{F}_{\underline{g}}} \doteq |Z(g)|$, where $Z(g)$ is the centralizer of g. Moreover, for any class function χ

$$\sum_{\underline{g}\in \mathrm{Conj}(\Gamma_F)} \frac{\chi(g)}{|Z(g)|} = \frac{1}{|\Gamma_F|} \sum_{g\in\Gamma_F} \chi(g) .$$

It remains to observe that because the natural immersion $\mu : \widehat{F}_{\underline{g}} \longrightarrow F$ is bijective, the integral over $\widehat{F}_{\underline{g}}$ coincides with the integral over F. □

Since the action of G is effective, we have $q = m - \gamma$, and hence $d - \gamma - p = q + \Sigma_j(n_j - 1)$. By Theorems 4.7, 4.12 and Remark 4.11, (8.4) is equal, up to an error of order $O(\frac{1}{n})$, to

(8.5)
$$n^{-q} \sum_k \mathcal{W}_{F,k} \operatorname{Res} \left[e^{\frac{k_1}{n} z_1 + \ldots + \frac{k_m}{n} z_m} \int_F \frac{\exp \omega_F}{\det(z_1 I + \Omega(\mathbf{E}_1^+)) \ldots \det(z_m I + \Omega(\mathbf{E}_m^+))} \right]$$

where "Res" takes the sum of finite residues and

$$\mathcal{W}_{F,k} = \frac{1}{|\Gamma_F|} \sum_{g\in\Sigma_F} \rho_{F,0}^{-n}(g) \rho_F^{\#}(g) \prod_j (\lambda_{F,j}(g))^{k_j} ,$$

$\Sigma_F \subseteq \Gamma_F$ being the abelian subgroup of those elements for which each $\rho_{F,j}^+(g)$ is a scalar multiple, $\lambda_{F,j}(g)$, of the identity. By the orthogonality relations for characters we know that

$$\sum_{g\in\Sigma_F} \left(\rho_{F,0}^{-n} \rho_F^{\#} \prod_j \lambda_{F,j}^{k_j} \right)(g) = 0$$

unless $\left(\rho_{F,0}^{-n}\rho_F^{\#}\prod_j \lambda_{F,j}^{k_j}\right)$ is the trivial character of Σ_F, in which case this sum is $|\Sigma_F|$. Hence, (8.5) is equal to

(8.6)

$$n^{-q}\sum_k \frac{|\Sigma_F|}{|\Gamma_F|}\operatorname{Res}\left[e^{\frac{k_1}{n}z_1+\ldots+\frac{k_m}{n}z_m}\int_F \frac{\exp\omega_F}{\det(z_1 I+\Omega(\mathbf{E}_1^+))\ldots\det(z_m I+\Omega(\mathbf{E}_m^+))}\right]$$

summed over all k satisfying

$$\Phi_F - \frac{k_1}{n}\alpha_1^+ - \ldots - \frac{k_m}{n}\alpha_m^+ - \frac{\alpha^\#}{n} = \alpha$$

and

$$\rho_{F,0}^{-n}\rho_F^{\#}\prod_j \lambda_{F,j}^{k_j} = \mathrm{id} \tag{8.7}$$

in the character group of Σ_F. By Theorems 5.1 and 5.2, equation (8.7) is picking up a sublattice of order $|\Sigma_F|$ inside $\mathbb{Z}^m$. Therefore, as n tends to infinity (8.6) tends to the integral

$$\int_{\Delta_F(\alpha)} \operatorname{Res}\left[e^{sz}\int_F \frac{1}{|\Gamma_F|}\cdot\frac{\exp\omega_F}{c_1(z_1)\ldots c_m(z_m)}\right] ds\ , \tag{8.8}$$

where $c_j(z_j) := \det\left(z_j I + \Omega(\mathbf{E}_j^+)\right)$ is the Chern polynomial of $\mathbf{E}_j^+$.

Remark 8.9. The integral of any continuous function f over an open set U, can be approximated by an average of f over a lattice inside U; by refining the lattice, the error can be made arbitrarily small. Consequently, when $q>0$ (8.6) converges pointwise to (8.8) for all rational α. However, when $q=0$ (8.6) converges to (8.8) only in a weak sense: for α rational and any continuous test function f, the distribution (8.6) evaluated at f tends to the evaluation of the distribution (8.8) at f, as n tends to infinity.

9. Proof of Theorem 3

By definition, the Duistermaat-Heckman measure is the "push-forward" by the moment map of the symplectic measure on M, *i.e.*, for a Borel subset B of $\mathfrak{g}^*$,

$$\mu_{DH}(B) = \int_{\Psi^{-1}(B)} \frac{\omega^d}{d!} \,.$$

If ν denotes the Radon-Nikodym derivative (2.5), let $\check{\nu}$ denote its inverse Fourier transform multiplied by $(2\pi)^\gamma$. Then $\check{\nu}$ evaluated at $2\pi\xi \in \mathfrak{g}$ is

$$\check{\nu}(2\pi\xi) = \int_{\mathfrak{g}^*} e^{i(\eta, 2\pi\xi)}\nu(\eta)d\mu_{Leb} = \int_M e^{i(\Psi,2\pi\xi)}\frac{\omega^d}{d!} = \int_M e^{i(\Phi,x)}e^{\omega}\,, \qquad \xi = ix\,,\ \Psi = \tfrac{i}{2\pi}\Phi$$

and by the orbifold abelian localization formula [BV1, BV2, M2] this is equal to the sum over fixed point components

$$\sum_F e^{i\Phi_F(x)}\frac{1}{|\Gamma_F|}\int_F \frac{\exp\omega_F}{\prod_j \det(i\alpha_{F,j}(x)I + \Omega(\mathbf{E}_{F,j}))} \tag{9.1}$$

provided $\alpha_{F,j}(x) \neq 0$ for all F and j.[12] Polarizing and dropping the F's in the double subscripts, the F summand becomes

$$\frac{(-1)^{\sigma_F}}{|\Gamma_F|}e^{i\Phi_F(x)}\int_F \frac{\exp\omega_F}{\prod_j \det(i\alpha_j^+(x)I + \Omega(\mathbf{E}_j^+))}\,.$$

By Theorem 4.1 this is equal to

(9.2)
$$\frac{(-1)^{\sigma_F}}{|\Gamma_F|}\cdot\frac{e^{i\Phi_F(x)}}{\prod_j\left(i\alpha_j^+(x)\right)^{n_j}}\sum_{k=0}^{\infty}\frac{1}{\prod_j\left(i\alpha_j^+(x)\right)^{k_j}}\int_F \exp\omega_F\prod_j \operatorname{trace}\mathcal{S}^{k_j}\left(-\Omega(\mathbf{E}_j^+)\right)\,.$$

(Note that this sum is finite: the terms on the right are zero if $2\sum n_jk_j > \dim F$.) By the Fourier inversion formula the Radon-Nikodym derivative (2.5) is the Fourier transform of (9.1) (divided by $(2\pi)^\gamma$), and we can compute this by computing the Fourier transforms of the summands in (9.2) and summing over k and the fixed point components. By formula (6.7), the Fourier transform of

$$\frac{e^{i\Phi_F(x)}}{\prod_j\left(i\alpha_j^+(x)\right)^{k_j+n_j}}$$

[12] In particular, when the fixed points are isolated we get the orbifold "exact stationary phase" formula:

$$\frac{1}{d_M}\int_M e^{i(\Phi,x)}\frac{\omega^d}{d!} = \sum_p \frac{1}{|\Gamma_p|}\cdot\frac{e^{i\Phi_p(x)}}{\prod_j i\alpha_{p,j}(x)}\,.$$

is (up to a constant) the function

$$f_k(\alpha) = \int_{\Delta_F(\alpha)} \frac{s_1^{k_1+n_1-1}}{(k_1+n_1-1)!} \cdots \frac{s_m^{k_m+n_m-1}}{(k_m+n_m-1)!} ds \, .$$

Substituting this into (9.2) one gets

$$(9.3) \qquad \frac{(-1)^{\sigma_F}}{|\Gamma_F|} \int_{\Delta_F(\alpha)} ds \left(\int_F \exp \omega_F \prod_j \frac{s_j^{k_j+n_j-1}}{(k_j+n_j-1)!} \operatorname{trace} \mathcal{S}^{k_j}(-\Omega(\mathbf{E}_j^+)) \right) \, .$$

However, by formula (4.5),

$$(9.4) \qquad \operatorname{trace} \mathcal{S}^{k_j}(-\Omega(\mathbf{E}_j^+)) = \frac{1}{2\pi i} \int_{\Gamma_j} \frac{z_j^{n_j+k_j-1}}{\det(z_j I + \Omega(\mathbf{E}_j^+))}$$

Γ_j being a contour about the origin in the z_j plane containing the zeroes of $(z_j I + \Omega(\mathbf{E}_j^+))$. If $-n_j < k_j < 0$ the integral on the right is zero, so by substituting (9.4) into (9.3) and summing over all $k_j \geq 0$ (or, equivalently, over all $k_j + n_j - 1 \geq 0$) one gets for the Fourier transform of (9.2):

$$\frac{(-1)^{\sigma_F}}{|\Gamma_F|} \int_{\Delta_F(\alpha)} ds \left[\operatorname{Res} e^{sz} \int_F \frac{\exp \omega_F}{\prod_j c_{F,j}(z_j)} \right] \, .$$

10. Multiplicity formulas when the fixed points are isolated

In this section, we assume that the action τ of G has only isolated fixed points. We write a more explicit expression for (2.1), and derive some particular cases interesting for applications.

10.1. General formula for isolated fixed points. When $F = p$ is an isolated fixed point with structure group Γ_p, then $\widehat{F}$ is a union of $|\mathrm{Conj}(\Gamma_p)|$ points. Let $g_l \in \Gamma_p$ represent the conjugacy class associated to $\tilde{p}_l \in \widehat{F}$ (see §A.10), and let $\Sigma_l = Z(g_l)$ be the structure group of $\tilde{p}_l$ ($Z(g_l)$ is the centralizer of g_l). $NF = T_pM$ decomposes into m orbifold vector subspaces

$$\mathbf{V}_{p,1} \oplus \ldots \oplus \mathbf{V}_{p,m} \ , \tag{10.1}$$

m depending on p, such that the isotropy representation of G on $\mathbf{V}_{p,j}$ has weight, $\alpha_{p,j}$ (where $\alpha_{p,j} \neq \alpha_{p,k}$ for $j \neq k$, and all $\alpha_{p,j} \neq 0$). Let $n_{p,j}$ be the dimension of $\mathbf{V}_{p,j}$. We fix a polarization as in §2. $\widehat{\mathbf{V}}_{p,j}$ and $\widehat{\mathbf{L}}_p$ are the pull-backs of $\mathbf{V}_{p,j}$ and $\mathbf{L}$ to $\widehat{F}$. Each $\widehat{\mathbf{V}}_{p,j}$ is a $n_{p,j}$-dimensional representation of Γ_p, which we will denote by $\rho_{p,j}$. These representations are polarized by setting $\rho^+_{p,j}$ to be the representation $\rho_{p,j}$ or its dual $\rho^*_{p,j}$ depending on whether $\epsilon_{p,j} = +1$ or -1. We also set $\rho^{\#}_p = \prod_{\epsilon_{p,j}=-1} \det \rho^+_{p,j}$. Let $\chi_{p,j}$, $\chi^+_{p,j}$, $\chi^{\#}_p$ be the characters of $\rho_{p,j}$, $\rho^+_{p,j}$, $\rho^{\#}_p$, respectively, and let $\chi_{p,0}$ be the character of the Γ_p-representation on $\widehat{\mathbf{L}}_p$. For the component $\tilde{p}_l$ of $\widehat{F}$, we have

$$m_{\tilde{p}_l} = |\Sigma_l| \ , \qquad A_j|_{\tilde{p}_l} = \rho_{p,j}(g_l) \ , \qquad A_0|_{\tilde{p}_l} = \chi_{p,0}(g_l) \ ,$$
$$A^+_j|_{\tilde{p}_l} = \rho^+_{p,j}(g_l) \ , \qquad A^{\#}|_{\tilde{p}_l} = \chi^{\#}_p(g_l) \ .$$

Formula (2.1) becomes:

$$\mathcal{N}_p(\alpha) = \sum_{k \in \Delta_p(\alpha + \alpha^{\#}_p)} \sum_l \frac{1}{|\Sigma_l|} \chi_{p,0}(g_l)^{-1} \chi^{\#}_p(g_l) \prod_j \left(\mathrm{trace}\, \mathcal{S}^{k_j} \rho^+_{p,j}(g_l)\right) \ . \tag{10.2}$$

Since there are $|\Gamma_p|/|\Sigma_l|$ elements in the conjugacy class of g_l, we can rewrite this as

$$\mathcal{N}_p(\alpha) = \sum_{k \in \Delta_p(\alpha + \alpha^{\#}_p)} \frac{1}{|\Gamma_p|} \sum_{g \in \Gamma_p} \chi_{p,0}(g)^{-1} \chi^{\#}_p(g) \prod_j \left(\mathrm{trace}\, \mathcal{S}^{k_j} \rho^+_{p,j}(g)\right) \ . \tag{10.3}$$

Remark 10.4. In order to deduce the multiplicity formula for the case of isolated fixed points, we could have applied the Fourier transform directly to the Atiyah-Bott-Lefschetz formula for orbifolds. The Atiyah-Bott-Lefschetz formula gives the character of the virtual linear representation of G on $\mathrm{Ind}(D_{\mathbf{L}})$ as a sum of contributions from the isolated fixed points, p. Namely, this character evaluated at an element $\exp ix$ of G, with $ix \in \mathfrak{g}$, is:

$$\mathcal{L}(x) = \sum_p \mathcal{L}_p(x) ,$$

where the contribution $\mathcal{L}_p(x)$ of p is:

$$\mathcal{L}_p(x) = \frac{1}{|\Gamma_p|} \sum_{g\in\Gamma_p} \frac{\chi_{p,0}^{-1}(g) \cdot e^{i\Phi_p(x)}}{\prod_j \det\left(I - \rho_{p,j}(g) e^{-i\alpha_{p,j}(x)}\right)} .$$

This formula is a special case of the equivariant index formula for orbifolds when all components of the fixed point set are isolated points.

Remark 10.5. If M is a manifold and M^G is finite, (10.3) reduces to

$$\mathcal{N}_p(\alpha) = \sum_{k\in\Delta_p(\alpha+\alpha_p^{\#})} \binom{k_1 + n_{p,1} - 1}{n_{p,1} - 1} \cdots \binom{k_m + n_{p,m} - 1}{n_{p,m} - 1} . \tag{10.6}$$

(Recall that $\dim \mathcal{S}^k V = \binom{k+n-1}{n-1}$, where $n = \dim V$.) This agrees with the result in [GLS].

Remark 10.7. For the case of isolated fixed points, the measure formula (2.3) becomes

$$\sum_{p\in M^G} \frac{(-1)^{\sigma_p}}{|\Gamma_p|} \int_{\Delta_p(\alpha)} \frac{s_1^{n_{p,1}-1} \cdots s_m^{n_{p,m}-1}}{(n_{p,1}-1)! \ldots (n_{p,m}-1)!} . \tag{10.8}$$

If M is a manifold, this agrees with the result in [GLS].

10.2. Special case of distinct weights. Suppose that the weights $\alpha_{p,j}$ are all distinct. Then the $\widehat{\mathbf{V}}_{p,j}$ are 1-dimensional representations of Γ_p and we have

$$\chi_p^{\#} = \prod_{\epsilon_{p,j}=-1} \chi_{p,j}^{-1} , \qquad \operatorname{trace} \mathcal{S}^{k_j}(\rho_{p,j}^{+}) = \left(\chi_{p,j}^{+}\right)^{k_j} .$$

Formula 10.3 can be written as

$$\mathcal{N}_p(\alpha) = \sum_{k\in\Delta_p(\alpha+\alpha_p^{\#})} \frac{1}{|\Gamma_p|} \sum_{g\in\Gamma_p} \left[\chi_{p,0}^{-1} \chi_p^{\#} \left(\prod_j (\chi_{p,j}^{+})^{k_j} \right) \right] (g) .$$

But by Frobenius' orthogonality relations for the particular case of 1-dimensional characters, we know that for any finite group Γ

$$\frac{1}{|\Gamma|} \sum_{g\in\Gamma} \left(\chi_0^{-1} \cdot \chi^{\#} \cdot \chi_1^{k_1} \cdots \chi_m^{k_m} \right)(g) = 0$$

unless $\left(\chi_0^{-1} \cdot \chi^{\#} \cdot \chi_1^{k_1} \ldots \chi_m^{k_m}\right)$ is the trivial character of Γ, in which case this sum is 1. Hence,

$$\begin{array}{rcl} \mathcal{N}_p(\alpha) & = & \text{the number of solutions of the equation} \\ & & \quad \Phi_p - \sum k_j \cdot \alpha_{p,j}^{+} = \alpha + \alpha_p^{\#} \\ (10.9) & & \text{where the } k_j\text{'s are non-negative integers satisfying} \\ & & \quad (\chi_{p,1}^{+})^{k_1} \ldots (\chi_{p,m}^{+})^{k_m} = \chi_{p,0} \prod_{\epsilon_{p,j}=-1} \chi_{p,j} \\ & & \text{in the character group of } \Gamma_p. \end{array}$$

Remark 10.10. In particular, when Γ_p is the cyclic group of order m, this equation can be simplified. Identifying Γ_p with a subgroup of S^1, we can write

$$\left(\chi_{p,0}^{-1} \cdot \chi_p^{\#}\right)(g) = g^{m_0} \qquad \text{and} \qquad (\chi_{p,j}^{+})(g) = g^{m_j}$$

for $g = e^{2\pi i \frac{r}{m}} \in \mathbb{Z}/m\mathbb{Z}$, $r \in \{0, 1, \ldots, m-1\}$. Then

$$\sum_{g \in \Gamma_p} \chi_{p,0}^{-1}(g) \cdot \chi_p^{\#}(g) \cdot \prod_j \left(\chi_{p,j}^{+}(g)\right)^{k_j} = \sum_{r=0}^{m-1} \exp\left(\frac{2\pi i}{m}\left(m_0 + \sum_j k_j m_j\right) r\right)$$

which is zero unless

$$m_0 + \sum_j k_j m_j \equiv 0 \mod m.$$

(This is a consequence of the formula for a geometric sum.) Therefore,

$$\begin{array}{rcl} \mathcal{N}_p(\alpha) & = & \text{the number of solutions of the equation} \\ & & \quad \Phi_p - \sum k_j \cdot \alpha_{p,j}^{+} = \alpha + \alpha_p^{\#} \\ (10.11) & & \text{where the } k_j\text{'s are non-negative integers} \\ & & \text{satisfying the congruence relation} \\ & & \quad m_0 + \sum_j k_j m_j \equiv 0 \mod m\ . \end{array}$$

An illustration of this case is given in the next section.

11. Application to a twisted projective space

In this section we illustrate formula (10.3) in the case of a special kind of toric varieties called twisted projective spaces.

11.1. Fixed point data. Our twisted (or "weighted") n-dimensional projective space, $\widetilde{\mathbb{CP}}^n$, is the orbifold obtained by taking the quotient of $\mathbb{C}^{n+1} - 0$ by the action of $\mathbb{C}^*$ given by

$$\rho(\omega)(z_0, \ldots, z_n) = (\omega^{q_0} z_0, \ldots, \omega^{q_n} z_n), \quad \omega \in \mathbb{C}^*, q_i \in \mathbb{Z}^+ .$$

We assume that the $q_0, \ldots, q_n$ are pairwise prime, and, for simplicity, that $q_0 > q_1 > \ldots > q_n > 0$. Hence, $\widetilde{\mathbb{CP}}^n$ is singular at the n+1 points:

$$\begin{aligned} p_0 &= [1,0,0,\ldots,0] \\ p_1 &= [0,1,0,\ldots,0] \\ &\vdots \\ p_n &= [0,0,\ldots,0,1] \end{aligned}$$

which have as stabilizers the cyclic groups $\mathbb{Z}/q_0, \ldots, \mathbb{Z}/q_n$.

The standard action of the torus T^{n+1} on $\mathbb{C}^{n+1} - 0$,

$$(e^{i\theta_0}, \ldots, e^{i\theta_n}) \cdot [z_0, \ldots, z_n] = [e^{i\theta_0} z_0, \ldots, e^{i\theta_n} z_n]$$

induces an action on the orbifold $\widetilde{\mathbb{CP}}^n$. Its fixed points are precisely $p_0, p_1, \ldots, p_n$. (This is not an effective action, since the diagonal circle acts trivially.)

Given $d \in \mathbb{Z}$, let $\mathbf{L}$ be the holomorphic orbifold line bundle over $\widetilde{\mathbb{CP}}^n$ associated with the representation

$$\gamma : \mathbb{C}^* \longrightarrow \mathrm{Aut}(\mathbb{C}), \quad \gamma(\omega)c = \omega^{-d} c ,$$

i.e., $\mathbf{L} = [(\mathbb{C}^{n+1} - 0) \times \mathbb{C}]/\{[\rho(\omega)z, c] \sim [z, \gamma(\omega)c], \omega \in \mathbb{C}^*\}$. In order for $\mathbf{L}$ to have no singular fibers, the condition

$$q_i | d, i = 0, \ldots, n, \quad \text{or equivalently,} \quad q_0 \cdots q_n | d$$

is required. We will write $d = l \cdot q_0 \cdots q_n$.

On the cross-section $z_n = 1$, $\omega \in \mathbb{Z}/q_n$ (that is, $\omega = e^{2\pi i \frac{q}{q_n}}, q = 0, \ldots, q_n - 1$) acts by

$$\rho(\omega)[z_0, \ldots, z_{n-1}, 1] = [\omega^{q_0} z_0, \ldots, \omega^{q_{n-1}} z_{n-1}, 1] ,$$

whereas

$$\begin{aligned} (e^{i\theta_0}, \ldots, e^{i\theta_n}) \cdot [z_0, \ldots, z_{n-1}, 1] &= [e^{i\theta_0} z_0, \ldots, e^{i\theta_{n-1}} z_{n-1}, e^{i\theta_n}] \\ &\sim [e^{i(\theta_0 - \frac{q_0}{q_n}\theta_n)} z_0, \ldots, e^{i(\theta_{n-1} - \frac{q_{n-1}}{q_n}\theta_n)} z_{n-1}, 1] . \end{aligned}$$

We define an action of T^{n+1} on $\mathbf{L}$ induced by letting T^{n+1} act trivially on the second factor of $(\mathbb{C}^{n+1} - 0) \times \mathbb{C}$. In particular:

$$(e^{i\theta_0}, \dots, e^{i\theta_n}) \cdot [(0, \dots, 0, 1), c] = [(0, \dots, 0, e^{i\theta_n}), c] \sim [(0, \dots, 0, 1), e^{-i\theta_n \frac{d}{q_n}} c] ,$$

so the action of $(e^{i\theta_0}, \dots, e^{i\theta_n}) \in T^{n+1}$ on the fiber of $\mathbf{L}$ above $[0, \dots, 0, 1]$ is given by multiplication by $e^{-i\theta_n \frac{d}{q_n}}$.

Let's now interpret these results in terms of isotropy weights. Consider the lift to the smooth $\mathbb{C}^n$ covering of the $z_n = 1$ cross-section (which roughly amounts to ignoring the last coordinate z_n when it's 1). The orbifold weights of the T^{n+1} representation on $T_0\mathbb{C}^n$ are:

$$\alpha_{n,j} = (0, \dots, 0, 1_{[j]}, 0, \dots, 0, -\frac{q_j}{q_n}), \qquad \text{with } j \neq n$$

(where the square brackets indicate the slot). The orbifold weight of the T^{n+1} representation on $\mathbf{L}_{p_n}$ is:

$$\mu_n = (0, \dots, 0, -\frac{d}{q_n}) .$$

The $\alpha_{n,j}$-weightspace inside $T_0\mathbb{C}^n \simeq \mathbb{C}^n$ is

$$V_{n,j} = \left\{ (0, \dots, 0, z_{[j]}, 0, \dots, 0) \in \mathbb{C}^n \right\} ,$$

hence the character, $\chi_{n,j}$, of the $\mathbb{Z}/q_n$ representation on it is:

$$\chi_{n,j}(\omega) = \omega^{q_j}, \qquad \omega \in \mathbb{Z}/q_n .$$

Finally the character of the $\mathbb{Z}/q_n$ representation on $\mathbf{L}_{p_n}$ is trivial.

In general we have the following data:

- fixed points and orbifold structure groups

$$\begin{array}{rcll} p_0 & = & [1, 0, 0, \dots, 0] , & \mathbb{Z}/q_0 \\ p_1 & = & [0, 1, 0, \dots, 0] , & \mathbb{Z}/q_1 \\ & \vdots & & \vdots \\ p_n & = & [0, 0, \dots, 0, 1] , & \mathbb{Z}/q_n \end{array}$$

- tangent weights

$$\text{at } p_i: \quad \begin{array}{ll} \alpha_{i,j} = (0, \dots, 0, 1_{[j]}, 0, \dots, 0, \left(-\frac{q_j}{q_i}\right)_{[i]}, 0, \dots, 0) , & j < i \\ \alpha_{i,j} = (0, \dots, 0, \left(-\frac{q_j}{q_i}\right)_{[i]}, 0, \dots, 0, 1_{[j]}, 0, \dots, 0) , & i < j \end{array}$$

- weight on $\mathbf{L}$

$$\text{at } p_i: \qquad \mu_i = (0, \dots, 0, \left(-\frac{d}{q_i}\right)_{[i]}, 0, \dots, 0) .$$

We are in the good situation where at each p_i all weights are different, so that the weight spaces, $V_{i,j}$, $i \neq j$, are 1-dimensional linear representations of $\mathbb{Z}/q_i$. (Anyway, as $\mathbb{Z}/q_i$ is abelian, all $V_{i,j}$ would necessarily break into 1-dimensional representations of $\mathbb{Z}/q_i$.) The corresponding characters, $\chi_{i,j}$, of $\mathbb{Z}/q_i$ on the $V_{i,j}$ are:

$$\chi_{i,j}(\omega) = \omega^{q_j} , \qquad \omega \in \mathbb{Z}/q_i ,$$

whereas the characters of $\mathbb{Z}/q_i$ on $\mathbf{L}_{p_i}$ are all trivial.

11.2. Multiplicity formula. For our polarization we take $v = (1, \dots, 1)$, so that (always with $i \neq j$):

$$\epsilon_{i,j} := \text{sign of } \left(1 - \tfrac{q_j}{q_i}\right) = \begin{cases} +1 & \text{if } i < j \\ -1 & \text{if } j < i \end{cases} \qquad \sigma_i := \sum_{\epsilon_{i,j}=-1} 1 = i$$

$$\alpha_{i,j}^{+} := |\alpha_{i,j}| = \begin{cases} \alpha_{i,j} & \text{if } i < j \\ -\alpha_{i,j} & \text{if } j < i \end{cases} \qquad \alpha_i^{\#} := -\sum_{j<i} \alpha_{i,j}$$

$$\chi_{i,j}^{+} := \begin{cases} \chi_{i,j}^{-1} & \text{if } j < i \\ \chi_{i,j} & \text{if } i < j \end{cases} \qquad \chi_i^{\#} := \prod_{j<i} \chi_{i,j}^{-1} .$$

Hence, in this case, we have that the multiplicity of the weight α in $\mathrm{Ind}(D_{\mathbf{L}})$ is:

$$\mathcal{M}(\alpha) = \sum_i (-1)^i \mathcal{N}_i(\alpha) ,$$

where, according to §10.2,

$$\mathcal{N}_i(\alpha) = \text{ the number of solutions of the equation}$$

$$\begin{cases} q_i\alpha_0 &= q_i(k_0 + 1) \\ &\vdots \\ q_i\alpha_{i-1} &= q_i(k_{i-1} + 1) \\ q_i\alpha_i &= -\sum_{j<i}(k_j + 1)q_j + \sum_{i<j} k_j q_j \\ q_i\alpha_{i+1} &= -q_i k_{i+1} \\ &\vdots \\ q_i\alpha_n &= -q_i k_n \end{cases}$$

where the k_j's are non-negative integers satisfying the congruence relation

$$d - \sum_{j<i}(k_j+1)q_j + \sum_{i<j} k_j q_j \equiv 0 \mod q_i \ ,$$

which is equivalent to the following:

$$\mathcal{N}_i(\alpha) = \begin{cases} 1 & \text{if } \alpha_0 q_0 + \ldots + \alpha_n q_n = -d \ , \ \alpha_j > 0 \text{ for } j < i \ \text{ and } \ \alpha_j \leq 0 \text{ for } j > i \\ 0 & \text{otherwise} \ . \end{cases}$$

11.3. Explicit multiplicity computation. Since $\widetilde{\mathbb{CP}}^n$ is a complex analytic orbifold, the virtual vector space $\mathrm{Ind}(D_{\mathbf{L}})$ is

$$\mathrm{Ind}(D_{\mathbf{L}}) = \bigoplus_{i=0}^{n}(-1)^i H^i(\widetilde{\mathbb{CP}}^n, \mathcal{O}(\mathbf{L}))$$

where $\mathcal{O}(\mathbf{L})$ is the sheaf of holomorphic sections of $\mathbf{L}$ (see remark 1.6).

By an equivariant Kodaira theorem, if $d \geq 0$ then $H^i(\widetilde{\mathbb{CP}}^n, \mathcal{O}(\mathbf{L})) = 0$ for $i > 0$. As for $H^0(\widetilde{\mathbb{CP}}^n, \mathcal{O}(\mathbf{L}))$ this is the global holomorphic sections of $\mathbf{L}$. A basis for these is given by monomials $z_0^{k_0} \ldots z_n^{k_n}$ on $\mathbb{C}^{n+1}$, with $k_0, \ldots, k_n \geq 0$. Only those monomials which transform under the action of $\mathbb{C}^*$ according to the law

$$\begin{aligned} \left[(z_0, \ldots, z_n), \left(z_0^{k_0} \ldots z_n^{k_n}\right)\right] &= \left[(\omega^{q_0} z_0, \ldots, \omega^{q_n} z_n), \omega^{\Sigma k_j q_j}\left(z_0^{k_0} \ldots z_n^{k_n}\right)\right] \\ &= \left[(z_0, \ldots, z_n), \omega^{-d+\Sigma k_j q_j}\left(z_0^{k_0} \ldots z_n^{k_n}\right)\right] \end{aligned}$$

represent sections of $\mathbf{L}$.

Therefore, $H^0(\widetilde{\mathbb{CP}}^n, \mathcal{O}(\mathbf{L}))$ is spanned by the elementary sections $\mathcal{Z}_k$, with graphs

$$\{ \ [(z_0, \ldots, z_n), \left(z_0^{k_0} \ldots z_n^{k_n}\right)] \ \}$$

such that

$$k_0 q_0 + \ldots + k_n q_n = d \qquad \text{and} \qquad k_0, \ldots, k_n \geq 0 \ .$$

In particular, the dimension of $H^0(\widetilde{\mathbb{CP}}^n, \mathcal{O}(\mathbf{L}))$ is the number of integer lattice points $(k_0, \ldots, k_n)$ satisfying $k_0 q_0 + \ldots + k_n q_n = d$, $k_0, \ldots, k_n \geq 0$.

T^{n+1} acts on the graph of $\mathcal{Z}_k$ by

$$\begin{aligned} (e^{i\theta_0}, \ldots, e^{i\theta_n}) \cdot \{[(z_0, \ldots, z_n), \left(z_0^{k_0} \ldots z_n^{k_n}\right)]\} &= \{[\left(e^{i\theta_0} z_0, \ldots, e^{i\theta_n} z_n\right), \left(z_0^{k_0} \ldots z_n^{k_n}\right)]\} \\ &= \{[(z_0, \ldots, z_n), e^{-ik_0\theta_0 - \ldots - ik_n\theta_n} \cdot \left(z_0^{k_0} \ldots z_n^{k_n}\right)]\} \end{aligned}$$

We conclude that each $\mathcal{Z}_k$ is an eigenvector with weight $(-k_0, \ldots, -k_n)$, and therefore

$$\mathcal{M}(\alpha) = \begin{cases} 1 & \text{if } \alpha_0 q_0 + \ldots + \alpha_n q_n = -d \ \text{ and } \ \alpha_0, \ldots, \alpha_n \leq 0 \\ 0 & \text{otherwise} \ . \end{cases}$$

APPENDIX A. ORBIFOLDS

In this appendix, we review some basic material on orbifolds, borrowing from Satake [S1], Haefliger-Salem [HS], Meinrenken [M2], Lerman-Tolman [LT] and Duistermaat [D]. For more details, please see these references.

A.1. Definition of orbifold. The notion of orbifold was introduced by Satake [S1] under the name of V-manifold, as a generalization of the notion of manifold.

An n-dimensional **orbifold** M is a Hausdorff topological space $|M|$, plus an **atlas of orbifold charts**, $\{(\widetilde{\mathcal{U}},\Gamma,\phi)\}$, where

- $\widetilde{\mathcal{U}}$ is a connected open subset of $\mathbb{R}^n$;
- Γ is a finite group acting on $\widetilde{\mathcal{U}}$ by linear transformations; we assume that the set of all fixed points of Γ has codimension at least two; we do *not* assume the action of Γ to be effective;
- $\phi : \widetilde{\mathcal{U}} \longrightarrow |M|$ is a continuous Γ-invariant map inducing a homeomorphism from $\widetilde{\mathcal{U}}/\Gamma$ to $\mathcal{U} := \phi(\widetilde{\mathcal{U}}) \subseteq |M|$;

such that

1. the $\{\mathcal{U}\}$ form a basis of open sets in $|M|$;
2. the $\{(\widetilde{\mathcal{U}},\Gamma,\phi)\}$ satisfy the following compatibility condition: If $\mathcal{U}_1 \subseteq \mathcal{U}_2$, then there is an **injection** $\kappa : (\widetilde{\mathcal{U}}_1,\Gamma_1,\phi_1) \longrightarrow (\widetilde{\mathcal{U}}_2,\Gamma_2,\phi_2)$ which, by definition, amounts to:

a diffeomorphism	$k : \widetilde{\mathcal{U}}_1 \longrightarrow k(\widetilde{\mathcal{U}}_1) \subseteq \widetilde{\mathcal{U}}_2$,
and a group isomorphism	$K : \Gamma_1 \longrightarrow K(\Gamma_1) \subseteq \Gamma_2$,
such that	$\phi_1 = \phi_2 \circ k$,
and k is K-equivariant	$k \circ g = K(g) \circ k$, for all $g \in \Gamma_1$.

Two such atlases are equivalent if their union is still an atlas. Notice that we do not require the action of Γ to be effective.

An ordinary manifold is a special case of orbifold where every group Γ is the identity group. Quotients of manifolds by locally free actions of Lie groups are orbifolds. In fact, any orbifold has a presentation of this form (see §A.6).

A.2. Stratification and structure groups. Given $p \in M$, let $(\widetilde{\mathcal{U}},\Gamma,\phi)$ be an orbifold chart for a neighborhood $\mathcal{U}$ of p. Then the **orbifold structure group** of p, Γ_p, is the isotropy group of a pre-image of p under ϕ. The group Γ_p is well defined up to isomorphism. We may choose an orbifold chart $(\widetilde{\mathcal{U}},\Gamma,\phi)$ such that $\phi^{-1}(p)$ is a single point (which is fixed by Γ). In this case $\Gamma \cong \Gamma_p$, and we say that $(\widetilde{\mathcal{U}},\Gamma_p,\phi)$ is a **structure chart** for p.

The **orbifold stratification** is the natural stratification of M into submanifolds, according to the type of the structure group. On each connected component of M, there is an open dense set of "regular" points in M for which the order of the structure group is minimal. This is called the **principal stratum** of M. Let M be connected. The (abstract) isotropy group of principal stratum is called the **structure group** of M and denoted by Γ_M. The order of Γ_M is called the **multiplicity** of M. When M is not connected the multiplicities of its connected components define a locally constant function $m_M : M \longrightarrow \mathbb{N}$, called the **multiplicity function**.

A.3. Suborbifolds. Let M and N be two orbifolds with a continuous inclusion of the underlying topological spaces $|\imath| : |N| \longrightarrow |M|$. Suppose there exists an atlas of orbifold charts $\{(\tilde{\mathcal{U}}, \Gamma, \phi)\}$ for M such that for each chart $(\tilde{\mathcal{U}}, \Gamma, \phi)$ intersecting N (*i.e.* $\phi(\tilde{\mathcal{U}}) \cap |\imath|(|N|) \neq$) the pre-image of N is given by the intersection of $\tilde{\mathcal{U}}$ with a linear subspace V of $\mathbb{R}^n$. Let Γ_V be the subgroup of those elements in Γ whose action preserves V. We say that N is a **suborbifold** of M if the collection of the $\{(\tilde{\mathcal{U}} \cap V, \Gamma_V, \phi|_{\tilde{\mathcal{U}} \cap V})\}$, together with the induced injections, forms an atlas of orbifold charts for N.

From the above charts $\{(\tilde{\mathcal{U}}, \Gamma, \phi)\}$ we can further extract the subgroup of those transformations in Γ which are the identity when restricted to $\tilde{\mathcal{U}} \cap V$. This subgroup is again well defined as an abstract group for each connected component of N; it is just the isotropy group of the corresponding principal stratum; when N is connected it is the structure group of N.

A.4. Maps and group actions on orbifolds. A **smooth map of orbifolds** $f : M \longrightarrow N$ is a continuous map between the underlying topological spaces satisfying the following condition:

> Let $p \in M$ and let $(\tilde{\mathcal{V}}, \Lambda, \psi)$ be a structure chart for $f(p)$. Then there exists a structure chart $(\tilde{\mathcal{U}}, \Gamma, \phi)$ for p and a *smooth* map $\tilde{f} : \tilde{\mathcal{U}} \longrightarrow \tilde{\mathcal{V}}$ such that $f \circ \phi = \psi \circ \tilde{f}$.

A **smooth function** on M is a collection of smooth invariant functions on each orbifold chart $(\tilde{\mathcal{U}}, \Gamma, \phi)$ which agree on overlaps of the images $\phi(\tilde{\mathcal{U}})$.

A **smooth action** τ of a Lie group G on an orbifold M is a smooth orbifold map $\tau : G \times M \longrightarrow M$ satisfying the usual laws: for all $g_1, g_2 \in G$ and $p \in M$ we have

$$\tau(g_1, \tau(g_2, p)) \doteq \tau(g_1 g_2, p)) \qquad \text{and} \qquad \tau(\mathrm{id}_G, p) \doteq p \ ,$$

where $\doteq$ means *equivalent as maps of orbifolds.*

An action of a Lie group G on an orbifold M induces an infinitesimal action of its Lie algebra $\mathfrak{g}$ on M. We will denote by ξ_M the vector field on M induced by $\xi \in \mathfrak{g}$ (see A.6 for the definition of vector field).

A.5. Orbifold fiber bundles. Given an orbifold M, an **orbifold fiber bundle** $\pi : \mathbf{E} \longrightarrow M$ is defined by a collection of Γ-equivariant fiber bundles

$$\begin{array}{ccc} Z & \hookrightarrow & \widetilde{\mathbf{E}}_{\mathcal{U}} \\ & & \downarrow \pi_{\widetilde{\mathcal{U}}} \\ & & \widetilde{\mathcal{U}} \end{array}$$

over each chart $(\widetilde{\mathcal{U}}, \Gamma, \phi)$, together with suitable compatibility conditions. Notice that the fibers $\pi^{-1}(p)$ are in general not diffeomorphic to Z, but only to some quotient of Z by an the action of the structure group Γ_p.

The fibers of an **orbifold vector bundle** are **vector orbispaces**, *i.e.* quotients of the form $\mathbf{V}/\Gamma$, where $\mathbf{V}$ is a vector space and Γ is a finite subgroup of $GL(\mathbf{V})$. (Γ is also a subgroup of the isotropy group of the base point of that fiber.) Let $N(\Gamma)$ be the normalizer of Γ in $GL(\mathbf{V})$. The group $GL(\mathbf{V}/\Gamma) := N(\Gamma)/\Gamma$ acts on the orbifold $\mathbf{V}/\Gamma$.

A **Riemannian metric** on an orbifold vector bundle $\mathbf{E}$ is a Γ-invariant smooth field of inner products on the fibers of $\widetilde{\mathbf{E}}_{\mathcal{U}}$ for each orbifold chart $(\widetilde{\mathcal{U}}, \Gamma_p, \phi)$, agreeing on overlaps.

An **orbifold complex vector bundle** is an orbifold vector bundle equipped with an almost complex structure. A **complex structure** on an orbifold vector bundle $\mathbf{E}$ is a Γ-invariant smooth field of linear operators J, with $J^2 = -\mathrm{id}$, on the fibers of $\widetilde{\mathbf{E}}_{\mathcal{U}}$ for each orbifold chart $(\widetilde{\mathcal{U}}, \Gamma_p, \phi)$, agreeing on overlaps.

An **orbifold Hermitian vector bundle** is an orbifold complex vector bundle equipped with a Hermitian structure. A **Hermitian structure** on an orbifold complex vector bundle $\mathbf{E}$ is a smooth field $(\cdot,\cdot)$ of positive definite Hermitian structures in the fibers of $\mathbf{E}$. That is, for smooth sections s, t of $\mathbf{E}$, (s,t) is a complex valued smooth function which is complex linear in s and satisfies

$$\overline{(s,t)} = (t,s) \qquad \text{and} \qquad (s,s) > 0 \quad \text{if} \quad s \neq 0\ .$$

We can apply to orbifold vector bundles *duals*, *tensor products*, *exterior products*, etc., by defining these constructions over each orbifold chart.

Orbifold sections of an orbifold fiber bundle E are defined by Γ-invariant sections on orbifold charts, agreeing on overlaps. It is *not* true that any orbifold mapping $\sigma : M \longrightarrow E$ with $\pi \circ \sigma = \mathrm{id}_M$ gives rise to a section of E. For example, $\mathbb{C}/\mathbb{Z}_2 \longrightarrow \mathrm{pt}$ does not have any nonvanishing sections.

Let $\mathbf{E}$ be an orbifold complex vector bundle over an orbifold M. A **connection** on $\mathbf{E}$ is a differential operator

$$\nabla : C^\infty(M, \mathbf{E}) \longrightarrow C^\infty(M, T^*M \otimes \mathbf{E})$$

which satisfies the condition

$$\nabla(fs) = df \otimes s + f\nabla s, \qquad f \in C^\infty(M)\ ,\ s \in C^\infty(M, \mathbf{E})\ .$$

Given an orbifold Hermitian vector bundle $\mathbf{E}$, we say that ∇ is a **Hermitian connection** if for any vector field ξ on M we have

$$\xi(s,t) = (\imath_\xi \nabla s, t) + (s, \imath_\xi \nabla t), \qquad s,t \in C^\infty(M,\mathbf{E}) \ .$$

If $(\mathbf{E},\nabla)$ is an orbifold Hermitian vector bundle $\mathbf{E}$ with a Hermitian connection ∇, there is a notion of **curvature form** generalizing the definition in the smooth case.[13] Likewise, we can define orbifold characteristic classes. Let F be the curvature form with respect to ∇. Then $\Omega(\mathbf{E}) = \frac{i}{2\pi}F$ is a real closed two-form. For instance, the **first Chern class** of $\mathbf{E}$ is the cohomology class represented by the trace of $\Omega(\mathbf{E})$:

$$c_1(\mathbf{E}) = [\operatorname{trace}\Omega(\mathbf{E})] \ ,$$

and the **Chern character** of $\mathbf{E}$ is

$$\operatorname{Ch}(\mathbf{E}) = [\operatorname{trace}(\exp \Omega(\mathbf{E}))] \ .$$

The **Todd class** is given by a polynomial in the Chern classes, which can be described more simply in the following way. By the orbifold version of the splitting principle, we can write any n-dimensional complex vector bundle $\mathbf{E}$ as a formal direct sum of line bundles:

$$\mathbf{E} = \mathbf{L}_1 \oplus \ldots \oplus \mathbf{L}_n \ .$$

The Todd class is then given by

$$\operatorname{Td}(\mathbf{E}) = \prod_i \frac{c_1(\mathbf{L}_i)}{1 - \exp(-c_1(\mathbf{L}_i))} \ .$$

$\operatorname{Td}(\mathbf{E})$ is well-defined, because it only depends on symmetric combinations of the $c_1(\mathbf{L}_i)$, which are determined by the Chern classes of $\mathbf{E}$.

A.6. Tangent bundles. Given a point p in an orbifold M, and a structure chart $(\tilde{\mathcal{U}}, \Gamma_p, \phi)$ for p, we define the **orbifold tangent space** at p, T_pM, to be the quotient of the tangent space to $\phi^{-1}(p)$ in $\tilde{\mathcal{U}}$ by its induced action of Γ:

$$T_pM := T_{\phi^{-1}(p)}\tilde{\mathcal{U}}/\Gamma_p \ .$$

The union of the orbifold tangent spaces at all p, with transition functions induced by the compatibility relations, build up the **orbifold tangent bundle** TM. TM is actually a smooth manifold outside the zero section. The general linear group $GL(n)$ acts locally freely on $TM - 0$, and $M \cong (TM - 0)/GL(n)$.[14]

A **vector field** ϑ on M is a Γ-invariant vector field $\vartheta_{\mathcal{U}}$ on each orbifold chart $(\tilde{\mathcal{U}}, \Gamma, \phi)$ agreeing on overlaps. Equivalently, it is a section of the orbifold tangent bundle. **Differential forms** can be similarly defined, as orbifold sections of the

[13]Please see §A.7 for the definition of curvature for *line* bundles. By the splitting principle, the case of line bundles is the most important.

[14]Choosing a Riemannian metric and taking the orthonormal frame bundle, $O(TM)$, we can present any orbifold M as $O(TM)/O(n)$, as remarked by Kawasaki.

exterior algebra of the orbifold cotangent bundle. A **Riemannian orbifold** is an orbifold equipped with a Riemannian metric on its tangent bundle. An **almost complex orbifold** is an orbifold equipped with a complex structure on its tangent bundle. The **Todd class** of an almost complex orbifold is, by definition, the Todd class of its tangent bundle. Continuing in this fashion, we can define the orbifold analogues of De Rham theory and Dolbeault theory.

An orbifold is **orientable** if we can assign an orientation of $\widetilde{\mathcal{U}}$ for each chart $(\widetilde{\mathcal{U}}, \Gamma, \phi)$ agreeing on overlaps. Let M be an orientable orbifold and let ω be a differential form of top degree. If ω has compact support on an open connected set $\mathcal{U}$ trivialized by an orbifold chart $(\widetilde{\mathcal{U}}, \Gamma, \phi)$, the **integral** of ω is

$$\int_M \omega = \frac{m_{\mathcal{U}}}{|\Gamma|} \int_{\widetilde{\mathcal{U}}} \widetilde{\omega}$$

where $m_{\mathcal{U}}$ is the multiplicity of the connected component of M containing $\mathcal{U}$, and $\widetilde{\omega}$ is the Γ-invariant form on $\widetilde{\mathcal{U}}$ representing ω. The integral of an arbitrary top form is then defined by using a partition of unity.

A.7. Connections on line bundles. Let $\mathbf{L} \xrightarrow{\pi} M$ be an orbifold complex line bundle over M, and let $\mathbf{L}^*$ denote $\mathbf{L}$ − zero section. Given a connection ∇ on $\mathbf{L}$, we can associate to it the unique 1-form $A \in \Omega^1(\mathbf{L}^*)$ which satisfies

- A is invariant under $\mathbb{C}^*$;
- for all $p \in M$, $A|_{\mathbf{L}^*_p} = \beta_p$ where β_p is the unique 1-form in $\mathbf{L}^*_p$ such that $\tau^*(\beta_p) = \frac{dz}{z}$ for any $\mathbb{C}^*$-map $\tau : \mathbb{C}^* \longrightarrow \mathbf{L}^*_p$;
- given a local section $s \in C^\infty(\mathcal{U}, \mathbf{L}^*|_{\mathcal{U}})$, ($\mathcal{U} \subseteq M$ open) one has $\frac{\nabla s}{s} = s^*A$.

A is called the **connection form** of $(\mathbf{L}, \nabla)$ (see [Ko2]). The kernel of A is called the **horizontal subspace** of $T\mathbf{L}^*$, whereas the **vertical subspace** is formed by the vectors tangent to the fibers. A **horizontal form** is a form on $\mathbf{L}^*$ which vanishes on vertical vectors. Given a vector field ξ_M on M, there is a unique horizontal vector field $\xi_M^\#$ on $\mathbf{L}$ such that $\pi_* \xi_M^\# = \xi_M$; $\xi_M^\#$ is called the **horizontal lift** of ξ_M by ∇. A **horizontal section** is a section $s \in C^\infty(M, \mathbf{L})$ which satisfies $\nabla s = 0$. dA is a $\mathbb{C}^*$-invariant horizontal 2-form. As a consequence, there is a unique closed 2-form on M, $F = F(A)$, such that $dA = \pi^* F$ (here π denotes the projection $\mathbf{L}^* \longrightarrow M$). F is called the **curvature** of $(\mathbf{L}, \nabla)$. It is the obstruction to finding a horizontal local section of $\mathbf{L}^*$.[15] Furthermore, the cohomology class of F is independent of the choice of connection on $\mathbf{L}$ (because any two connections differ by a 1-form on M).

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[15] If we start from any local section s and try to modify it into a new section $t = fs$ which is horizontal, we need to solve the following differential equation for f

$$0 = \nabla t = df \otimes s + f \nabla s$$

or equivalently

$$\frac{df}{f} = -\frac{\nabla s}{s} .$$

When ∇ is Hermitian and $(s,s)=1$, the connection form satisfies

$$s^*A + \overline{s^*A} = 0 \ .$$

Proof. For all $\xi \in TM$,

$$0 = \xi(s,s) = (\imath_\xi \nabla s, s) + (s, \imath_\xi \nabla s) = \imath_\xi \left((\nabla s, s) + \overline{(s, \nabla s)} \right) = \imath_\xi \left(s^*A + \overline{s^*A} \right) \ .$$

□

As a consequence, we have $F + \bar{F} = 0$. Therefore, $\frac{i}{2\pi}F$ is a real "integral" closed two-form on M.[16] $\Omega(\mathbf{L}) := \frac{i}{2\pi}F$ is a **Chern form** for $\mathbf{L}$, and the cohomology class represented by it is the **Chern class** of $\mathbf{L}$

$$c(\mathbf{L}) := \left[\tfrac{i}{2\pi}F\right] \ .$$

An orbifold Hermitian line bundle $\mathbf{L}$ with a Hermitian connection is equivalent to an orbifold principal circle bundle $\mathbf{P}$ with a connection, such that $\mathbf{L} = \mathbf{P} \times_{S^1} \mathbb{C}$, and such that the connection on $\mathbf{L}$ is induced from a connection on $\mathbf{P}$. The corresponding connection form A_P on $\mathbf{P}$ satisfies $A_P(\frac{\partial}{\partial\theta}) = i$, where $\frac{\partial}{\partial\theta}$ is the vector field that generates the principal circle action with a period 2π.

A.8. Symplectic orbifolds. A **symplectic orbifold** is an orbifold M equipped with a closed non-degenerate two-form ω. We say that an almost complex structure J is **compatible** with ω if for all $p \in M$ the bilinear form

$$g_p(v,w) = \omega_p(J_p v, w) \ ; \quad v, w \in T_pM \ ,$$

is symmetric and positive definite.

A group G acts **symplectically** on (M,ω) if the action preserves ω. A **moment map** for a symplectic action of a group G is an equivariant map $\Psi : M \longrightarrow \mathfrak{g}^*$ such that

$$\imath_{\xi_M}\omega = -\langle d\Psi, \xi \rangle \qquad \text{for all } \xi \in \mathfrak{g} \ .$$

When a moment map exists, we say that the action of G on (M,ω) is **Hamiltonian**.

On symplectic orbifolds the strata are *symplectic* manifolds, hence even dimensional, hence the principal stratum is connected.

For a symplectic action of a *connected* group G on an orbifold M, it follows from the existence of slices [LT], that the fixed point set M^G is a suborbifold.

Let G act symplectically on (M,ω). At a fixed point p, there is a local action of G on $\widetilde{\mathcal{U}}_p$. If G is compact, this local action gives rise to an action of some cover

Then f exists if and only if $\frac{\nabla s}{s} = s^*\alpha$ is closed.

[16] $\frac{i}{2\pi}F$ is a *rational* class in the sense that integrated over homology 2-cycles yields a rational number. However, any compact orbifold M can be presented as a global quotient of a compact manifold X by a locally free action of a Lie group K. The cohomology of M is isomorphic to the K-equivariant cohomology of X. We call $\frac{i}{2\pi}F$ integral because it is obtained by the Weil recipe as an element in the equivariant cohomology of X with *integral* coefficients, which is the image of an Ad-invariant polynomial in the Lie algebra of K.

$\widetilde{G}$ of the identity component of G, commuting with the action of Γ_p. The group $\widetilde{G}$ is an extension of G of degree not greater than the order of Γ_p. The action of $\widetilde{G}$ induces by its derivative a linear representation of $\widetilde{G}$ on $T_{\phi^{-1}(p)}\widetilde{\mathcal{U}}_p$, with isotropy weights $\alpha_{p,j}$, with $j = 1, 2, \ldots, m$ (these weights are taken with respect to an almost complex structure compatible with the symplectic structure). We will call the $\alpha_{p,j}$ the **orbifold weights** of the G action at p. Notice that it is only the $|\Gamma_p| \cdot \alpha_{p,j}$ that need lie in the weight lattice $\mathbb{Z}^T$ of G, while the $\alpha_{p,j}$ themselves can be rational. The weights $\alpha_{p,j}$ are well-defined since they are independent of the choice of the orbifold chart and of the choice of the compatible almost complex structure.

A.9. Equivariant prequantization. Let $(\mathbf{L}, \nabla)$ be an orbifold Hermitian line bundle over M with a Hermitian connection. Let A and F be the corresponding connection and curvature forms. Suppose there is an action of a torus G on M which lifts to $\mathbf{L}$. (For $\gamma = \dim G$, we fix isomorphisms $G \cong (\mathbb{R}/2\pi\mathbb{Z})^\gamma$, $\mathfrak{g} \cong (i\mathbb{R})^\gamma$, so that $\exp : \mathfrak{g} \longrightarrow G$, $(i\theta_1, \ldots, i\theta_\gamma) \mapsto (e^{i\theta_1}, \ldots, e^{i\theta_\gamma})$ has kernel $(2\pi i\mathbb{Z})^\gamma$.) By averaging if necessary, we can assume that A (and hence F) is an invariant form. (This is equivalent to ∇ commuting with the action of G.) In this case we say that ∇ is an **invariant connection**. Let ξ_L be the vector field on $\mathbf{L}^*$ generated by $\xi \in \mathfrak{g}$. Since $A(\xi_L)$ is constant on fibers, we can define a map $\Psi : M \longrightarrow \mathfrak{g}^*$ by the condition

$$\pi^*\langle\Psi, \xi\rangle = \tfrac{i}{2\pi} A(\xi_L) \ .$$

Ψ is G-invariant because A is G-invariant. Furthermore, this definition implies that

$$\langle d\Psi, \xi\rangle = -\tfrac{i}{2\pi} \imath_{\xi_M} F \ .$$

Proof. This follows from

$$\pi^*\langle d\Psi, \xi\rangle = \tfrac{i}{2\pi} d\imath_{\xi_L} A = -\tfrac{i}{2\pi}\imath_{\xi_L} dA = -\tfrac{i}{2\pi}\imath_{\xi_L}\pi^* F = -\pi^*\imath_{\xi_M}(\tfrac{i}{2\pi}F)$$

and the fact that π^* is injective. □

Hence, Ψ is a **moment map** for $(M, G, \frac{i}{2\pi}F)$ in the sense of §A.8.

$\xi_L - \xi_M^\#$ is a vertical vector field, where $\xi_M^\#$ is the horizontal lift of the vector field on M generated by $\xi \in \mathfrak{g}$. Denote by ϑ the vector field on $\mathbf{L}^*$ which satisfies $A(\vartheta) = 1$. ($\vartheta|_p$ can be obtained by any $\mathbb{C}^*$-map $\tau : \mathbb{C}^* \longrightarrow \mathbf{L}_p^*$ as the push-forward of $z\frac{\partial}{\partial z}$ on $\mathbb{C}^*$.) On the level set $\Psi^{-1}(a)$, $a \in \mathfrak{g}^*$, we have

$$\xi_L = \xi_M^\# - 2\pi i\langle a, \xi\rangle\vartheta \ .$$

If $p \in M$ is a fixed point, then $\xi_M = 0$, ξ_L is vertical for all $\xi \in \mathfrak{g}$, and $\mathbf{L}_p$ is a linear (orbifold) representation of G given by some character $\rho : G \longrightarrow S^1$, $\exp\xi \mapsto e^{\langle\alpha_p, \xi\rangle}$ for $\xi \in \mathfrak{g} \cong (i\mathbb{R})^n$ and a fixed rational weight $\alpha_p \in \mathbb{Q}^\gamma$. We have

$$\xi_L(z) = \langle\alpha_p, \xi\rangle\vartheta$$

where z is a coordinate on $\mathbf{L}_p$.

Proof. The corresponding Lie algebra representation $\rho' = (d\rho)_{id} : \mathfrak{g} \longrightarrow i\mathbb{R}$ is given by $\xi \mapsto \langle \alpha_p, \xi \rangle$. Then

$$\xi_L(z) = \left(\frac{d}{dt} \rho(\exp t\xi) z \right) \bigg|_{t=0} = \rho'(\xi)\vartheta = \langle \alpha_p, \xi \rangle \vartheta \ .$$

□

Therefore

$$\langle \Psi_p, \xi \rangle = \tfrac{i}{2\pi} A\left(\langle \alpha_p, \xi \rangle \vartheta \right) = \tfrac{i}{2\pi} \langle \alpha_p, \xi \rangle \ ,$$

that is,

$$\alpha_p = -2\pi i \Psi_p \ .$$

A.10. Associated orbifolds. Given an orbifold M we define an **associated** orbifold $\widehat{M}$ (see [Ka, F]) by orbifold charts $(\widetilde{\mathcal{V}}, \Gamma, \psi)$ as follows: For each orbifold chart for M, $(\widetilde{\mathcal{U}}, \Gamma, \phi)$, let

$$\widetilde{\mathcal{V}} = \{ (u, g) \in \widetilde{\mathcal{U}} \times \Gamma | g \cdot u = u \} \ .$$

Γ acts on $\widetilde{\mathcal{V}}$ via

$$h \cdot (u, g) = (h \cdot u, hgh^{-1})$$

so that

$$\mathcal{V} := \widetilde{\mathcal{V}}/\Gamma \ .$$

The orbifold charts $(\widetilde{\mathcal{V}}, \Gamma, \psi)$ inherit the compatibility conditions from the $(\widetilde{\mathcal{U}}, \Gamma, \phi)$. In general, $\widehat{M}$ will have various connected components of different dimension. As a set,

$$\widehat{M} = \bigcup_{p \in M} \mathrm{Conj}(\Gamma_p) \ ,$$

where $\mathrm{Conj}(\Gamma_p)$ is the set of conjugacy classes in Γ_p.

Example A.1. Suppose M is the teardrop orbifold having one singularity, P, with structure group $\mathbb{Z}/3$. Then $\widehat{M}$ has three components: two points with structure group $\mathbb{Z}/3$ (corresponding to P paired with the two non-identity elements of $\mathbb{Z}/3$), and one component diffeomeorphic to M.

Remark A.2. Let $m = X/\Gamma$ be the quotient of a compact connected manifold X by a finite group Γ. Let $D : C^\infty(X; \mathbf{E}) \longrightarrow C^\infty(X; \mathbf{F})$ be a Γ-equivariant elliptic differential operator, where $\mathbf{E}$ and $\mathbf{F}$ are smooth Γ-equivariant complex vector bundles. $Q(X) := \mathrm{Ind}(D) = \mathrm{kernel}\, D - \mathrm{cokernel}\, D$ is a finite dimensional virtual representation of Γ. Then

$$\begin{aligned} Q(M) := Q(X)^\Gamma &= \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \mathrm{trace}(g : Q(X) \longrightarrow Q(X)) \\ &= \sum_{\underline{g} \in \mathrm{Conj}(\Gamma)} \frac{1}{|Z(g)|} \mathrm{trace}(g : Q(X) \longrightarrow Q(X)) \ , \end{aligned}$$

$Z(g)$ being the centralizer of g. By the Atiyah-Segal-Singer equivariant index theorem, trace$(g : Q(X) \longrightarrow Q(X))$ equals the evaluation of a certain characteristic class on the set fixed by Γ, X^g.

On the other hand, the associated orbifold is

$$\widehat{M} = \bigcup_{g\in\Gamma} (X^g \times \{g\})/\Gamma = \coprod_{\underline{g}\in\mathrm{Conj}(\Gamma)} X^g/Z(g)\ .$$

$\frac{1}{|Z(g)|}$ trace$(g : Q(X) \longrightarrow Q(X))$ can then be written just in terms of data on the component of $\widehat{M}$ associated to g, $X^g/Z(g)$.

Similar arguments naturally lead to associated orbifolds in other index formulas for arbitrary compact orbifolds.

There is a *canonical* bundle automorphism of any orbifold vector bundle $\mathbf{E}$ over $\widehat{M}$, given on each orbifold chart $(\widetilde{\mathcal{V}}, \Gamma, \psi)$ by the natural action of $g \in \Gamma$ on the fiber of $\widetilde{\mathbf{E}}_{\mathcal{V}}$ above $(u, g) \in \widetilde{\mathcal{V}}$. We will call it the **canonical automorphism** of $\mathbf{E}$ and denote it by $A(\mathbf{E})$.

The natural mapping $\mu : \widehat{M} \longrightarrow M$ is an immersion (since it is an immersion on each chart). Let $\mathbf{N}_{\widehat{M}}$ be the normal bundle of this immersion. When M has an almost complex structure, $\mathbf{N}_{\widehat{M}}$ can be endowed with a Hermitian structure. Let $-2\pi i\Omega(\mathbf{N}_{\widehat{M}})$ be the curvature of $\mathbf{N}_{\widehat{M}}$ with respect to a Hermitian connection. We define the following **twisted** characteristic form

$$D_{\widehat{M}} = \det\left(I - A(\mathbf{N}_{\widehat{M}})\exp(-\Omega(\mathbf{N}_{\widehat{M}}))\right)$$

which appears in our fixed point fromulas.

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