Competitive Multi-period Pricing with Fixed Inventories

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***Abstract*— This paper studies the problem of multi-period pricing for perishable products in a competitive (oligopolistic) market. We study non cooperative Nash equilibrium policies for sellers. At the beginning of the time horizon, the total inventories are given and additional production is not an available option. The analysis for periodic production-review models, where production decisions can be made at the end of each period at some production cost after incurring holding or backorder costs, does not extend to this model. Using results from game theory and variational inequalities we study the existence and uniqueness of equilibrium policies. We also study convergence results for an algorithm that computes the equilibrium policies. The model in this paper can be used in a number of application areas including the airline, service and retail industries. We illustrate our results through some numerical examples.**

I. Introduction

The aim of this paper is to propose and analyze a model of competitive non-collusive oligopoly for sellers competing to sell given inventories of a perishable product over a finite multi-period time horizon. We start by describing the terminology used and giving example of practical situations where such a model would be more applicable than other proposed models in the existing literature.

A *perishable* product is defined as a product that has a finite life, or equivalently, loses its value if not sold before a preset deadline. A *market* is an oligopoly if there are more than one participating seller competing with each other. Each of the participating sellers affects but does not control the market. The competition is *non-collusive* if the sellers do not (or, by law, are not allowed to) enter collusive agreements with each other, but instead compete to capture demand. A *multi-period time horizon* implies that the sellers have a fixed timetable for changing their prices and we characterize the behavior of demand within these discrete intervals. The following examples help explain these concepts.

Consider the problem of pricing advance reservations for *Hotel StayHere* in Atlanta for a particular weekend in October. For simplicity, assume that there is only one type of room offered by the hotel. The hotel starts making reservations for this weekend three months in advance but is unsure of the actual demand that will materialize. Apart from the demand uncertainty, another reason behind this uncertainty is the presence of competing hotels in the same area offering similar rooms. In this paper we will focus on the latter and will assume that the demand is deterministic. The hotel competes with other hotels on the basis of price and offers incentives for early reservations for purposes of customer differentiation (revenue management). While it wants to make sure that it does not end up with too many vacancies, it also also wants to prevent selling too many rooms at low rates early on. There are several settings which have similar characteristics to the one described above. For example, those involving competing airlines selling airfares for their own flights leaving within a small time window with the same origin and destination, are very similar to the one described above. Instead of hotel reservations or airline fares, problems with the same structure could involve shipping containers, broadband communication links or any capacity constrained industry product.

Several models have been proposed for monopolistic versions of this problem. McGill and van Ryzin [31] and the references therein also provide a thorough review of revenue management and pricing models. Bitran and Caldentey [6] provide an overview of pricing models for the monopolistic version of the revenue management problem in which a perishable and non-renewable set of resources satisfy stochastic price-sensitive demand processes over a finite period of time. They survey results on deterministic as well as non-deterministic, single as well as multi-product, and static as well as dynamic pricing cases. Elmaghraby and Keskinocak [20] review the literature and current practices in dynamic pricing in industries where capacity or inventory is fixed in the short run and perishable. They classify monopolistic models on the basis of whether inventory can be replenished or not, whether demand is dependent over time or not, and whether customers are myopic or strategic optimizers. Yano and Gilbert [41] review models for joint pricing and production under a monopolistic setup.

On the competitive side, Vives [40] discusses the development of oligopoly pricing models. A survey by Chan et al [10] summarizes research on joint pricing, inventory control and production decisions in a supply chain. They also survey literature on price and quantity competition in supply chain settings. Cachon and Netessine [8] also survey the problem of competition from a supply chain perspective where the problem is characteristically a periodic production-review model. They discuss both non-cooperative and cooperative games in

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static and dynamic settings.

Our model differs from the competitive supply chain models since we have rigid inventory constraints over the entire horizon and the flexibility to replenish inventory between periods through production is not an available option. Under these modelling restrictions, we lose the convenient structure of the problem which would otherwise allow us to analyze equilibrium with standard techniques. In this paper we take and alternate approach using ideas from variational inequalities. To the best of our knowledge, no general results which could be used to analyze such a model have been presented. A more detailed discussion of this follows in Section II.

The paper is organized as follows. In Section II we review some of the relevant literature in the field and discuss how the results in this paper could not be achieved under the framework of other papers. In Section III we describe the model for the problem and the notation used. Sections III-A and III-B describe the best-response policy problem and the market equilibrium problem respectively. In Section IV we give some theoretical results for these two problems and the conditions under which those results hold. In Section V we analyze an algorithm that can be used to compute the equilibrium policies and provide sufficient conditions for convergence. In Section VI we illustrate these results through some numerical examples.

II. Literature Review

Cournot and Bertrand established the foundations for the analysis of oligopolistic competition between sellers with their quantity and price competition models. Cournot [11] proposed a solution concept for oligopolistic interaction, stability of the resulting solutions, phenomena of collusion, and compared perfect competition with oligopolies. These models were based on quantity competition: The competing sellers controlled their individual production and the prevailing market clearing price was determined by the net production. Bertrand [3] proposed an alternate price competition model where the competing sellers controlled prices while quantities for each seller were determined by the prevailing prices. In Bertrand's model, each seller was assumed to supply the entire quantity demanded at their set price, at an increasing production cost. Edgeworth [17] proposed a price competition model where no seller is required to supply all the forthcoming demand at the set price. In such a case, the residual demand is split amongst all sellers on some rational rule.

We refer the reader to the surveys mentioned in Section I for a comprehensive overview of literature in the area, but we would like to bring special attention to some particularly relevant papers. Rosen [38] proves existence and uniqueness results for general oligopolistic games. The paper shows existence under concavity of the payoff to a seller with respect to it's own strategy space and convexity of the joint strategy space and uniqueness under strict diagonal dominance of the payoff function. Murphy et al [32] analyze equilibrium in a single-period quantity competition model using mathematical programming results. Harker [27] analyze the same model using variational inequalities. Eliashberg and Jeuland [19] model a two stage problem. The market in the first stage is a monopoly and becomes a duopoly in the second stage with the entry of a second seller. The sellers dynamically price their product. The paper analyzes the pricing behavior under the cases that the incumbent seller foresees or does not foresee the entrant.

Pricing models in traditional revenue management research can be classified into two broad categories: static and dynamic. Static pricing models are based on aggregated demand distributions and can be seen as a special case of the multi-product newsvendor problem with fixed production costs and perishable product with no salvage. The extension of the newsvendor problem with price as a decision variable was studied by Zabel [43], Young [42], Dada and Petruzzi [14], etc. Other relevant research includes Zabel [44], Thomas [39], Dada and Petruzzi [13] and Federgruen and Heching [22] who study the single-product, multi-period combined pricing and inventory control problem that is typically solved by dynamic programming.

Dynamic pricing models represent demand as a controllable stochastic point process with price dependent intensity. Gallego and van Ryzin [24] and Zhao and Zheng [45] consider the problem of optimally pricing a given inventory of a single product over a finite planning period before it perishes or is sold at salvage value. There is no reordering. Gallego and van Ryzin [25] and Paschalidis and Tsitsiklis [35] extend this type of model to the dynamic pricing of multiple products whose production draws from a shared supply of resources. Kleywegt [30] gives an optimal control formulation of the multi-period dynamic pricing problem. Kachani and Perakis [28] propose a deterministic fluid model for dynamic pricing and inventory management for non-perishable products in capacitated and competitive make-to-stock manufacturing systems.

Some work has been done recently that explicitly considers the presence of competition within the pricing framework. Dockner and Jørgensen [16] provide a treatment of the optimal pricing strategies for oligopolistic markets from a marketing perspective but not a computational perspective. Bernstein and Federgruen [2] develop a stochastic general equilibrium inventory model for supply chains in an oligopoly environment where the policies involve prices, service level targets and inventory control with linear models of demand. Bertsimas and Perakis [4] propose an optimization approach for jointly learning the demand as a function of price and the competitor response by dynamically setting prices of products in a duopoly environment.

Previously published results prove existence and uniqueness for equilibrium strategies for pricing games under various conditions. We found that none of these conditions hold for our model, hence requiring a new approach for analysis. Results for games with supermodular payoff functions and lattice strategy spaces are well known (See Vives [40]). The problem in this paper does not fulfill the latter requirement. Other results require the payoff function to be concave over a convex strategy space. The problem discussed in this paper can be reformulated so that the strategy space is a lattice and nicely convex but the resulting objective function to be maximized is neither concave nor supermodular. Alternatively, it can be formulated to have a concave objective function, but then the resulting strategy space is no longer convex. We study the uniqueness of equilibrium prices using variational inequalities since this analysis does not require the payoff functions to be concave. To the best of our knowledge, no such analysis for multi-period price competition models for perishable products has been done before.

III. Model Formulation

Consider a market of a single product with a set of competing sellers. Each seller has a given inventory of the product. The time horizon over which each seller wants to sell her inventory of product is divided into discrete intervals. We make the following assumptions regarding the model and policies.

1) **Perfect Information:** We assume that each seller has perfect information about the structure of demand, impact of their prices on their competitors demand and the starting inventory levels of each of her competitors.
2) **Consumer Choice:** We assume that the demand for each seller is a function only of the current period prices (the seller's and her competitors') and that is the only distinguishing factor between products from different sellers.
3) **Demand:** We assume that the demand that each seller will see is a deterministic function of the prices set by all sellers in that period.
4) **Product:** We assume that there is only a single product and the inventory that is saleable over all time periods is perishable at the end of the time horizon.
5) **Objectives:** We assume that the sellers are objectively maximizing their respective revenue over the time horizon of the problem and practices like short-squeezing competitors out of the market by short-term price-cutting, introductory discount pricing to capture market share, etc. are absent.

The notation we use is as follows. A single seller is denoted by $i \in \mathbf{I}$, where $\mathbf{I}$ is the set of all sellers. For ease of notation, we denote the set of all competitors of i by $-i$. The inventory of products belonging to seller i is denoted by C_i. A time interval is denoted by $t \in \mathbf{T}$. A price is set by seller i in every period t and is denoted by p_i^t. Thus, we denote a pricing policy for seller i by $\mathbf{p}_i = (p_i^1, p_i^2, \dots, p_i^T)$ and a set of pricing policies for all sellers by $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_I)$. In period t, the resulting number of buyers who wish to purchase from seller i is denoted by $h_i^t(\mathbf{p}^t)$ (observed demand) and is a function of the price levels set by all sellers in that period. We assume that the current demand is not affected by the previous history of prices. The demand that seller i realizes (i.e the sale made) in period t is denoted by d_i^t. Note the implication that $d_i^t \leq h_i^t(\mathbf{p}^t)$ since the sale made (realized demand) cannot be greater than the demand (observed demand). The relation is not an equality since the seller might be restricted by the actual inventory level available. We use the notation $\mathbf{d}_i = (d_i^1, d_i^2, \dots, d_i^T)$ and $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_I)$ to denote the realized demand. We denote the strategy of the sellers by the prices set, together with the maximum realized demand using $\mathbf{z}_i = (\mathbf{p}_i, \mathbf{d}_i)$ and $\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_I)$. As before, we use the notation $\mathbf{h}_i(\mathbf{p}) = (h_i^1(\mathbf{p}^1), \dots, h_i^T(\mathbf{p}^T))$ and $\mathbf{h}(\mathbf{p}) = (\mathbf{h}_1(\mathbf{p}), \dots, \mathbf{h}_I(\mathbf{p}))$. Given the inventory information $\mathbf{C} = (C_1, \dots, C_I)$, the total payoff to seller i, over the entire time horizon as a function of the sellers' policies $\mathbf{p}$, is denoted by $J_i(\mathbf{p})$. The corresponding payoff in any single period t is given by π_i^t.

We denote the best response policy that maximizes the payoff of seller i over the entire time horizon given that her competitors have adopted policies $\mathbf{p}_{-i}$ by $\mathcal{BR}_i(\mathbf{p}_{-i})$. This will be obtained by solving the best response multi-period pricing problem. We define this problem in section III-A and formulate it as an optimization problem. We denote the resulting best response policy $\mathcal{BR}_i(\mathbf{p}_{-i})$ for seller i by $\mathbf{p'}_i$. In the special case where the pricing policies are at equilibrium, we denote them by $\mathbf{p}^* = (\mathbf{p}^*{}_i, \mathbf{p}^*{}_{-i})$. In what follows, we define the concept of Nash equilibrium policies:

Definition 3.1: The pricing policies for each seller are Nash equilibrium pricing policies if no single seller can increase her payoff by unilaterally changing her policy.

This definition implies that each seller sets her equilibrium pricing policy as the best response to the equilibrium pricing policies of her competitors. This set of policies would then, by definition, be a Nash equilibrium set of policies. See Nash [34] for further details on the notion of a Nash equilibrium in non-cooperative games.

A. Best Response Policy

The best response pricing policy for seller i is the policy that maximizes seller i's payoff in response to all others sellers' pricing policies. We first define the multi-period pricing problem followed by the formulation of the best response problem for this problem.

Definition 3.2: **Multi-period Pricing Problem** Consider a set of sellers $\mathbf{I}$ with inventories $\mathbf{C}$ and time horizon $\mathbf{T}$. The strategy of each seller consists of setting her price levels $\mathbf{p}_i$ optimally, i.e. as best response prices arising from formulation (1) below. The demand observed by seller i in any period is equal to the number of buyers who are willing to buy from her given the price levels for all sellers. Seller i will realize that demand if she has enough inventory.

The best response policy $\mathbf{p}'_i$ of seller i, given all her competitors' policies $\bar{\mathbf{p}}_{-i}$ is the solution of the following optimization problem:

$$\begin{array}{llll} \text{argmax}_{\mathbf{d}_i,\mathbf{p}_i} & \sum_{t=1}^{T} d_i^t p_i^t & & (1) \\ \text{such that} & d_i^t \leq h_i^t(p_i^t, \bar{p}_{-i}^t) & \forall t \in \mathbf{T} & \\ & \sum_{t=1}^{T} d_i^t \leq C_i & & \\ & p_{\text{min}}^t \leq p_i^t \leq p_{\text{max}}^t & \forall t \in \mathbf{T} & \\ & d_i^t \geq 0 & \forall t \in \mathbf{T}. & \end{array}$$

In compact notation, the above can be rewritten as:

$$\begin{array}{ll} \max\limits_{\mathbf{z}_i=(\mathbf{d}_i,\mathbf{p}_i)} & J_i(\mathbf{z}_i) = \frac{1}{2}\mathbf{z}_i'\mathbf{Q}\mathbf{z}_i \\ \text{such that} & \mathbf{d}_i \leq \mathbf{h}_i(\mathbf{p}_i, \bar{\mathbf{p}}_{-i}) \\ & \mathbf{1} \cdot \mathbf{d}_i \leq C_i \\ & \mathbf{p}_{\text{min}} \leq \mathbf{p}_i \leq \mathbf{p}_{\text{max}} \\ & \mathbf{d}_i \geq \mathbf{0}, \end{array}$$

where $\mathbf{Q} = \begin{pmatrix} \mathbf{0} & \mathcal{I} \\ \mathcal{I} & \mathbf{0} \end{pmatrix}$, $\mathcal{I}$ denotes a square identity matrix of suitable dimension.

Note that in this optimization problem, given a $\bar{\mathbf{z}}_{-i}$, seller i selects that $\bar{\mathbf{z}}_i$ that maximizes the objective function $J_i(\mathbf{d}_i, \mathbf{p}_i) = \sum_{t=1}^{T} d_i^t p_i^t$ within the feasible space $\mathcal{K}_i(\bar{\mathbf{z}}_{-i}) = \{(\mathbf{d}_i, \mathbf{p}_i) \mid d_i^t \leq h_i^t(p_i^t, \bar{p}_{-i}^t), \sum_{t=1}^{T} d_i^t \leq C_i, p_{\text{min}}^t \leq p_i^t \leq p_{\text{max}}^t, d_i^t \geq 0, \ \forall t \in \mathbf{T}\}$.

In Section IV-A we show that the solution $\mathbf{z}_i'$ to the best response problem for seller i, given $\bar{\mathbf{z}}_{-i}$, will also satisfy the following variational inequality problem:

$$-\nabla J_i(\mathbf{z}_i') \cdot (\mathbf{z}_i - \mathbf{z}_i') \geq 0 \qquad \forall \mathbf{z}_i \in \mathcal{K}_i(\bar{\mathbf{z}}_{-i}) \tag{2}$$

B. Market Equilibrium Model

The definition of a Nash equilibrium (Definition 3.1) implies that, at equilibrium, each seller would select a pricing policy that optimally solves her own best response problem. Notice that all competitors solve their best response problems simultaneously. Therefore each of them solve variational inequality (2) given their competitors policy $\mathbf{z}_{-i}$.

Given a potential candidate for an equilibrium set of pricing policies for her competitors, $\mathbf{z}_{-i}$, seller i sets her equilibrium pricing policy by solving variational inequality problem (2). Thus the equilibrium set of policies will solve the following set of variational inequality problems:

$$-\nabla J_i(\mathbf{z}_i^*) \cdot (\mathbf{z}_i - \mathbf{z}_i^*) \geq 0 \quad \forall \mathbf{z}_i \in \mathcal{K}_i(\mathbf{z}_{-i}^*) \quad i \in \mathbf{I} \tag{3}$$

The next result shows how this set of variational inequality problems can be combined into a single variational inequality problem. This new problem solves the best response problem for each seller simultaneously, hence determining a set of equilibrium pricing policies.

Proposition 3.1: The equilibrium set of prices satisfies the following variational inequality formulation:

$$F(\mathbf{z}^*) \cdot (\mathbf{z} - \mathbf{z}^*) \geq 0 \qquad \forall \mathbf{z} \in \mathcal{K}, \tag{4}$$

where $F_i(\mathbf{z}^*) = -\nabla J_i(\mathbf{z}_i^*)$, $\forall i \in \mathbf{I}$ and

$$\mathcal{K} \;\; = \;\; \{\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_I) | \mathbf{z}_i \in \mathcal{K}_i(\mathbf{z}_{-i}) \forall i \in \mathbf{I}\}.$$

Proof: *See Perakis and Sood [36].* □

In the remainder of the paper, we refer to variational inequality (4) as the market equilibrium model.

IV. Analysis of Equilibrium

In this section we examine the analytical properties of the best response model (2) and the market equilibrium model (4). These properties hold under certain conditions. We state these conditions and try to provide some intuition on when these conditions hold.

First we impose a condition that ensures that the space of allowed prices is bounded. One way to achieve this boundedness property would be to constrain the prices between some *allowable* upper and lower limits. Under this condition, we can eliminate strategies involving infinitely high price levels. Note that the lower limit could be the zero price level and the higher limit could be the price level at which the demand function vanishes.

Condition 4.1: There exists a minimum and maximum allowable price level. We denote this by p_{min} and p_{max} for each period respectively.

The next condition ensures that the demand for a seller is concave in the seller's price for each period. This condition ensures that the strategy space in the best response problem is convex. The linear demand model trivially satisfies this condition. From a demand elasticity point of view, this condition holds for demand functions which have an increasing price elasticity as price increases. This holds for products where demand decreases faster as price increases.

Condition 4.2: The demand function $h_i^t(p_i^t, \bar{p}_{-i}^t)$ is a concave function of p_i^t over the set of feasible prices for all $i \in \mathbf{I}, t \in \mathbf{T}$ for a fixed $\bar{p}_{-i}^t$.

We also require that the demand is strictly monotonic in price in order to ensure that the best response policy is uniquely defined. For a linear demand case, this implies that the demand function is downward sloping with respect to price as is true for normal goods.

Condition 4.3: For any period t, for any fixed $\bar{p}_{-i}^t$, the function $h_i^t(p_i^t, \bar{p}_{-i}^t)$ is strictly decreasing with respect to p_i^t over the set of feasible prices. Mathematically,

$$\begin{aligned}\left(-h_i^t(\hat{p}_i^t, \bar{p}_{-i}^t) + h_i^t(\check{p}_i^t, \bar{p}_{-i}^t)\right) \cdot (\hat{p}_i^t - \check{p}_i^t) > 0 \\ \forall (\hat{p}_i^t, \check{p}_i^t) \ , \ \hat{p}_i^t \neq \check{p}_i^t \ , \ i \in \mathbf{I}.\end{aligned}$$

The next condition ensures that the market equilibrium model has a unique solution by requiring strict monotonicity on the demand function as a whole. For a two seller linear demand case this is equivalent to saying that the sensitivity of seller i's demand to seller i's price is higher than the sensitivity of seller $-i$'s demand to seller i's price and the sensitivity of seller i's demand to seller $-i$'s price. This makes intuitive sense since we expect the decrease in demand seen by seller i when she raises prices to be more than the resulting increase in demand seen by her competitor. This can be interpreted as saying that upon seeing an increase in seller i's price, some of her customers will prefer to switch to her competitor and some will prefer not to buy at all.

Condition 4.4: The function $-\mathbf{h}(\mathbf{p})$ is strictly monotone with respect to $\mathbf{p}$, over the set of feasible pricing policies $\mathcal{K}$. That is,

$$([-\mathbf{h}(\hat{\mathbf{p}}) + \mathbf{h}(\check{\mathbf{p}})] \cdot (\hat{\mathbf{p}} - \check{\mathbf{p}})) > 0 \qquad \forall \hat{\mathbf{p}}, \check{\mathbf{p}} \in \mathcal{K} \ , \ \hat{\mathbf{p}} \neq \check{\mathbf{p}}.$$

Note that Condition 4.3 refers to the strict monotonicity of $\mathbf{h}_i(\cdot)$ with respect to only $\mathbf{p}_i$, while Condition 4.4 refers to the strict monotonicity of $\mathbf{h}(\cdot)$ with respect to $\mathbf{p}$ and is thus a stronger condition. The following lemmas are consequences of these conditions:

Lemma 4.1: Under Condition 4.2, the constraints

$$d_i^t \leq h_i^t(p_i^t, \bar{p}_{-i}^t) \quad \forall t \in \mathbf{T}, \ i \in \mathbf{I}$$

define a convex set for any given $\bar{p}_{-i}^t$.

Lemma 4.2: Under Condition 4.3, for any fixed $\bar{\mathbf{p}}_{-i}$, the function $-\mathbf{h}_i(\mathbf{p}_i, \bar{\mathbf{p}}_{-i})$ is strictly monotone with respect to $\mathbf{p}_i$, over the set of feasible pricing policies $\mathcal{K}_i(\bar{\mathbf{p}}_{-i})$. That is,

$$[-\mathbf{h}_i(\hat{\mathbf{p}}_i, \bar{\mathbf{p}}_{-i}) + \mathbf{h}_i(\check{\mathbf{p}}_i, \bar{\mathbf{p}}_{-i})] \cdot (\hat{\mathbf{p}}_i - \check{\mathbf{p}}_i) > 0$$
$$\forall \hat{\mathbf{p}}_i, \check{\mathbf{p}}_i \in \mathcal{K}_i(\bar{\mathbf{p}}_{-i}) \ , \ \hat{\mathbf{p}}_i \neq \check{\mathbf{p}}_i \ , \ i \in \mathbf{I}.$$

Lemma 4.3: Under Condition 4.4, the function $-\mathbf{h}^t(\mathbf{p}^t)$ is strictly monotone with respect to $\mathbf{p}^t$, over the set of feasible pricing policies. That is,

$$\left[-\mathbf{h}^t(\hat{\mathbf{p}}^t) + \mathbf{h}^t(\check{\mathbf{p}}^t)\right] \cdot (\hat{\mathbf{p}}^t - \check{\mathbf{p}}^t) > 0$$
$$\forall (\hat{\mathbf{p}}^t, \check{\mathbf{p}}^t) \ , \ \hat{\mathbf{p}}^t \neq \check{\mathbf{p}}^t \ , \ t \in \mathbf{T}.$$

A. Best Response Problem

We start with an intuitive result that characterizes a solution to the best response problem. We use this result while proving the uniqueness of the solution to the best response.

Lemma 4.4: Given a competitor strategy $(\bar{\mathbf{d}}_{-i}, \bar{\mathbf{p}}_{-i})$, the solution $\mathbf{z}'_i = (\mathbf{d}'_i, \mathbf{p}'_i)$ to variational inequality problem (5) satisfies the following relation:

$$d_i'^t \quad = \quad h_i^t(p_i'^t, \bar{p}_{-i}^t), \quad \text{if } d_i'^t > 0.$$

If $d_i'^t = 0$ then there exists $\mathbf{z}''_i$ that also solves variational inequality problem (5) and satisfies the above relation.

Proof: *See Perakis and Sood [36].* □

1) Existence and Uniqueness of Solution to the Best Response Problem: To show that the variational inequality (5) has a solution we first show that a best response strategy which solves model (1) exists. We then show that it satisfies the variational inequality.

Proposition 4.1: For any fixed $\bar{\mathbf{p}}_{-i}$, there exists a solution $\mathbf{z}'_i = (\mathbf{d}'_i, \mathbf{p}'_i)$ to the Best Response optimization model (1).

Proof: *It is easy to show that the feasible space is non-empty and compact and the objective function is continuous. Under these conditions the result follows from the well known Weierstrass theorem (See Bazaraa, Sherali and Shetty [1]).* □

Proposition 4.2: Let $\mathbf{z}'_i = (\mathbf{d}'_i, \mathbf{p}'_i)$ be a solution to Best Response optimization model (1). Condition 4.2 implies that $\mathbf{z}'_i$ also solves the following variational inequality problem:

$$-\nabla J_i(\mathbf{z}'_i) \cdot (\mathbf{z}_i - \mathbf{z}'_i) \geq 0 \qquad \forall \mathbf{z}_i \in \mathcal{K}_i(\bar{\mathbf{z}}_{-i}) \qquad (5)$$

Proof: *See Perakis and Sood [36].* □

Theorem 4.1: Under Condition 4.3, for fixed $\bar{\mathbf{z}}_{-i}$, there exists a unique solution to best response problem (1).

Proof: *See Perakis and Sood [36].* □

B. Market Equilibrium Prices

1) Existence and Uniqueness of Equilibrium Prices:

Theorem 4.2: Condition 4.2 implies that there exists at least one solution to variational inequality (4).

Proof: *The result follows from Kinderlehrer and Stampacchia (1980) since the feasible set is compact (closed and bounded) and (under Condition 4.2) convex, and function $-\nabla J_i(\cdot)$ is continuous over the feasible set.* □

Theorem 4.3: Under Conditions 4.2 and 4.3, there exists a unique solution to the Market Equilibrium Model (3).

Proof: *See Perakis and Sood [36].* □

V. Computation of Market Equilibrium Prices

A. Iterative Learning Algorithm

In this section we study an algorithm for computing the market equilibrium prices arising from variational inequality (4). A number of algorithms proposed for solving variational inequalities exist in literature. The algorithm we study is based on a simple intuitive process inspired by the concept of *fictitious play*, first introduced by Brown [7] and Robinson [37]. The tatônnement process described in Vives [40] is very similar in nature and is shown to converge (tatônnement stability) for supermodular games. We give sufficient conditions for convergence of the algorithm for the multi-period pricing game discussed in this paper and discuss how these conditions can be interpreted for the linear demand case.

Consider the market we described in Section III, consisting of several sellers pricing a product in a multi-period setting. Assume that the process is repeated under the same conditions of initial inventory and period-wise demand. The sellers do not start with the equilibrium policies but rather follow a naive myopic optimization approach: They price using the best response policy given all competitors' prices from the previous instance of the process. The key question is that if this process is repeated sufficiently many times, under what conditions will the sellers' prices converge to the equilibrium prices, irrespective of the assumed starting policies?

The outline of the general algorithm is as follows. Start by considering an initial estimate for the solution denoted by $\mathbf{z}^0 \in \mathcal{K}$ and set $k = 1$. Compute $\mathbf{z}^k$ by solving the following set of separable variational inequality subproblems for each $i \in \mathbf{I}$:

$$F_i(\mathbf{z}_i^k) \cdot (\mathbf{z}_i - \mathbf{z}_i^k) \geq 0 \qquad \forall \mathbf{z}_i \in \mathcal{K}_i(\mathbf{z}_{-i}^{k-1}). \qquad (6)$$

For our problem, this iteration step corresponds to each seller setting the best response policy to her competitors' strategies from the last iteration. We discuss the details of this computation step in Perakis and Sood [36]. We check for convergence (if the policies from two successive iterations are the same or ϵ-close to each other) and stop; otherwise we repeat with an incremented value for k. This algorithm is formally presented in Algorithm 1 below.

B. Convergence of the Iterative Learning Algorithm

In this section we study the convergence of Algorithm 1. We prove that the following set of conditions are sufficient for convergence.

Condition 5.1: For any given $\bar{\mathbf{p}}_i$, $\mathbf{h}_i(\bar{\mathbf{p}}_i, \mathbf{p}_{-i})$ is Lipschitz continuous with respect to $\mathbf{p}_{-i}$ with parameter $\mathcal{L}$.

$$\|\mathbf{h}_i(\bar{\mathbf{p}}_i, \hat{\mathbf{p}}_{-i}) - \mathbf{h}_i(\bar{\mathbf{p}}_i, \check{\mathbf{p}}_{-i})\| \leq \mathcal{L} \ \|\hat{\mathbf{p}}_{-i} - \check{\mathbf{p}}_{-i}\|$$

Condition 5.2: For any given $\bar{\mathbf{p}}_{-i}$, $-\mathbf{h}_i(\mathbf{p}_i, \bar{\mathbf{p}}_{-i})$ is strongly monotone with respect to $\mathbf{p}_i$ with parameter $\mathcal{A}$.

$$(-\mathbf{h}_i(\hat{\mathbf{p}}_i, \bar{\mathbf{p}}_{-i}) + \mathbf{h}_i(\check{\mathbf{p}}_i, \bar{\mathbf{p}}_{-i})) \cdot (\hat{\mathbf{p}}_i - \check{\mathbf{p}}_i) \geq \mathcal{A} \ \|\hat{\mathbf{p}}_i - \check{\mathbf{p}}_i\|^2$$

Condition 5.3: $\mathcal{A} > \mathcal{L}$ where $\mathcal{A}$ and $\mathcal{L}$ are defined as above.

For the two seller, linear demand case, the above conditions hold when for all $i \in \mathbf{I}$, the minimum sensitivity of seller i's demand to seller i's price over all periods, is greater than the

Algorithm 1

for $i = 1 \dots N$ **do**
  ${}^0p_i^t \leftarrow p_{\text{initial}}^t$
end for
for $i = 1 \dots N$ **do**
  ${}^1\mathbf{p}_i \leftarrow \mathcal{BR}_i({}^0\mathbf{p}_{-i})$
end for
$k \leftarrow 1$
while ${}^k\mathbf{p}_i \neq {}^{k-1}\mathbf{p}_i$ **do**
  for $i = 1 \dots N$ **do**
    ${}^{k+1}\mathbf{p}_i \leftarrow \mathcal{BR}_i({}^k\mathbf{p}_{-i})$
  end for
  $k \leftarrow k + 1$
end while
$\mathbf{p}^* \leftarrow {}^k\mathbf{p}$
RETURN $\mathbf{p}^*$

maximum sensitivity of her demand to her competitor's price over all periods. In particular, if the demand for a two seller market is given by

$$h_i^t(p_i^t, p_{-i}^t) = D_{\text{base}i}^t - \beta_i^t p_i^t + \alpha_i^t p_{-i}^t$$

then $\mathcal{A} = \min_t(\beta_i^t)$ and $\mathcal{L} = \max_t(\alpha_i^t)$.

In order to formally prove the convergence of Algorithm 1, we introduce the following reformulation of the best response problem. We aim to move the constraint involving observed demand and realized demand into the objective function. We first define the relaxed strategy space: $\bar{\mathcal{K}}_i = \{\mathbf{z}_i = (\mathbf{p}_i, \mathbf{d}_i) \mid \sum_{t=1}^T d_i^t \leq C_i, p_{\text{min}}^t \leq p_i^t \leq p_{\text{max}}^t, d_i^t \geq 0, \ \forall t \in \mathbf{T}\}$, and

$$\bar{\mathcal{K}} \quad = \quad \{\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_I) \mid \mathbf{z}_i \in \bar{\mathcal{K}}_i \quad \forall i \in \mathbf{I}\}.$$

Note that these spaces are different from $\mathcal{K}$ in that the constraint involving $d_i^t \leq h_i^t(\mathbf{p}^t)$ is missing. To move the constraint into the objective function, we introduce a dummy variable $\mathbf{v}$ in addition to the price $\mathbf{p}$ and allocation variables $\mathbf{d}$, where $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_I) \in \mathbb{R}^{\mathbf{T}\times\mathbf{I}}$ and $\mathbf{v}_i = (v_i^1, v_i^2, \dots, v_i^T) \in \mathbb{R}^{\mathbf{T}}$. The complete variable space is thus defined in terms of $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_I)$ where $\mathbf{w}_i = (\mathbf{z}_i, \mathbf{v}_i) \in \bar{\mathcal{K}}_i \times \mathbb{R}^{\mathbf{T}}$. The feasible strategy space in terms of this variable is

$$\mathcal{K}_w \quad = \quad \left\{\mathbf{w} = (\mathbf{z}, \mathbf{v}) | \mathbf{z} \in \bar{\mathcal{K}}, \mathbf{v} \in \mathbb{R}^{I\times T}\right\}.$$

The variational inequality problem can then be stated as:

$$G(\mathbf{w}^*) \cdot (\mathbf{w} - \mathbf{w}^*) \geq 0 \qquad \forall \mathbf{w} \in \mathcal{K}_w, \tag{7}$$

where

$$\begin{aligned} G_i(\mathbf{w}) \quad &= \quad (-\nabla_{\mathbf{z}_i} J_i(\mathbf{z}_i), \mathbf{d}_i - \mathbf{h}(\mathbf{z}_i, \mathbf{z}_{-i})) \\ &= \quad (-\mathbf{Q}\mathbf{z}_i, \mathbf{d}_i - \mathbf{h}(\mathbf{z}_i, \mathbf{z}_{-i})). \end{aligned}$$

The function $G(\mathbf{w})$ in variational inequality problem (7) is non-separable (ie. depending on the seller's and her competitors' strategies) making the problem difficult to solve. Algorithm 1 considers an approximation of (7) which modifies the problem into a separable variational inequality problem which is easier to solve. This separable problem is actually nothing but the best-response problem for an individual seller. We apply Algorithm 1 by solving this separable version at each step for each individual seller. For a given $\bar{\mathbf{w}}$, solve

$$G(\mathbf{w}', \bar{\mathbf{w}}) \cdot (\mathbf{w} - \mathbf{w}') \geq 0 \qquad \forall \mathbf{w} \in \mathcal{K}_w \tag{8}$$

where

$$\begin{aligned} G_i(\mathbf{w}_i, \bar{\mathbf{w}}) \quad &= \quad (-\nabla_{\mathbf{z}_i} J_i(\mathbf{z}_i), \mathbf{d}_i - \mathbf{h}(\mathbf{z}_i, \bar{\mathbf{z}}_{-i})) \\ &= \quad (-\mathbf{Q}\mathbf{z}_i, \mathbf{d}_i - \mathbf{h}(\mathbf{z}_i, \bar{\mathbf{z}}_{-i})). \end{aligned}$$

This variational inequality problem can be separated into smaller sub-problems (is separable) and for each $i \in \mathbf{I}$, we find $\mathbf{w}_i'$ satisfying

$$G_i(\mathbf{w}_i', \bar{\mathbf{w}}) \cdot (\mathbf{w}_i - \mathbf{w}_i') \geq 0, \qquad \forall \mathbf{w}_i \in \mathcal{K}_{w_i}. \tag{9}$$

The following lemma and proposition prove that the variational inequality in $\mathbf{w}$ and the variational inequality in $\mathbf{z}$ are equivalent.

Lemma 5.1: Any $\mathbf{w}'$ that solves variational inequality (8) satisfies

$$d_i'^t = h_i^t(p_i'^t, \bar{p}_{-i}^t) \qquad \forall i \in \mathbf{I}, t \in \mathbf{T}$$

Proof: *The result follows from the fact that $v_i^t \in \mathbb{R}$, $\forall i \in \mathbf{I}, t \in \mathbf{T}$.* □

Remark: From Proposition 5.1 it follows that if vector $\mathbf{w}^* = (\mathbf{z}^*, \mathbf{v}^*)$ solves variational inequality (8), $\mathbf{v}^*$ can be any vector in $\mathbb{R}^{\mathbf{T}\times\mathbf{I}}$. Without loss of generality, we select $\mathbf{v}^* = \mathbf{0}$ in our solution.

Proposition 5.1: The Best Response Problem (6) and variational inequality problem (8) are equivalent.

Proof: *It is easy to see that any solution to the Best Response Problem (6) is also a solution to the formulation (8). To show the converse, we use Lemma 5.1.* □

The next theorem proves that Conditions 5.1, 5.2 and 5.3 are sufficient conditions for convergence.

Theorem 5.1: Under Conditions 5.1, 5.2 and 5.3, Algorithm 1 converges to an equilibrium pricing policy.

Proof: *See Perakis and Sood [36].* □

VI. Numerical Examples

In this section, we examine the results presented in this paper numerically. Note that the results presented hold for general demand functions though in this section we use the linear demand case for illustration. We study the nature of the resulting equilibrium pricing policies when the initial inventories $\{C_1, C_2\}$ and price sensitivities are varied. The general trends observed are as expected:

1) The higher the inventory that any seller has available for sale over the entire horizon, the lower the prices that she sets. The revenue earned, however, is higher even though the prices set are lower.
2) Correspondingly, an increase in the inventory of a competitor results in lower revenues for the seller since the competitor reduces prices.
3) Prices are higher in periods with lower price sensitivities.

We also examine the convergence behavior of Algorithm 1 numerically as the relative ratio of price sensitivities is varied and also as the initial estimate of prices used in the algorithm is varied. In general, numerical experience led us to the following conclusions regarding the practical convergence of the algorithm:

1) The algorithm converges to the equilibrium policies rapidly in practice.
2) The numerical results verify the theoretical analysis regarding convergence of the algorithm to the unique equilibrium pricing policies when starting from different starting points.
3) The number of iterations taken to converge were dependent on the starting point. Convergence was tested by initializing the algorithms with different initial prices. In general, numerical experience led us to conclude that the number of iterations required to converge were smallest for cases where the starting prices were taken close to the equilibrium prices for all sellers. However, the rate of convergence did not depend on the starting point.
4) Changing the relative ratio of demand sensitivities to price affected the rate of convergence in accordance to Theorem 5.1. The prices converged to the equilibrium prices at a geometric rate roughly proportional to the theoretically predicted rate.

For illustration purposes, we consider a two-seller multi-period, symmetric linear demand example. For this example, $\mathbf{I} = \{1, 2\}$ and $\mathbf{T} = \{1, 2, \cdots, 10\}$. The demand is linear in prices and symmetric with respect to both sellers and varies with time: $\forall i \in \mathbf{I}$, the demand function $h_i^t = D_{\text{base}}^t - \beta^t p_i^t + \alpha^t p_{-i}^t$. We assume the symmetry of demand for the sake of convenience. Note that the results hold in general for asymmetric demand. We consider markets where customers with lower price sensitivities typically arrive in later periods. As a result the sensitivity of the demand to the seller's price (and also to her competitor's price) in the examples decreases towards the end of the time horizon.

In Table I we study the trend in pricing policies with varying inventory balances. We consider three cases with different inventories for each of the two sellers. In the first case both players are over-inventoried with $\{C1, C2\} = \{3000, 2000\}$ and the optimal equilibrium policy results in neither of them selling their entire inventory. This case is effectively equal to the uncapacitated case. Figure 1 shows the evolution of the pricing policies as the algorithm iterates, the resulting equilibrium prices, the remaining inventory over the time horizon under the equilibrium prices, and the cumulative revenue from those sales. In the second case, only one of them is over-inventoried ($\{C1, C2\} = \{3000, 500\}$). Figure 2 shows the results from this case. Note that the seller with less inventory sets prices higher than the seller with higher inventory. Even though the average price is lower for the latter, her total revenues are higher. The prices in general are also higher than in the previous case. Finally, in the third case, neither has sufficient inventory ($\{C1, C2\} = \{1000, 500\}$) so the demand supply imbalance results in a general price hike (Figure 3).

In Table II we study the movement of pricing policies as Algorithm 1 iterates with varying initial estimates for starting

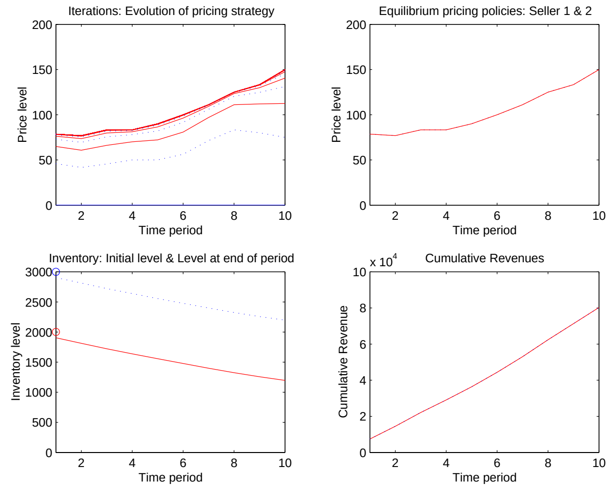

Fig. 1. Both sellers have excess inventory. $\{C1, C2\} = \{3000, 2000\}$

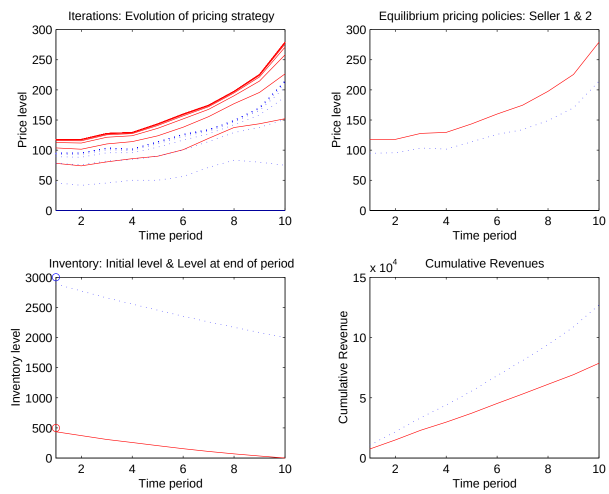

Fig. 2. One seller has excess inventory. $\{C1, C2\} = \{3000, 500\}$

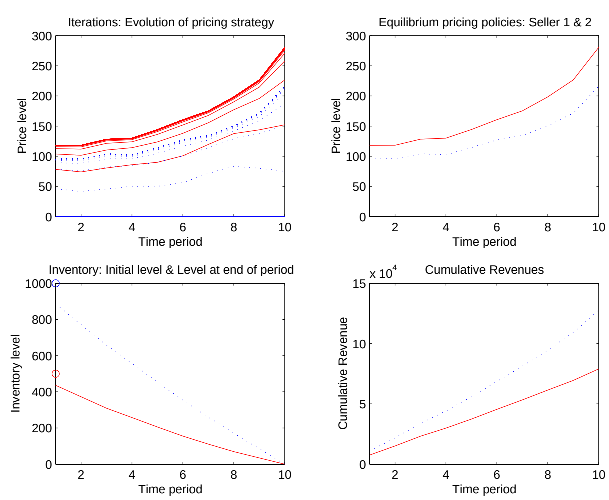

Fig. 3. Neither of the sellers have excess inventory. $\{C1, C2\} = \{1000, 500\}$

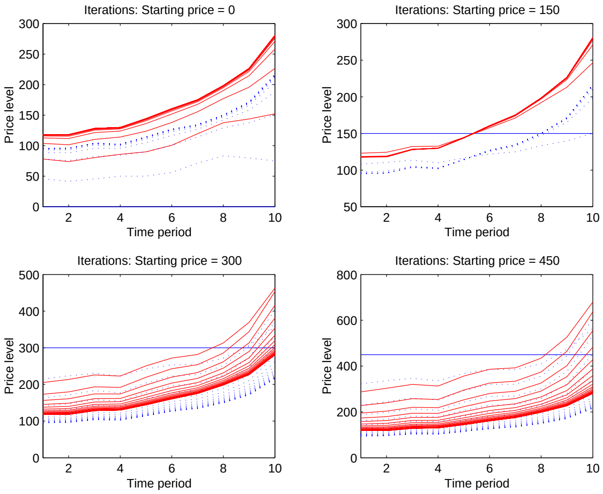

Fig. 4. Actual trend of pricing policies over successive iterations of Algorithm 1 when starting with different initial prices.

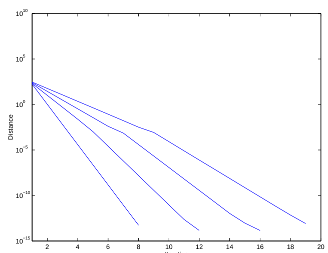

Fig. 5. Example: Convergence starting with different α.

prices. Figure 4 shows how Algorithm 1 converges to the equilibrium pricing policy when starting from four different starting points. We consider prices which are constant over all time periods as our initial estimates. We find that the convergence occurs fastest when the starting point is close to the equilibrium price. In Table IV we look at the same issue by measuring the 2-norm distance between the price policy vector from successive iterations.

In Table III we study the practical convergence behavior of Algorithm 1 with varying relative price sensitivities. Figure 5 shows the 2-norm distance between the price vectors $\mathbf{p}$ in the current iteration and the previous iteration of Algorithm 1. The four cases correspond to the choice of different ratios of the sensitivity of seller's demand to her own price and her competitor's price. The steepest line occurs for the smallest ratio and vice versa.

In Table IV we study the practical convergence behavior of Algorithm 1 with varying initial estimates for starting prices. Figure 6 shows the 2-norm distance between the price vectors $\mathbf{p}$ in the current iteration and the previous iteration of Algorithm 1. The four cases correspond to the choice of different initial estimates of the $\mathbf{p}$. We observe that the slope of the line is the same and the different cases just result in parallel displaced lines.

TABLE I

TREND IN PRICING POLICIES WITH VARYING INVENTORY BALANCES.

Model parameters held constant		
D_{base}	$=$	$\{110, 100, 100, 100, 90, 90, 100, 100, 80, 60\}$
β	$=$	$\{1.2, 1.2, 1.1, 1.0, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4\}$
α	$=$	$\{1.0, 1.1, 1.0, 0.8, 0.8, 0.7, 0.5, 0.4, 0.4, 0.4\}$
Model parameters varied		
$\{C1, C2\}$	$=$	$\{3000, 2000\}$, $\{3000, 500\}$, $\{1000, 500\}$

TABLE II

MOVEMENT OF PRICING POLICIES IN ITERATIONS OF ALGORITHM 1 WITH VARYING INITIAL ESTIMATES FOR STARTING PRICES.

Model parameters held constant		
D_{base}	$=$	$\{110, 100, 100, 100, 90, 90, 100, 100, 80, 60\}$
β	$=$	$\{1.2, 1.2, 1.1, 1.0, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4\}$
α	$=$	$\{1.0, 1.1, 1.0, 0.8, 0.8, 0.7, 0.5, 0.4, 0.4, 0.4\}$
$\{C_1, C_2\}$	$=$	$\{1000, 500\}$
Model parameters varied		
Starting estimate of prices $\forall i \in \mathbf{I}$ and $t \in \mathbf{T}$ $p_i^t = 0, 150, 300, 450$		

TABLE III

PRACTICAL CONVERGENCE BEHAVIOR OF ALGORITHM 1 WITH VARYING RELATIVE PRICE SENSITIVITIES.

Model parameters held constant		
D_{base}	$=$	$\{110, 105, 100, 95, 90, 85, 80, 75, 75, 85\}$
β	$=$	$\{1.2, 1.15, 1.1, 1.05, 1, .95, .9, .85, .8, 0.75\}$
$\{C1, C2\}$	$=$	$\{1000, 500\}$
Model parameters varied		
α	$=$	$k\beta$
where, k	$=$	$\frac{20}{32}, \frac{15}{32}, \frac{10}{52}, \frac{5}{32}$

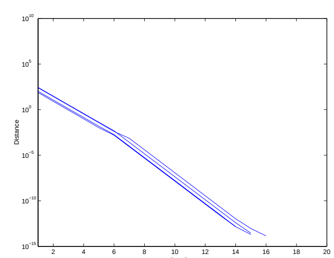

Fig. 6. Convergence behavior starting with different initial prices.

TABLE IV

PRACTICAL CONVERGENCE BEHAVIOR OF ALGORITHM 1 WITH VARYING INITIAL ESTIMATES FOR STARTING PRICES.

Model parameters held constant		
D_{base}	$=$	$\{110, 105, 100, 95, 90, 85, 80, 75, 75, 85\}$
β	$=$	$\{1.2, 1.15, 1.1, 1.05, 1, .95, .9, .85, .8, 0.75\}$
α	$=$	$\frac{15}{32}\beta$
$\{C1, C2\}$	$=$	$\{1000, 500\}$
Model parameters varied		
Starting estimate of prices $\forall i \in \mathbf{I}$ and $t \in \mathbf{T}$ $p_i^t = 0, 50, 100, 150$		

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