Quantum and Floer cohomology have the same ring structure

by

Serguei Piunikhin

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 1996

Author ....
Department of Mathematics
May 17, 1996

Certified by ....
I. M. Singer
Institute Professor
Thesis Supervisor

Accepted by ....
David Vogan
Chairman, Departmental Committee on Graduate Students

Quantum and Floer cohomology have the same ring structure

by

Serguei Piunikhin

Submitted to the Department of Mathematics
on May 17, 1996, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Abstract

The action of the total cohomology space $H^*(M)$ of an almost-Kahler manifold M on its Floer cohomology, introduced originally by Floer, gives a new ring structure on the cohomology of the manifold. In this thesis we prove that the total cohomology space $H^*(M)$, provided with this new ring structure, is isomorphic to the quantum cohomology ring. As a special case, we prove the the formula for the Floer cohomology ring of the complex grassmanians conjectured by Vafa and Witten.

Thesis Supervisor: I. M. Singer
Title: Institute Professor

Acknowledgments

I am greatly indebted to my thesis advisor I.M. Singer and to my old friend Vladimir Sadov for introducing me to the problem and for many helpful remarks without which this work would not be possible.

This work would not also be possible without Gang Tian who helped me to fix some gaps in the analysis present in the early versions of the manuscript.

Edward Witten explained to me his work on quantum cohomology of Grassmanians which inspired me to work out this example rigorously.

Michael Gromov, Dusa McDuff, Dietmar Salamon and Clifford Taubes taught me about analysis relevant to Floer-Gromov theory of pseudoholomorphic curves. Dietmar Salamon and Clifford Taubes helped me to fix analytic problems in the proof of the "gluing theorem" for J-holomorphic curves.

Clifford Taubes helped me with the literature on this subject at the early stage of the work.

Scott Axelrod, David Kazhdan, Maxim Kontsevich, Gang Liu, Shing-Tung Yau and expecially Yongbin Ruan and Gang Tian read carefully the early versions of my manuscript and made many suggestions towards its improvement.

I also benefited very much from discussions with Vladimir Igorevich Arnold, Alex Astashkevich, Raoul Bott, Michael Callahan, Sheldon Chang, Radu Constantinescu, Robert Dijkgraaf, Simon Donaldson, Kenji Fukaya, Alexander Givental, Alexander Goncharov, Ken Intrilligator, Ai-Ko Liu, Gang Liu, Yurii Ivanovich Manin, David Morrison, Sergei Petrovich Novikov, Graeme Segal, Cumrun Vafa and Richard Wentworth, to whom I express my sincere gratitude.

Chapter 1

Introduction

CONTENTS

Floer cohomology $HF^*(M)$ of the free loop-space of the symplectic manifold M is under extensive study by both mathematicians and physicists.

Originally the machinery of Floer cohomology was developed in [F1] in order to prove the classical Arnold's Conjecture giving a lower bound on the number of fixed points of a symplectomorphism in terms of Morse theory. Floer cohomology appeared also in "topological sigma-models" of string theory [Wi1],[BS].

Quantum cohomology rings of Kahler (or more generally, almost-Kahler) manifolds were introduced by Witten [Wi2] and Vafa [Va1], [LVW] using moduli spaces of holomorphic curves [CKM],[Gr],[McD1],[Ru2].

It is difficult to give a rigorous and self-consistent mathematical definition of quantum cohomology rings because the moduli spaces of holomorphic curves are non-compact and may have singularities, facts often ignored by physicists.

Recently Ruan and Tian [RT1] by adjusting Witten's degeneration argument, proved that these rings are associative. The proof of associativity for the quantum cohomology ring uses a definition of multiplication that involves the moduli spaces of solutions of inhomogeneous Cauchy-Riemann equations [Ru1].

Quantum cohomology rings have been computed for:

a) complex projective spaces [Wi2]

b) complex Grassmanians (the formula was conjectured by Vafa [Va1], proved in the present paper, and also independently in [AS] and [ST])

c) toric varieties [Ba]

d) flag varieties [GK],[GFP]

e) more general hermitian symmetric spaces [AS]

Quantum cohomology rings of Calabi-Yau 3-folds, important in physics, have been "computed" in several examples by "mirror symmetry " [COGP]. These computations are not considered rigorous. To justify these "computations" is a very interesting problem for algebraic geometers.

The linear map $m_F : H^*(M) \to End(HF^*(M))$ or, equivalently, the action of the classical cohomology of the manifold M on its Floer cohomology, was defined by Floer himself. He computed this action for the case $M = \mathbf{CP^n}$ and noticed the following fact:

For any two cohomology classes A and B in $H^*(\mathbf{CP^n})$ the product $m_F(A)m_F(B)$ of the linear operators $m_F(A)$ and $m_F(B)$ acting on Floer cohomology $HF^*(\mathbf{CP^n})$ has the form $m_F(C)$ for some cohomology class C in $H^*(\mathbf{CP^n})$. This gives us a new ring structure on $H^*(\mathbf{CP^n})$, which is known to be different from the classical cup-product. We will call this new multiplication law **Floer multiplication**.

Floer conjectured that the same phenomenon might be true for all symplectic (or at least Kahler) manifolds with a technical condition of being "positive", thus providing a new ring structure on the total cohomology $H^*(M)$ of a symplectic manifold M.

Using path-integral arguments, V.Sadov [S] "proved" that when M is a "positive" or "semi-positive" symplectic manifold (see chapter 2 for the definition), then:

1) "The operator algebra should close", i.e., for any two cohomology classes A and B in $H^*(M)$ the product $m_F(A)m_F(B)$ always has the form $m_F(C)$ for some cohomology class C in $H^*(M)$.

and

2) **Floer multiplication** coincides with **quantum multiplication**.

There exists a "pair of pants" cup-product in Floer cohomology which is (formally) different from the cup-product introduced by Floer himself and discusssed here. However, [Fu1], [CJS], [GK], [McD S] conjectured and [PSS], [RT2] and [Liu] announced a proof that this other "pair of pants cup-product" in Floer cohomology also coincides with the quantum cup-product.

The purpose of this paper is to give a rigorous proof of Sadov's statements. We prove

The Main Theorem. *For a semi-positive symplectic manifold M, the ring structure on $H^*(M)$ inherited from its action m_F on the Floer cohomology coincides with the the quantum multiplication on $H^*(M)$ defined as in [RT1].*

Chapter 2

Moduli spaces of J-holomorphic spheres and their compactification.

2.1 Definitions

Definition. *The manifold M is called an almost-Kahler manifold if it admits an almost-complex structure J and a symplectic form ω such that for any two tangent vectors x and y to M, $\omega(x;y) = \omega(J(x);J(y))$ and for any non-zero tangent vector x to M the following inequality holds:*

$$\omega(x;J(x)) > 0 \tag{2.1}$$

Definition. *An almost-complex structure J and a closed 2-form ω on M are called compatible if* (2.1) *holds.*

Let M be a compact almost-Kahler manifold of dimension $2n$ which we assume (for simplicity) to be simply-connected. Let us fix an almost-complex structure J_0 on M and let us consider the space $\tilde{K}$ of all J_0-compatible symplectic forms and its image K in the cohomology $H^2(M,R)$.

If it will not lead to confusion, we will denote the closed J_0-compatible two-form and the corresponding cohomology class by the same symbol.

It follows directly from the definitions that if M is an almost-Kahler manifold (which is equivalent to the fact that $\tilde{K}$ is non-empty), then:

1) $\tilde{K}$ is an open convex cone in the space of all closed 2-forms on M. The set $\tilde{K}$ does not contain any nontrivial linear subspace (otherwise ω and $-\omega$ would be simultaneously J_0-compatible which is impossible).

2) K is an open convex cone in $H^2(M,R)$ which does not contain any nontrivial linear subspace.

To prove openness, let us consider the functional $\omega(x; J(x))$ defined on the product of the space $\Omega^2(M)$ of C^1-smooth 2-forms times the space $S(TM)$ of the unit sphere bundle in the tangent space to M. This functional is continuous and bounded from below by a positive constant, if restricted to $\{\omega\} \times S(TM)$. Then any small perturbation of ω preserves this property, and leaves the resulting 2-form non-degenerate, which proves openness of K .

Since a symplectic form compatible with J_0 (and in fact any symplectic form) on a compact oriented manifold M cannot be cohomologically trivial, then K cannot contain a nontrivial linear subspace in $H^2(M,R)$.

Let us consider symplectic forms $\{\omega_1, ..., \omega_s\}$ such that:

1) they lie inside $\tilde{K}$.

2) their cohomology classes form a basis in $H^2(M,R)$.

3) the elements of this basis are represented by integral cohomology classes

4) $\{\omega_1, ..., \omega_s\}$ generate $H^2(M,Z)$ as an abelian group.

We can always find such a collection of symplectic forms since any open convex cone in $H^2(M,R)$ contains such a collection.

In the case when M is a Kahler manifold, it sometimes appears to be useful to perturb the complex structure on M (or on $CP^1 \times M$) and to work with non-integrable almost-complex structures. It is easier to prove transversality results if we are allowed to work in this larger category.

Let us consider the complex projective line CP^1 with its standard complex structure i and Fubiny-Study Kahler form Ω. Let us take the product $CP^1 \times M$ in the almost-Kahler category. Let $\mathcal{J}$ be the space of all C^{10n}-smooth almost-complex structures on $CP^1 \times M$ such that the projection on the first factor $CP^1 \times M \to CP^1$ is holomorphic. Let us equip this space with the C^{10n}-norm topology.

Comment. *The space of almost-complex structures on $CP^1 \times M$ lies inside the space of (1,1)-tensors on $CP^1 \times M$ and thus, we can talk about almost-complex structures of class C^{10n}.*

Let $\mathcal{J}_0$ be a neighborhood of $i \times J_0$ in $\mathcal{J}$ consisting of almost-complex structures compatible with symplectic forms $\{1 \otimes \omega_1, ..., 1 \otimes \omega_s\}$ and $\Omega \otimes 1$. Since the notion of compatibility with a 2-form is an open condition in $\mathcal{J}$, such a neighborhood always exists.

Let us consider the vector bundle over the product $CP^1 \times M$ consisting of $i \times J_0$-antilinear maps from $T(CP^1)$ to TM. "$i \times J_0$-antilinear" means that for any $g \in \mathcal{G}$ we have $J_0 g = -gi$. Let $\mathcal{G}$ be the space consisting of all C^{10n-1}-sections of the above-defined vector bundle. Equivalently, $\mathcal{G}$ can be thought as a space of all $(0,1)$-forms on CP^1 with the coefficients in the tangent bundle to M.

If g is any such $(0,1)$-form, we can construct an almost-complex structure J_g on $CP^1 \times M$ which is written in coordinates as follows:

$$J_g = \begin{pmatrix} i & -g \\ 0 & J_0 \end{pmatrix} \tag{2.2}$$

Here we wrote the matrix of J_g acting on $T(CP^1) \oplus TM$.

Thus, we have an embedding $\mathcal{G} \subset \mathcal{J}$. Let $\mathcal{G}_0$ be the intersection of $\mathcal{G}$ and $\mathcal{J}_0$.

We will assume both $\mathcal{J}_0$ and $\mathcal{G}_0$ to be contractible.

Presumably, the introduction of almost-complex structures can be avoided [K3]. We use them to modify the proofs of some analytic lemmas. What we really needed is one fixed almost-complex structure J_0 on M (which in all examples will be an actual complex structure) and perturbations of the product (almost)-complex structure on $CP^1 \times M$ of the form (2.2). We decided to use more complicated notations in order to simplify the proofs.

Let $J \in \mathcal{J}_0$ be an almost-complex structure on $CP^1 \times M$.

Definition (Gromov). *A J-holomorphic sphere in M is any almost-complex submanifold in $CP^1 \times M$ of real dimension two (or "complex dimension one") which projects isomorphically onto the first factor CP^1.*

Equivalently, a J-holomorphic sphere in M can be defined as a pseudo-holomorphic section of the the (pseudo-holomorphic) bundle $M \times CP^1$ over CP^1 where the almost-complex structure J on $M \times CP^1$ is a perturbation of the product almost-complex structure. Topologically this is the trivial bundle over CP^1 with the fiber M but (pseudo)-holomorphically it is not trivial.

Any J-holomorphic sphere is the graph of a map φ from CP^1 to M which satisfies a non-linear PDE

$$\bar{\partial}_J \varphi = 0 \tag{2.3}$$

If our almost-complex structure J_g has the form (2.2) than the equation of a J_g-holomorphic sphere φ can be rewritten as

$$\bar{\partial}_{J_0} \varphi = g \tag{2.4}$$

Here $\bar{\partial}_{J_0}$ is the usual $\bar{\partial}$-operator on M associated with our original (almost)-complex structure J_0

Let $C \subset H_2(M, \mathbf{R})$ be the closure of the convex cone generated by the images of homology classes of J-holomorphic spheres (for all $J \in \mathcal{J}_0$). Then C will lie in the closure of the convex dual of the cone $K \subset H^2(M, \mathbf{R})$.

Following [Ru1], we will call a non-zero homology class $A \in H_2(M, Z)$ **an effective class** if A lies inside the closed cone C.

Let $q_1, ..., q_s$ be the dual to $\omega_1, ..., \omega_s$ basis in $H_2(M)$. We will write the elements of $H_2(M) = H_2(M, Z)$ in multiplicative notation. The monomial $q^d = q_1^{d_1}...q_s^{d_s}$ is by definition the sum $\sum_{i=1}^s d_i q_i \in H_2(M)$. Here d is a vector of integers $(d_1, ..., d_s)$ and $q = (q_1, ..., q_s)$ is a multi-index. Then the group ring $Z_{[H_2(M)]}$ is a commutative ring generated (as an abelian group) by monomials of the form $q^d = q_1^{d_1}...q_s^{d_s}$.

The group ring $Z_{[H_2(M)]}$ which is isomorphic to the ring $Z_{[q_1^{\pm 1}, ..., q_s^{\pm 1}]}$ of Laurent polynomials, has an important subring $Z_{[C]}$. The fact that symplectic forms $\{\omega_1, ..., \omega_s\}$ have positive integrals over all J-holomorphic curves implies that

$$Z_{[C]} \subset Z_{[q_1, ..., q_s]} \subset Z_{[q_1^{\pm 1}, ..., q_s^{\pm 1}]}$$

i.e., that monomials $q_1^{d_1}...q_s^{d_s}$ may appear in $Z_{[C]}$ only if all $(d_1, ..., d_s)$ are non-negative.

The ring $Z_{[C]}$ has a natural augmentation $I : Z_{[C]} \to Z$ which sends all non-constant monomials in $\{q_i\}$ to zero. Thus, we can consider its completion $Z_{<C>}$ with respect to the I-adic topology. This completion lies naturally in the ring $Z_{<q_1, ..., q_s>}$ of formal power series in $\{q_i\}$.

Following Novikov [No], let us introduce the following subring Λ_ω in $Z_{<q_1, ..., q_s>} \otimes_{Z_{[q_1, ..., q_s]}} Z_{[H_2(M)]}$ as a subring consisting of formal Laurent power series of the form $\sum_d c_d q^d$ such that:

1) There exists a number N such that all $\{c_d\}$ are zero if $(\omega; q^d) < -N$;

2) For any number M there exists only finitely many non-zero $\{c_d\}$ such that $(\omega; q^d) < M$

The ring Λ_ω is called **the Novikov ring** which appeared in Novikov's study of Morse theory of multivalued functions [No]. Novikov's refinement of Morse theory is exactly the kind of Morse theory we need in our study of of Floer homology (see also [HS]).

Let us consider the abelian group $H^*(M, Z) \otimes \Lambda_\omega$. It has an obvious structure of a Z-graded ring inherited from the usual grading in cohomology, provided that all the elements of the augmentation ideal $I(\Lambda_\omega)$ have degree zero.

The same abelian group $H^*(M) \otimes \Lambda_\omega$ has **another** Z-graded ring structure which can be constructed as a q-deformation of the classical cohomology ring $H^*(M)$ with non-trivial grading of the "deformation parameters" $\{q_i\}$. To be more concrete, let us define a Z-grading on $H^*(M) \otimes \Lambda_\omega$ as follows: any element A from $H^*(M) \otimes \Lambda_\omega$ can be obtained as a (possibly infinite) sum of "bihomogenous pieces" $A = \sum_{m,d} A^{m,d} \otimes q^d$ where $A^{m,d} \in H^m(M, Z)$. Then let us define

$$deg[A^{m,d} \otimes q^d] = m + 2 < c_1(TM); q^d > \tag{2.5}$$

where the last term means evaluation of the 2-cocycle $c_1(TM)$ on the 2-cycle q^d.

The previous formula can be rewritten in more elegant way:

$$deg[A^{m,d} \otimes q^d] = m + 2 \sum_{i=1}^{s} d_i < c_1(TM); q_i >$$

Definition. *For each multi-index* $d = (d_1, ..., d_s)$ *let* Map_d *be the space of all* $W^{1,p}$*-Sobolev maps from* CP^1 *to* M *of a given homotopy type specified by "the generalized degree"* $d = (d_1, ..., d_s)$.

$W^{1,p}$-"Sobolev" means that the derivative of the map $\varphi \in Map_d$ should lie in the space L^p. (The first derivative of the map φ from CP^1 to M is a one-form on CP^1 with the values in $\varphi^*(TM)$). We will fix oncer and for all the value of p ¿ 2.

"Homotopy type specified by the generalized degree" $d = (d_1, ..., d_s)$ means that $\int_{\varphi(CP^1)} \omega_i = d_i$ for each $\varphi \in Map_d$ and for each $i = 1, ..., s$.

The space Map_d thus has a natural structure of a (connected) Banach manifold which is homotopically equivalent to the space of all smooth (or all continuous) maps from CP^1 to M of a given homotopy type. This space is a connected component of the larger space $Map = \bigcup_d Map_d$ of all Sobolev maps from CP^1 to M (regardless of homotopy type) which is also a Banach manifold.

Let us introduce an infinite-dimensional Banach bundle $\mathcal{H}$ over $Map \times \mathcal{J}_0$. The fiber $\mathcal{H}_{J,\varphi}$ of the bundle $\mathcal{H}$ over the "point" $(\varphi, J) \in Map \times \mathcal{J}_0$ will be the space of all $(0,1)$-forms of the type L^p on CP^1 with the values in the complex n-dimensional vector bundle $\varphi^*(TM)$. The almost-complex structure J on M provides the tangent bundle TM with the structure of the complex n-dimensional vector bundle.

The bundle $\mathcal{H}$ is provided with a section $\bar{\partial}$, given by the formula

$$(\varphi, J) \to \bar{\partial}_J(\varphi) \tag{2.6}$$

The above-defined section $\bar{\partial}$ is actually a nonlinear $\bar{\partial}$-operator

Proposition 2.1. *The zero set $\bar{\partial}^{-1}(0)$ consists of the pairs (φ, J) where φ is a J-holomorphic map.*

Definition. *For each multi-index d let $\mathcal{M}_{J,d} \subset Map_d$ be the space of all solutions of (2.4) of homotopy type specified by d. Let us and call $\mathcal{M}_{J,d}$ the moduli space of J-holomorphic maps from CP^1 to M of "the generalized degree" d.*

The above defined Banach bundle $\mathcal{H}$ over $Map \times \mathcal{J}_0$ can be (trivially) extended to a Banach bundle over the product of $Map \times \mathcal{J}_0$ and $\mathcal{G}_0$. It can also be trivially extended to a Banach bundle over the product of $Map \times \mathcal{J}_0$ and $\mathcal{G}_0 \times \mathcal{G}_0 \times [0;1]$.

We will denote these three Banach bundles by the same symbol $\mathcal{H}$. We will also denote by $\mathcal{H}$ the restriction of these Banach bundles to connected components $Map_d \times$ [auxilliary space] of their bases.

Since $\mathcal{G}_0$ is an open subset in the vector space $\mathcal{G}$ which has has a base-point (zero), it makes sense to speak about extension of smooth sections of $\mathcal{H}$ from $Map \times \mathcal{J}_0$ to the larger spaces $Map \times \mathcal{J}_0 \times \mathcal{G}_0$ and $Map \times J_0 \times \mathcal{G}_0 \times \mathcal{G}_0 \times [0;1]$

We assume that $Map \times \mathcal{J}_0$ is embedded as $Map \times \mathcal{J}_0 \times \{0\}$ into the product with the auxiliary spaces.

We have an obvious

Proposition 2.2. *If restricted to the subspace $Map \times \mathcal{G}_0$ in $Map \times \mathcal{J}_0$ the zero* *set $\bar{\partial}^{-1}(0)$ consists of the pairs (φ, g) where φ is a solution of the inhomogenous Cauchy-Riemann equation* (2.4).

Definition. *A section Φ of the Banach bundle $\mathcal{H}$ over some base Banach manifold B is called regular if its derivative $D\Phi$ at each point in the zero-locus $\Phi^{-1}(0)$ is a surjective linear map from the tangent space to B to the tangent space to the fiber of $\mathcal{H}$.*

The section $\bar{\partial}$ is regular since its derivative in $\mathcal{J}_0$-directions is already surjective linear map from $T\mathcal{G}_0 \subset T\mathcal{J}_0$ to $T\mathcal{H}$.

Thus, $\bar{\partial}^{-1}(0)$ is a smooth Banach manifold and by an infinite-dimensional version of Sard Theorem, we have that for "generic" $g \in \Omega^{0,1}(TM)$ the space of solutions of the inhomogenous Cauchy-Riemann equation (2.4) is a smooth finite-dimensional manifold.

By the same reason, for "generic" almost-complex structure $J \in \mathcal{J}_0$ the moduli space $\mathcal{M}_{J,d}$ of J-holomorphic spheres of "degree d" is a smooth finite-dimensional manifold.

We now note that the dimension of this manifold is given by the index of the Fredholm linear operator $D\bar{\partial}$ which acts from $T(Map_d)$ to $T\mathcal{H}$. The operator $D\bar{\partial}$ is defined as a derivative of the section $\bar{\partial}$ in Map_d-directions.

"Generic" here means "is a Baire second category set".

Proposition 2.3 (Gromov). *For the "generic" choice of J the moduli space $\mathcal{M}_{J,d}$ will be a smooth manifold of dimension*

$$dim \mathcal{M}_{J,d} \;=\; dim M \;+\; \sum_{i=1}^{s} d_i \;\, deg[q_i] \tag{2.7}$$

Ruan [Ru1] and McDuff and Salamon [McD S] proved that the moduli space $\mathcal{M}_{J,d}$ carries a canonical orientation coming from determinant line bundle of $D\bar{\partial}$.

The idea of the proof of (2.7) is as follows. The operator $D\bar{\partial}$ is actually a (twisted) $\bar{\partial}$-operator on CP^1. Then the Atiyah-Singer index theorem, applied to any of our " $\bar{\partial}$-operators", gives us the r.h.s. of (2.7).

To prove that the actual dimension of the moduli space $\mathcal{M}_{J,d}$ is equal to its "virtual dimension" given by the index calculation in the r.h.s. of (2.7), we need several analytic lemmas. These lemmas were first proved by Freed and Uhlenbeck [FU]. We are referring to the book [DK] which is better adjusted for our purposes.

Proposition 4.3.11 of [DK]. *Let B and S be Banach manifolds;$\mathcal{H}$ be a Banach bundle over $B \times S$, Φ is the regular section of $\mathcal{H}$, which is Fredholm in B-directions (when we restrict it to $B \times \{g\}; g \in S$). Then for "generic" values of the parameter g in S, the zero-set of Φ restricted to $B \times \{g\}$ will be a smooth submanifold of dimension equal to "the virtual dimension".*

By applying this proposition to our case when $B = Map_d$, $S =$ is some "auxiliary space" we obtain

Lemma 2.4. *If Φ is the regular section of the Banach bundle $\mathcal{H}$ over $Map_d \times S$. Then for "generic" values of the parameter g in the auxiliary space S, the zero-set of Φ restricted to $Map_d \times \{g\}$ will be a smooth submanifold of dimension equal to "the virtual dimension".*

Here "the virtual dimension" means the index of the derivative of the section Φ in Map_d-directions (these operators are always Fredholm).

In the terminology of [DK] let $\pi : B \times S \to S$ be the "projection operator" onto "the auxiliary space" S.

Proposition 4.3.10 of [DK]. *If $\pi : \mathcal{P} \to S$ is a Fredholm map between paracompact Banach manifolds, and $h : R \to S$ is a smooth map from a finite-dimensional manifold R, there exists a map $h' : R \to S$, arbitrary close to h in the topology of C^∞-convergence on the compact sets and transverse to π. If h is already transverse to π on a closed subset $G \subset R$ we can take $h' = h$ on G.*

By applying this proposition to our case when $P = \Phi^{-1}(0) \subset B \times S$, π is a projection operator to S, we obtain

Lemma 2.5. *Any finite-dimensional pseudo-manifold of parameters in S can be* *perturbed to be made transversal to the projection operator π.*

Here the projection operator projects $\Phi^{-1}(0) \subset Map_d \times S$ to the second factor (the auxilliary space S).

The particular case of this lemma is

Lemma 2.6. *For the pair g^1 and g^2 of "the regular values" of parameters in the auxilliary space any path γ joining them can be perturbed to be made transversal to the projection operator.*

Lemma 2.6 implies that the inverse image of this "transversal path" γ gives us a smooth cobordism between $\Phi^{-1}(0) \bigcap Map_d \times \{g^1\}$ and $\Phi^{-1}(0) \bigcap Map_d \times \{g^2\}$.

Using Lemmas 2.4 and 2.6 we have that there exists a smooth cobordism $\mathcal{M}^t$ inside $Map_d \times \mathcal{J}_0$ between the moduli spaces $\mathcal{M}_{J_{g^1},d}$ and $\mathcal{M}_{J_{g^2},d}$ constructed using different "regular" almost-complex structures J_{g^1} and J_{g^2}.

2.2 Semi-Positivity

Let Σ be a two-dimensional Kahler manifold (in most applications, Σ is 2-sphere, cylinder or half-cylinder) and let us fix an almost-complex structure J on $\Sigma \times M$ compatible with ω. Then we have a fibration $\Sigma \times M \to \Sigma$ such that each fiber (a copy of M) is equipped with a Riemannian metric $g(x, y) = \omega(x; Jy)$ where x, y are tangent vectors to M.

Definition. *Let us define the energy of the smooth map $\phi : \Sigma \to M$ to be the L^2-norm of the 1-form $d\phi \in \Omega^1(\phi^* TM)$:*

$$E(\phi) = \frac{1}{2} \int_\Sigma |d\phi|^2 dA$$

with respect to the metric defined above (this metric depends on J only). Here dA is the Kahler area of Σ .

Lemma 4.1.2 of [McD S].

$$E(\phi) \geq \int_{\Sigma} \phi^*(\omega) \tag{2.8}$$

with the equality if and only if Σ is J-holomorphic.

Let K be a positive real number.

Definition. *An almost-complex structure $J \in \mathcal{J}$ on $CP^1 \times M$ is called K-semi-positive if every J-holomorphic sphere ϕ in M with the energy $E(\phi) \leq K$ has non-negative Chern number ($\int \phi^*(c_1(TM)) \geq 0$.) An almost-complex structure $J \in \mathcal{J}$ on $CP^1 \times M$ is called semi-positive if it is K-semi-positive for every K.*

Lemma 5.1.2 of [McD S]. *For all $K > 0$ the space $\mathcal{J}_+(M, \omega, K)$ of K-semi-positive almost-complex structures is an open subspace in $\mathcal{J}$.*

Definition. *An almost-Kahler manifold M is called semi-positive if the space $\mathcal{J}_+(M, \omega, K)$ is non-empty for every K.*

From now on we will always assume M to be semi-positive. The semi-positivity is required in order to have a good compactification and a well-defined intersection theory on the moduli spaces of J-holomorphic spheres.

We will also always assume that for any "generalized degree" $d = (d_1, ..., d_s)$ we are considering only the almost-complex structures from $\mathcal{J}_+(M, \omega, K)$ where $K > \sum_{1=1}^{s} d_i < \omega; q_i >$ and are chosing "generic" ones (to prove transversality results) only from that range. The range of allowed almost-complex structures changes when we change the degree d of the map (but still remains opened according to Lemma 5.1.2 of [McD S]).

The moduli spaces $\mathcal{M}_{J,d}$ of J-holomorphic spheres are not compact. There are two (closely related) sources of non-compactness of these moduli spaces:

1) The sequence of unparametrized J-holomorphic spheres may "split" into two J-holomorphic spheres by contracting of some loop on CP^1. The resulting "split J-holomorphic sphere" is (formally) not in our space which means that the above

sequence diverges. This "degeneration" may occur only if both spheres which appear after this "splitting process" have non-trivial homotopy type (and cannot be contracted to a point).

2) The sequence of parametrized J-holomorphic spheres in $\mathcal{M}_d$ may diverge by "splitting off" a J_0-holomorphic sphere of lower (or the same) degree at some point on CP^1. This means that the curvature of our sequence of maps "blows up" at some point on CP^1. This phenomenon is so-called "bubbling off" phenomenon (which takes place for the moduli space of solutions of any conformally-invariant elliptic PDE. This phenomenon was first observed in [SU]).

The bubbling off may be possible even when the classical splitting is impossible. For example, let us consider the simplest case when $M = CP^1$ with the standard complex structure and $d = 1$. Then the sequence of holomorphic degree-one maps from CP^1 to itself may diverge by "bubbling off" any any point on CP^1. This will compactify the non-compact space $\mathcal{M}_{J,1}$ (which is diffeomorphic to $PSL(2, C)$ in this example) to a compact space CP^3.

2.3 Compactification of Moduli Spaces

In order to compactify the moduli space $\mathcal{M}_{J_g,d}$ in the sense of Gromov, we should, roughly speaking, add to it the spaces of J-holomorphic maps of the connected sum of several copies of CP^1 to M of total degree d.

In other words, the space of "non-degenerate" J-holomorphic spheres in M is non-compact but it will be compact if we add to it "degenerate J-holomorphic spheres".

Ruan [Ru1] gave an explicit description how to stratify the compactified moduli spaces $\bar{\mathcal{M}}_{J_g,d}$.

Definition (Ruan). *Let us call degeneration pattern the following set of data DP1) - DP3) :*

DP1) The class $d^0 \in C$, the set $\{d^1; ...; d^k\} \subset C \subset H^2(M)$ of non-zero effective classes , and the set $\{a_1; ...; a_k\}$ of positive integers, such that the following identity holds: $d = d^0 + \sum_{i=1}^{k} a_i d^i$

DP2) The set $\{I_1; ...; I_t\}$ of subsets in the set $\{d^0; d^1; ...; d^k\}$. We do not allow one of $\{I_1; ...; I_t\}$ to be the proper subset of another.

Using the set of data $\{d^0; d^1; ...; d^k; I_1; ...; I_t\}$, we can construct a graph T with $k+1+t$ vertices $\{d^0; d^1; ...; d^k;\ I_1; ...; I_t\}$ as follows:

If the class d^i lies in the set I_j then we join the vertices d^i and I_j by an edge.

DP3) The graph T obtained by above prescription is a tree.

Definition. *We will call a J_0-holomorphic sphere $C_i \in \mathcal{M}_{J_0,d^i}$ simple if C_i cannot be obtained as a branched cover of any other J_0-holomorphic sphere.*

Definition. *If the J_0-holomorphic sphere C_i is not simple then we will call it multiple-covered.*

We will denote $\mathcal{M}^*_{J_0,d^i}$ the space of all simple J_0-holomorphic spheres of "degree d" in M. According to the theorem of McDuff [McD1] if the almost-complex structure J_0 on M is "generic" then $\mathcal{M}^*_{J_0,d^i}$ is a smooth manifold of dimension given by the formula (2.7)

Let $D_d = \{\ \{d^0; d^1; ...; d^k\};\ \{a_1; ...; a_k\};\ \{I_1; ...; I_t\};\ T\ \}$ be some degeneration pattern. Then let us define $\mathcal{N}_{J_g,D_d}$ as a topological subspace in

$\mathcal{M}_{J_g,d^0} \times \Pi_{i=1}^{k} [\mathcal{M}^*_{J_0,d^i}\ /\ PSL(2,C)]$ as follows:

An element φ in $\mathcal{N}_{J_g,D_d}$ consists of one parametrized J_g-holomorphic sphere $C_0 \in \mathcal{M}_{J,d^0}$ and k unparametrized J_0-holomorphic spheres

$\{C_i \in [\mathcal{M}^*_{J_0,d^i}/PSL(2,C)]\ \}$. We require that for any subset

$I_j = \{d^{j_1}; ...; d^{j_{n_j}}\}$ from $\{I_1; ...; I_t\}$ the spheres $\{C_{j_1}; ...; C_{j_{n_j}}\}$ have a common intersection point. We do not allow this intersection point to lie on any other sphere $C_i \subset M$ in our collection[1]

[1]If $C_{j_1} \bigcap ... \bigcap C_{j_{n_j}} \bigcap C_i \neq \emptyset$ then our "degenerate J-holomorphic sphere" would lie in the other

Comment 1. *We can think about parametrized spheres in* M *as about unparametrized spheres in* $M \times CP^1$ *which have degree one in* CP^1*-directions.*

Comment 2. *Degeneration of parametrized* J_g*-holomorphic sphere of degree* d *in* M *can be translated in this language as splitting of unparametrized* J_g*-holomorphic sphere of degree* $d+[CP^1]$ *in* $M \times CP^1$ *in connected sum of several unparametrized* J_g*-holomorphic spheres of total degree* $d + [CP^1]$.

One of these spheres has degree one in CP^1-directions (and should lie in $\mathcal{M}_{J,d^0}$). All the other spheres have degree zero in CP^1-directions. Each of these spheres maps to a point under projection $M \times CP^1 \to M$ and thus, it should lie in $\mathcal{M}_{J_0,a_i d^i}/PSL(2,C)$.

Comment 3. *The energy* $E(C_i)$ *of the "pieces"* $\{C_i\}$ *is strictly positive and* $\sum_i a_i E(C_i) \leq E(\phi) < K$ *which allows us to choose the same range of allowed almost-complex structures for all degeneration patterns.*

Comment 4. *The numbers* $\{a_i\}$ *respect the fact that the some of* J_0*-holomorphic spheres which appear in "the degeneration process" are* $\{a_i\}$*-fold branched covers of other* J_0*-holomorphic spheres* $\{C_i \in \mathcal{M}^*_{J_0,d^i}/PSL(2,C)\}$.

The topological space $\mathcal{N}_{J_g,D}$ is not a smooth manifold. However, it admits a smooth desingularization $\mathcal{M}_{J_g,D}$ constructed as follows [Ru1]:

For each point $z \in CP^1$ let ev_z be the evaluation at the point z map from Map to M defined as follows: $ev_z(\varphi) = \varphi(z)$

We also have a more general evaluation map from $Map \bar{\times} (CP^1)^m$ to M^m:

$$ev(\varphi, z_1, ..., z_m) = \{\varphi(z_1), ..., \varphi(z_m)\}$$

Here the symbol $\bar{\times}$ means taking the product and then moding out by the action of $PSL(2,C)$. The group element $g \in PSL(2,C)$ acts on $Map \times (CP^1)^m$ by the formula:

stratum governed by another "degeneration pattern".

$$g \cdot (\varphi, z_1, ..., z_m) \;=\; (\varphi \cdot g^{-1} \;,\; g \cdot z_1,\; ...,\; g \cdot z_m)$$

To construct the desired desingularization, we also need "the product evaluation map" , which we will also define by ev. This "product map"

$$ev: \;\; Map \times (CP^1)^{m_0} \times Map\bar{\times}(CP^1)^{m_1} \times ... \times Map\bar{\times}(CP^1)^{m_k} \;\rightarrow$$

$$\rightarrow M^{m_0+...+m_k} \times (CP^1)^{m_0} \tag{2.9}$$

acts as identity from the factor $(CP^1)^{m_0}$ in the l.h.s. of (2.9) to the factor $(CP^1)^{m_0}$ in the r.h.s. of (2.9).

For any degeneration pattern D_d let us consider the evaluation map

$$ev: \bigcup_{g \in \mathcal{G}_0} \mathcal{M}_{J_g, d^0} \times (CP^1)^{m_0} \times \mathcal{M}^*_{J_0, d^1} \bar{\times} (CP^1)^{m_1} \times ... \times \mathcal{M}^*_{J_0, d^k} \bar{\times} (CP^1)^{m_k} \rightarrow$$

$$\rightarrow \;\; M^{m_0+...+m_k} \times (CP^1)^{m_0} \times \mathcal{G}_0 \tag{2.10}$$

Here m_i is the valency of the vertex d^i of the "degeneration graph" T of our degeneration pattern (how many other components the given component C_i intersects)

Let us observe that the factors of M in the r.h.s. of (2.10) are in one-to-one correspondence with the edges of the "degeneration graph" T . The set of these edges can be divided in the union of groups in two different ways:

The first way is to consider two edges lying in the same group iff they have the common vertex of the type $\{d^0; d^1; ...; d^k\}$ This corresponds to the grouping the factors of M as in the r.h.s. of (2.10) .

The second way is to consider two edges lying in the same group iff they have the common vertex of the type $\{I_1; ...; I_t\}$. Using this way of grouping the edges, we can regroup the factors of M in $M^{m_0+...+m_k} \times (CP^1)^{m_0}$ and rewrite the r.h.s. of

(2.10) as

$$M^{m_0+\ldots+m_k} \times (CP^1)^{m_0} \times \mathcal{G}_0 \;=\; M^{n_0+\ldots+n_t} \times (CP^1)^{m_0} \times \mathcal{G}_0 \qquad (2.11)$$

For each index $j = 1, ..., t$ let us take the diagonal $\Delta_j = M \subset M^{n_j}$ and take the product $\Delta = \Pi_{j=0}^{t} \Delta_j \subset M^{m_0+\ldots+m_k}$ of these diagonals

Let π be the projection from

$$\mathcal{M}_{J_g,d^0} \times (CP^1)^{m_0} \times \mathcal{M}^*_{J_0,d^1} \bar{\times} (CP^1)^{m_1} \times \ldots \times \mathcal{M}^*_{J_0,d^k} \bar{\times} (CP^1)^{m_k}$$

to $\mathcal{M}_{J_g,d^0} \times \mathcal{M}^*_{J_0,d^1} \times \ldots \times \mathcal{M}^*_{J_0,d^k}$

It follows directly from the definition of $\mathcal{N}_{J_g,D}$ that $\pi^{-1}(\mathcal{N}_{J_g,D})$ lies inside $ev^{-1}[\Delta \times (CP^1)^{m_0} \times \{g\}]$ (both topological spaces lie inside the manifold

$$\mathcal{M}_{J_g,d^0} \times (CP^1)^{m_0} \times \mathcal{M}^*_{J_0,d^1} \bar{\times} (CP^1)^{m_1} \times \ldots \times \mathcal{M}^*_{J_0,d^k} \bar{\times} (CP^1)^{m_k} \;).$$

Moreover, dimension-counting [Ru1] implies that the map π restricted to $ev^{-1}[\Delta \times (CP^1)^{m_0} \times \{g\}]$ is a branched covering. Let us denote the topological space $ev^{-1}[\Delta \times (CP^1)^{m_0} \times \{g\}]$ by $\mathcal{M}_{J_g,D}$.

The proposition 6.3.3 of [McD S] states that the image of the evaluation map ev is transversal to the product of diagonals Δ .

McDuff and Salamon stated this proposition in slightly different terms without working with inhomogenous Cauchy-Riemann equations and without including an additional factor of $(CP^1)^{m_0}$. However, the transversality result stated here reduces to their result by replacing M by $M \times CP^1$.

It follows from the lemma 2.4 that for generic value of $g \in \mathcal{G}_0$ the space $\mathcal{M}_{J_g,D}$ is a smooth manifold which gives the desired smooth desingularization of $\mathcal{N}_{J_g,D}$

At this point we make a

2.4 List of analytic lemmas about the compactification

Let M be a semi-positive almost-Kahler manifild; $d = (d_1, ..., d_s)$ be a vector of integers.

Lemma 2.7. *For the "generic" choice of* $g \in \mathcal{G}_0 \bigcap \mathcal{J}_+(M, \omega, K)$ *the moduli space* $\mathcal{M}_{J_g,d}$ *can be compactified as a stratified space* $\bar{\mathcal{M}}_{J_g,d}$ *such that each stratum is a smooth manifold.*

Lemma 2.8. *The strata of* $\bar{\mathcal{M}}_{J_g,d}$ *are labelled by degeneration patterns* $\{D_d\}$ *and are diffeomorphic to the manifolds* $\{\mathcal{M}_{J_g,D_d}\}$.

The stratum $\mathcal{M}_{J_g,D^\beta}$ lies inside the closure of another stratum $\mathcal{M}_{J_g,D^\alpha}$ if the degeneration pattern D^β is a **subdivision** of the degeneration pattern D^α.

Definition. *A degeneration pattern*

$$D^\beta \;=\; \{\;\{(d^0)^\beta; (d^1)^\beta; ...; (d^{k^\beta})^\beta; I_1^\beta; ...; I_{t^\beta}^\beta\;\; T^\beta\;\}$$

is called a **subdivision** of a degeneration pattern

$$D^\alpha \;=\; \{\;\{(d^0)^\alpha; (d^1)^\alpha; ...; (d^{k^\alpha})^\alpha; I_1^\alpha; ...; I_{t^\alpha}^\alpha\;\; T^\alpha\;\}$$

if there is a system of maps

$$\psi_d: \;\{(d^0)^\beta; (d^1)^\beta; ...; (d^{k^\beta})^\beta\} \to \{(d^0)^\alpha; (d^1)^\alpha; ...; (d^{k^\alpha})^\alpha\}$$

$\psi_I: \;\{I_1^\beta; ...; I_{t^\beta}^\beta\} \to \{I_1^\alpha; ...; I_{t^\alpha}^\alpha\}$ and

$$\psi_T: \;T^\beta \to T^\alpha$$

which are consistent in an obvious sense and satisfy an additional property

$$\sum_{(d^i)^\beta \in \psi_d^{-1}(d^{i^\alpha})} a_{i^\beta} d^{i^\beta} = a_{i^\alpha} d^{i^\alpha}$$

Lemma 2.9. *$\langle c_1(TM); C_i\rangle \geq 0$ for all i and $\langle c_1(TM); q^d\rangle \geq \sum_i \langle c_1(TM); C_i\rangle$ where the sum is taken over bubbled-off J-holomorphic spheres $\{C_i\}$ in our degeneration pattern.*

Lemma 2.10. *The codimension of the stratum $\mathcal{M}_{J_g,D_d}$ is always greater or equal to $2k$ where $\{d^0; d^1; ...; d^k\}$ is from the degeneration pattern D_d*

Lemma 2.11. *For any two generic g_1 and g_2 in $\mathcal{G}_0 \bigcap \mathcal{J}_+(M,\omega,K)$ there exists a smooth path $\gamma : [0;1] \to \mathcal{G}_0 \bigcap \mathcal{J}_+(M,\omega,K)$ joining them, such that for any degeneration pattern D_d the manifold $\bigcup_{g\in\gamma} \mathcal{M}_{J_g,D_d}$ gives a smooth cobordism between $\mathcal{M}_{J_{g_1},D_d}$ and $\mathcal{M}_{J_{g_2},D_d}$. This cobordism has dimension at least one smaller than the moduli space $\mathcal{M}_{J_g,d}$ istelf.*

We are not proving lemmas 2.7 - 2.11 in our paper, although the results of these lemmas are necessary to justify our considerations.

The proof of Lemmas 2.7 - 2.11 can be found either in [RT] or in [McD S].

Remark. *The choice of the compactification of the moduli space $\mathcal{M}_{J_g,d}$ of J-holomorphic curves which we use following [Ru1] is not the only possible one. Kontsevich [Ko2] introduced "the moduli space of stable maps" which*

1) Maps to the compactification we choose,

2) Has all the desired properties (i.e., lemmas 2.7 - 2.11 still hold for "the moduli space of stable maps" as well).

Chapter 3

Quantum Cup-Products

The total cohomology group $H^*(M)$ has a natural bilinear form given by Poincare duality . We will denote this bilinear form by $<;>$ i.e., $\eta^{AB} = <A;B>$ where $A \in H^m(M)$; $B \in H^{2n-m}(M)$.

In order to determine the structure constants $(Q^D_{AC})_q$ of the quantum cohomology ring it is sufficient to define "quantum tri-linear pairings" $<A;B;C>_q$ and then put

$$(Q^D_{AC})_q = \eta^{BD} < A;B;C >_q \tag{3.1}$$

where we use Einstein notation and sum over the repeated index B.

Definition A (Witten).

Let $A, B, C \in H^*(M, Z) \otimes Z_{<C>}$ Then

$$< A;B;C >^{Wi}_q = \sum_d q^d \int_{\mathcal{M}_{J,d}} ev_0^*(A) \bigwedge ev_\infty^*(B) \bigwedge ev_1^*(C) \tag{3.2}$$

Strictly speaking, the r.h.s. does not make sense because the moduli space $\mathcal{M}_{J,d}$ is non-compact and the notion of its top-dimensional homology class is not well-defined.

In order to make it well-defined, the integral in the r.h.s. of (3.2) should be considered as an integral over the **compactified** moduli space.

Since the evaluation maps ev_0, ev_1 and ev_∞ do not extend to the compactification divisor, in order to define the integral in the r.h.s. of (3.2), we should make some choices of differential forms on M representing cohomology classes A, B and C.

In addition we need $ev_0^*(A)$, $ev_\infty^*(B)$ and $ev_1^*(C)$ to be differential forms on $\mathcal{M}_{J,d}$ which should extend (at least as continious differential forms) to the compactification divisor.

In order to show that the integral (3.2) over the compactified moduli space is well-defined, one must prove that it converges and is independent of the choice of differential form representatives of cohomology classes A, B and C and on the choice of J, assuming the latter to be "generic".

This analytic problem has not been solved. It is likely that the analogous construction of Taubes in gauge theory [Ta2] can be adjusted to this situation.

In order to handle analytic problems related to the non-compactness of the moduli spaces $\bar{\mathcal{M}}_{J,d}$, it is more convenient to work with cycles on M and and their intersections instead of forms on M and their wedge product (if we choose our cycles to be "generic").

The two approaches are related by Poincare duality $A \to \widehat{A}$
where $A \in H^m(M)$, $\widehat{A} \in H_{2n-m}(M)$.

Let M be a smooth compact $2n$-dimensional manifold. A d-dimensional pseudocycle of M is a smooth map

$$f : V \to M$$

where $V = V_1 \cup \ldots \cup V_d$ is a disjoint union of oriented σ-compact manifolds without boundary[1] such that

$$\overline{f(V_d)} - f(V_d) \subset \bigcup_{j=0}^{d-2} f(V_j), \qquad \dim V_j = j, \qquad V_{d-1} = \emptyset$$

Every d-dimensional singular homology class α can be represented by a pseudo-

[1]A finite dimensional manifold V is called σ-compact if it is a countable union of compact sets.

cycle $f : V \to M$. To see this represent it by a map $f : P \to M$ defined on a d-dimensional finite oriented simplicial complex P without boundary. This condition means that the oriented faces of its top-dimensional simplices cancel each other out in pairs.[2]

Thus P carries a fundamental homology class $[P]$ of dimension d and α is by definition the class $\alpha = f_*[P]$. Now approximate f by a map which is smooth on each simplex. Finally, consider the union of the d and $(d-1)$-dimensional faces of P as a smooth d-dimensional manifold V and approximate f by a map which is smooth across the $(d-1)$-dimensional simplices.

Pseudo-cycles of M form an abelian group with addition given by disjoint union. The neutral element is the empty map defined on the empty manifold $V = \emptyset$. The inverse of $f : V \to M$ is given by reversing the orientation of V. A d-dimensional pseudo-cycle $f : V \to M$ is called **cobordant to the empty set** if there exists a $(d+1)$-dimensional

pseudo-cycle with boundary $F : W \to M$ with $W = \cup_j W_j$ such that

$$\delta W_{j+1} = V_j, \qquad F|_{V_j} =.f|_{V_j}$$

for $j = 0, \ldots, d$. Two d-dimensional pseudo-cycles $f : V \to M$ and $f' : V' \to M$ are called **cobordant** if $f \cup f' : (-V) \cup V' \to M$ is cobordant to the empty set.

Two pseudo-cycles $e : U \to M$ and $f : V \to M$ are called **transverse** if $e_i : U_i \to M$ is transverse to $f_j : V_j \to M$ for all i and j.

Lemma 3.1 (McDuff and Salamon). *Let $e : U \to M$ be an $(m-d)$-dimensional singular submanifold and $f : V \to M$ be a d-dimensional pseudo-cycle.*

If e is transverse to f then the set $\{(u,x) \in U \times V \mid e(u) = f(x)\}$ is finite. In this

[2]To avoid some technicalities with jiggling (i.e. making maps transverse) caused by the fact that P is not a manifold, one could equally well work with elements in the rational homology $H_*(M,Q)$. Because rational homology is isomorphic to rational bordism $\Omega_*(M)\otimes Q$, there is a basis of $H_*(M,Q)$ consisting of elements which are represented by smooth manifolds. Thus we may suppose that P is a smooth manifold, if we wish.

case define

$$e \cdot f = \sum_{\substack{u \in U, x \in V \\ e(u)=f(x)}} \nu(u,x)$$

where $\nu(u,x)$ is the intersection number of $e_{m-d}(U_{m-d})$ and $f_d(V_d)$ at the point $e_{m-d}(u) = f_d(x)$.

The intersection number $e \cdot f$ depends only on the cobordism classes of e and f.

Every $(2n-d)$-dimensional pseudo-cycle $e : W \to M$ determines a homomorphism

$$\Phi_e : H_d(M,Z) \to Z$$

as follows. Represent the class $\alpha \in H_d(M,Z)$ by a pseudo-cycle $f : V \to M$. Any two such representations are cobordant and hence, by Lemma 2.5. , the intersection number

$$\Phi_e(\alpha) = e \cdot f$$

is independent of the choice of f representing α. The next assertion also follows from Lemma 2.5.

Lemma 3.2 (Lemma 7.1.3 of [McD S]). *The homomorphism Φ_e depends only on the cobordism class of e.*

Using this isomorphism, "q-deformed tri-linear pairings" $< A; B; C >_q$ can be defined as follows:

Definition B (Vafa,Ruan).

$$< A; B; C >_q^{VR} = \sum_d q^d \sum_{[\varphi \in \mathcal{M}_{J_g,d} \bigcap ev_0^{-1}(\widehat{A}) \bigcap ev_\infty^{-1}(\widehat{B}) \bigcap ev_1^{-1}(\widehat{C})]} \pm 1 \qquad (3.3)$$

Here the sum in the r.h.s. of (3.3) is over those values of d that

$dimA + dimB + dimC = dim\mathcal{M}_{J,d}$ and over components of the intersection $\mathcal{M}_{J,d} \bigcap ev_0^{-1}(\widehat{A}) \bigcap ev_\infty^{-1}(\widehat{B}) \bigcap ev_1^{-1}(\widehat{C})$ (all these components are zero-dimensional)

The sign ± 1 is taken according to the orientation of intersection

$\mathcal{M}_{J,d} \bigcap ev_0^{-1}(\widehat{A}) \bigcap ev_\infty^{-1}(\widehat{B}) \bigcap ev_1^{-1}(\widehat{C})$. This intersection index is unambigously defined since the moduli space $\mathcal{M}_{J,d}$ is provided with its canonical orientation using the determinant line bundle of the $\bar{\partial}$-operator [Ru1],[McD S].

The above definition requires several comments:

1) The almost-complex structure J involved in the definition of $\mathcal{M}_{J,d}$ in the r.h.s. of (3.3) may depend on d. For each particular d the range of allowed almost-complex structures is an open dense set in $\mathcal{J}_+(M,\omega,K)$ where the value of K depends on d. The intersection of the allowed ranges for different d may be empty.

2) We should make some "clever choice of cycles" representing the homology classes $\widehat{A}, \widehat{B}, \widehat{C}$ in order the r.h.s. of (3.3) to be defined (i.e., the intersection of the cycles to be transverse)

3) We should prove that the r.h.s. of (3.3) is independent of this choice

4) We should prove that the r.h.s. of (3.3) is independent of the choice of J as long as J is "regular" and varies in the allowed range $\mathcal{J}_+(M,\omega,K)$.

"Regular" means that J is a regular value of the projection map π from $\bar{\partial}^{-1}(0) \in Map_d \times \mathcal{J}_+(M,\omega,K)$ to $\mathcal{J}_+(M,\omega,K)$.

5) We should prove that the r.h.s. of (3.3) lies in the Novikov ring Λ_ω.

"The clever choice of cycles" means that these cycles should be realized by "pseudo-manifolds".

The proof of "independence of the choices" is given in [RT]. This proof uses cobordism arguments and relies on the Lemmas 2.7 - 2.11 about compactification.

Lemma 3.3. *The r.h.s. of* (3.3) *lies in the Novikov ring* Λ_ω.

To prove the lemma, it is sufficient to prove that

$$Z_{<C>} \subset Z_{q_1,\ldots,q_s} \subset \Lambda_\omega \tag{3.4}$$

The first inclusion in (3.4) was proved in section two. To prove the second inclusion, it is sufficient to show that $<\omega, q_i>= a_i > 0$ for all $i = 1,\ldots,s$ which is equivalent to the fact that we can choose the basis of symplectic forms $\{\omega_i\} \in \tilde{K}$ such that $\omega = \sum_i a_i\omega_i$ with all $\{a_i\} > 0$.

To construct such a collection $\{\omega_i\} \in \tilde{K}$, let us introduce an inner product in $H^2(M, R)$, orthogonal complement to ω with respect to this inner product and take a $s-1$-dimensional simplex in this orthogonal complement with vertices $\{\epsilon_1, ..., \epsilon_s\}$ of norm less than sufficiently small number ϵ.We will assume that this is a regular simplex with center zero.Then we claim that the cohomology classes $\{\omega+\epsilon_1, ..., \omega+\epsilon_s\}$ will do the job (since ω will be ain a convex hull of $\{\omega_1, ..., \omega_s\}$. We can adjust $\{\epsilon_i\}$ such that $\{\omega + \epsilon_i\}$ are rational cohomology classes and multiply them by a common factor to make them integer cohomology classes. The lemma is proved.

The formula (3.3) for the "q-deformed tri-linear pairings" was first written by Vafa [Va1].

But in [Va1] only "unperturbed" holomorphic maps were considered. This makes the formula (3.3) incorrect when the dimension formula (2.7) does not hold for some components of the moduli space $\mathcal{M}_{J,d}$.

Lemma 3.4. *There exist choices of smooth differential form representatives of cohomology classes A, B and C such that*

$< A; B; C >_q^{VR} = < A; B; C >_q^{Wi}$.

If we take differential forms with supports near $\widehat{A}, \widehat{B}, \widehat{C}$ respectively then the integral in the r.h.s. of (3.2) is well-defined.

If A and $C \in H^*(M, Z) \otimes Z_{<C>}$ be $Z_{<C>}$-valued cohomology classes of M and let $A * B$ be their quantum cup-pruduct. Then we have:

$$deg(C * A) \ = \ deg(C) \ + \ deg(A) \tag{3.5}$$

Thus, we have a new Z-graded ring structure on $H^*(M, Z) \otimes Z_{<C>}$. We will call this new ring **the quantum cohomology ring** of M and we will denote it $HQ^*(M)$.

Let us define the homomorphism $l^* : HQ^*(M) \to H^*(M)$ as tensor multiplication on the ring Z over the ring $Z_{<C>}$ which is induced by the augmentation $I : Z_{<C>} \to Z$.

Lemma 3.5. *l^* is a ring homomorphism which preserves the grading.*

Before going to the Floer cohomology ring and proving that it is isomorphic to the quantum cohomology ring let me say once more.

Only Definition B of the quantum cup-product has well-defined mathematical objects on its r.h.s.

In Floer theory which will be discussed in the next section there is a linear map $m_F : H^*(M) \to End(H^*(M) \otimes \Lambda_\omega)$ (Floer multiplication).

To formulate the main result of our paper, that the quantum and Floer cohomology have the same ring structure, we should define an analog of this Floer's map m_F in quantum cohomology: namely, an operation $m_Q(C)$ of quantum multiplication (from the left) on the cohomology class $C \in H^*(M) \otimes \Lambda_\omega$. This operation acts as

$$m_Q(C) : H^*(M) \otimes \Lambda_\omega \to H^*(M) \otimes \Lambda_\omega$$

If we fix some (homogenous) basis $\{A, B, ...\}$ in $H^*(M, Z)$ then the matrix elements $< B|m_Q(C)|A >$ of the operator $m_Q(C)$ in this basis can be written as:

$$< B|m_Q(C)|A >=< \hat{A}; \eta(B); \hat{C} >_q \tag{3.6}$$

Here $\eta : H^m(M) \to H_m(M)$ is a duality isomorphism determined by the choice of a basis.

Chapter 4

Review of Symplectic Floer Cohomology

Let $\mathcal{L}M$ be the free loop-space of our (compact, simply-connected semi-positive) almost-Kahler manifold M and let $\widehat{\mathcal{L}M}$ be its universal cover. The points in $\widehat{\mathcal{L}M}$ can be described as pairs $(\gamma; z)$ where $\gamma : S^1 \to M$ be a free-loop in M and $z : D^2 \to M$ be a smooth map from 2-disc D^2 which coincides with γ at the boundary of the disc $\partial D^2 = S^1$. The two maps z_1 and z_2 of the disc are considered to be equivalent if they are homotopic to each other and the corresponding homotopy leaves their common boundary loop γ fixed.

Following Floer [F1-F8] we can define "the symplectic action functional" $S_\omega : \widehat{\mathcal{L}M} \to R$ as follows:

$$S_\omega(\gamma; z) = \int_{D^2} z^*(\omega) \tag{4.1}$$

where ω is the symplectic form on M and $z^*(\omega)$ is its pull-back to the 2-disc D^2.

The tangent vectors to the free loop-space at the point $\gamma \in \mathcal{L}M$ can be described as C^{7n-1}-smooth vector fields $\{\xi, \eta, ...\}$ on M along the loop γ. The free loop-space $\mathcal{L}M$ (and its universal cover) has a natural structure of (infinite-dimensional) almost-

Kahler Banach manifold described as follows:

Let g and ω be the Riemannian metric and the symplectic form on M. Then we can define the Riemannian metric $\tilde{g}$ and the symplectic form $\tilde{\omega}$ on the loop-space $\mathcal{L}M$ by the formulas:

$$\tilde{g}(\xi,\eta) = \int_{S^1} g(\xi(\gamma(\theta)); \eta(\gamma(\theta))d\theta \tag{4.2A}$$

and

$$\tilde{\omega}(\xi,\eta) = \int_{S^1} \omega(\xi(\gamma(\theta)); \eta(\gamma(\theta))d\theta \tag{4.2B}$$

where θ is the natural length parameter on the circle S^1 defined modulo 2π

The Riemannian metric $\tilde{g}$ and the symplectic form $\tilde{\omega}$ on the loop-space $\mathcal{L}M$ are related through the almost-complex structure operator $\tilde{J}$. Action of this almost-complex structure operator $\tilde{J}$ on the tangent vector ξ to the loop γ (which is the vector field restricted to the loop γ) is defined as the action of the almost-complex structure operator J on the base manifold M on our vector field ξ.

Lemma 4.1. *(Givental). The following statements hold:*

A) S_ω is a Morse-Bott function on $\widehat{\mathcal{L}M}$

B) All the critical submanifolds of the "symplectic action" S_ω on the universal cover of $\mathcal{L}M$ are obtained from each other by the action of the group $\pi_1(\mathcal{L}M) = \pi_2(M) = H_2(M)$ of covering transformations. The image of (any of) these critical submanifolds under the universal covering map $\pi : \widehat{\mathcal{L}M} \to \mathcal{L}M$ is the submanifold $M \subset \mathcal{L}M$ of constant loops.

If we consider $\widehat{\mathcal{L}M}$ as a symplectic manifold with the symplectic form $\tilde{\omega}$ given by $(3.2B)$ then:

C) The hamiltonian flow of the functional S_ω generates the circle action on $\widehat{\mathcal{L}M}$ and

D) This circle action is just rotation of the loop $\gamma(\theta) \to \gamma(\theta + \theta^0)$

Let us choose (once and for all) one particular critical submanifold $M \subset \widehat{\mathcal{L}M}$ of the symplectic action S_ω. Then any other critical submanifold of S_ω has the form $q^d M$ which means that it is obtained from M by the action of the element q^d of the group $H_2(M)$ of covering transformations.

Lemma 4.2. *The gradient flow of the symplectic action functional S_ω on the universal cover of the loop-space (which is provided with its canonical Riemannian metric $\tilde{g}$) depends only on the almost-complex structure J and does not depend on the symplectic form ω.*

We assume that the metric g and the symplectic form ω are related in the standard way through the almost-complex structure J .

Let $\dot{\gamma}(\theta)$ be unit the tangent vector field to the loop $\gamma \in \mathcal{L}M$ (this tangent vector coincides with the generator of the circle action rotating the loop). Then we have

Lemma 4.3.

$$grad \ \ S_\omega(\gamma(\theta)) = J(\dot{\gamma}(\theta))$$

Let $\{H_\theta\} : M \to R$ be some (smooth) family of functions on M parametrized by $\theta \in S^1$. This family of functions on M is usually called "periodic time-dependent Hamiltinian" where θ is "time". The fact that $\theta \in S^1$ reflects the fact that the time-dependence of our Hamiltonian is periodic. Let $S_{\omega,H} : \widehat{\mathcal{L}M} \to R$ be a functional on $\widehat{\mathcal{L}M}$ defined as follows:

$$S_{\omega,H}(\gamma; z) \ = \ S_\omega(\gamma; z) - \int_{S^1} H_\theta(\gamma(\theta))d\theta \tag{4.3}$$

Theorem 4.4 (Floer). *For "generic" choice of H and J the functional $S_{\omega,H}$ is a Morse functional on $\widehat{\mathcal{L}M}$ (wchich is usually called "the symplectic action functional perturbed by a Hamiltonian term")*

"Generic" here means that the statement is true for the Baire second category set in the product of the space of all C^{7n}-functions on $M \times S^1$ and the space of all C^{10n}-almost-complex structures on M.

The idea of the proof is as follows: The critical points of the functional $S_{\omega,H}$ are in one-to-one correspondence with the 2π-periodic trajectories of the θ-dependent Hamiltonian flow on M

$$\frac{dx(\theta)}{d\theta} = grad_\omega(H(\theta, x)) \tag{4.4}$$

If we denote $\phi^H_{2\pi}$ the operator of shift for the time 2π along the trajectories of the vector field (4.4) then the critical points of $S_{\omega,H}$ on $\mathcal{L}M$ are in one-to-one correspondence with the fixed-points of the diffeomorphism $\phi^H_{2\pi} : M \to M$. By varying the "Hamiltonian" H, we can arrange these fixed-points to be isolated and non-degenerate.

The gradient flow trajectory of "the perturbed symplectic action functional"on the universal cover of the loop-space can be defined as a solution of the following PDE:

$$\frac{\partial \gamma_\tau(\theta)}{\partial \tau} = J\frac{\partial \gamma_\tau(\theta)}{\partial \theta} - grad\ H_\theta \tag{4.5}$$

where τ is the parameter on the gradient flow line, varying from minus infinity to plus infinity, and θ be the parameter on the loop.

We will consider only those solutions of (4.5) which have bounded energy, i.e. satisfy the estimate

$$E(\gamma_\tau(\theta)) = \frac{1}{2}\int_R d\tau \int_{S^1} d\theta || \frac{\partial \gamma_\tau(\theta)}{\partial \tau}\ ||^2 + ||J\frac{\partial \gamma_\tau(\theta)}{\partial \theta} - grad\ H_\theta||^2 < \infty \tag{4.6}$$

The L^2-boundedness condition (4.6) implies that

$$\gamma_\tau(\theta) \to \gamma_-(\theta) \qquad \tau \to -\infty \tag{4.7A}$$

and

$$\gamma_\tau(\theta) \to \gamma_+(\theta) \qquad \tau \to +\infty \tag{4.7B}$$

where $\gamma_-(\theta)$ and $\gamma_+(\theta)$ are some "critical loops" or, in another words, critical points of the perturbed symplectic action functional on the universal cover of the loop-space.

It follows from (4.6) that

$$E(\gamma_\tau(\theta)) \geq S_{\omega,H}(\gamma_+) - S_{\omega,H}(\gamma_-) \tag{4.8}$$

with the equality if and only if $\gamma_\tau(\theta)$ is a solution to the gradient flow equation (4.5).

We can identify $S^1 \times R$ with C^* by the map

$$(\theta(mod 2\pi); \tau) \quad \rightarrow \quad exp(\tau + i\theta) \tag{4.9}$$

which allows us to study $\bar{\partial}$-operators on the cylinder $S^1 \times R$..

Let $\gamma_+, \gamma_- \in \widehat{\mathcal{L}M}$ be two such critical points of $S_{\omega,H}$.

Let us define $\mathcal{M}_{J,H}(\gamma_-, \gamma_+)$ as the space of all L^2-bounded trajectories of the gradient flow of $S_{\omega,H}$, joining the critical point γ_- and the critical point γ_+

In more down-to earth terms, the space $\mathcal{M}_{J,H}(\gamma_-, \gamma_+)$ can be defined as the space of all solutions of (4.5) , 2π-periodic in θ with the assymptotics given by (4.7A) and (4.7B)

The space $\mathcal{M}_{J,H}(\gamma_-, \gamma_+)$ admits an obvious free $\mathbf{R}$-action (translation in τ-direction). Let us denote the quotient by $\tilde{\mathcal{M}}_{J,H}(\gamma_-, \gamma_+)$.

The space $\mathcal{M}_{J,H}(\gamma_-, \gamma_+)$ can be thought as union of all loops lying on the gradient flow trajectories, and thus, as a topological subspace in $\widehat{\mathcal{L}M}$

Definition. *A simply-connected symplectic manifold M is called weakly monotone if for any homology class $A \in H_2(M)$ the two conditions $\omega(A) > 0$ and $c_1(TM)(A) \geq 3 - n$ implies $c_1(TM)(A) \geq 0$.*

Lemma 5.1.3 of [McD S]. *Let M be weakly monotone compact simply-connected symplectic manifolfd. Then the set $\mathcal{J}_+(M,\omega) = \bigcap_K \mathcal{J}_+(M,\omega,K)$ contains a path-connected dense subset. The set $\mathcal{J}_+(M,\omega,K)$ is open, dense and path-connected for every K.*

From now on we will assume M to be weakly monotone.

Theorem 4.6 (Floer, Hofer,Salamon). *For the any "generic" choice of the function H on $S^1 \times M$ and for any pair $\{\gamma_+, \gamma_-\}$ of the critical points of $S_{\omega,H}$ in $\widehat{\mathcal{L}M}$ and for any "generic" choice of $J \in \mathcal{J}_+(M, \omega, K)$ the following statements hold:*

A) The space $\mathcal{M}_{J,H}(\gamma_-, \gamma_+)$ is a smooth submanifold in $\widehat{\mathcal{L}M}$

B) The dimension of this submanifold is equal to the spectral flow of the family $\{D_\tau = -J\frac{\partial}{\partial\theta} + grad(H_\theta)\}$ $(-\infty < \tau < \infty)$ of the operators acting from the space $W^2_{5n}(S^1, \gamma^*_\tau(TM)$ to the space $W^2_{5n-1}(S^1, \gamma^*_\tau(TM)$

C) For any element $q^d \in \pi_2(M)$ we have

$$dim(\mathcal{M}_{J,H}(\gamma_-, q^d\gamma_+)) = dim(\mathcal{M}_{J,H}(\gamma_-, \gamma_+)) + 2 < c_1(TM); q^d > \qquad (4.10)$$

(this formula follows from the computation of the spectral flow)

We will reprove this theorem (in a more general setting adjusted for our purposes) in the next section.

Since the Hessian of $S_{\omega,H}$ at any of its critical points has infinitely many positive and infinitely many negative eigenvalues, the usual Morse index of the critical point is not well-defined.

But the relative Morse index of the pair γ_- and γ_+ of the critical points is well-defined as $vdim(\mathcal{M}(\gamma_-, \gamma_+))$

Here by $vdim(\mathcal{M}(\gamma_-, \gamma_+))$ we denote "the virtual dimension" of the manifold $\mathcal{M}(\gamma_-, \gamma_+)$ which is defined as a spectral flow of the family of operators $\{D_\tau\}(-\infty < \tau < +\infty)$

In the case when J and H are "generic" (or "regular" in the sence specified below), this virtual dimension $vdim$ is equal to actual dimension $dim(\mathcal{M}_{J,H}(\gamma_-, \gamma_+))$ of this manifold.

Floer and Hofer [FH] and also McDuff and Salamon [McD S] proved that the moduli space $\mathcal{M}(\gamma_-,\gamma_+)$ of gradient flow trajectories carries a canonical orientation coming from determinant line bundle of $\bar{\partial}$-operator $\{D_\gamma = \bar{\partial} - grad(H_\theta)\}$

This operator acts from the tangent space to $Map(\gamma_-,\gamma_+)$ of all maps from the cylinder to M which are locally (on compact sets in $S^1 \times R$) lie in $W^{1,p}$-Sobolev space and which tend to $\gamma_\pm$ in $W^{1,p}$-Sobolev norm as $\tau \to \pm\infty$ to the space of all L^p-integrable (0,1)-forms on $S^1 \times R$ with the coefficients in $\gamma^*(TM)$.

Remark. *The above choice of "boundary conditions" at $\tau \to \pm\infty$ automatically implies exponential decay at $\tau \to \pm\infty$ and the boundary conditions in the sense of Atiyah-Patodi-Singer (This was proved already in [APS]).*

But for some choices of H (like $H = 0$) the virtual dimension of $\mathcal{M}_{J,H}(\gamma_-,\gamma_+)$ might not be equal to the actual dimension. In these cases $\mathcal{M}_{J,H}(\gamma_-,\gamma_+)$ is usually not smooth. Different components of $\mathcal{M}_{J,H}(\gamma_-,\gamma_+)$ are allowed to have different dimensions and to meet each other nontransversally.

Lemma 4.7. *Let $\gamma_1,\gamma_2,\gamma_3$ be three "critical loops" in $\widehat{\mathcal{L}M}$ Then*

$$vdim(\mathcal{M}(\gamma_1,\gamma_3)) = vdim(\mathcal{M}(\gamma_1,\gamma_2)) + vdim(\mathcal{M}(\gamma_2,\gamma_3)) \tag{4.11}$$

This formula follows from the spectral flow calculations and from the fact that we are working on the simply-connected space $\widehat{\mathcal{L}M}$.

It is worth mentioning that the formula (4.11) is not true if we do not go from $\mathcal{L}M$ to its universal cover $\widehat{\mathcal{L}M}$. Without going to the universal cover the formula (4.11) is only true modulo 2Γ where Γ is the least common multiple of the numbers $\{< c_1(TM); q_i >\}$

Lemma 4.8.

$$\mathcal{M}_{J,H}(q^d\gamma_-, q^d\gamma_+) = q^d[\mathcal{M}_{J,H}(\gamma_-,\gamma_+)]$$

Although the Morse index of the critical points $\{\gamma_i\}$ of $S_{\omega,H}$ is not defined in the usual sense, the formulas (4.10) and (4.11) allow us to define it by hands.

Let us fix some "basic critical point" $\gamma_0 \in \widehat{\mathcal{L}M}$

For any other critical point $\gamma \in \widehat{\mathcal{L}M}$ we can always find $q^d \in H_2(M)$ such that either the manifold $\mathcal{M}_{J,H}(\gamma_0, q^d\gamma)$ or the manifold $\mathcal{M}_{J,H}(\gamma, q^d\gamma_0)$ is non-empty. Then we can define

$$deg[\gamma] \;=\; deg[\gamma_0] \;+\; vdim(\mathcal{M}_{J,H}(\gamma_0, q^d\gamma)) \;-\; deg[q^d] \tag{4.12A}$$

$$deg[\gamma] \;=\; deg[\gamma_0] \;-\; vdim(\mathcal{M}_{J,H}(\gamma, q^d\gamma_0)) \;+\; deg[q^d] \tag{4.12B}$$

Here $deg[q^d]$ is defined by (2.4)

The formulas $(4.12A)$ and $(4.12B)$ for different $\{d\}$ are consistent with each other.

So, our grading on the set of critical points of $S_{\omega,H}$ is defined uniquely up to an additive constant $deg[\gamma_0]$

The manifolds $\{\mathcal{M}_{J,H}(\gamma_-, \gamma_+)\}$ of the gradient flow trajectories are non-compact. There are two basic reasons of their non-compactness:

A) The gradient flow trajectory may go through the intermediate critical point, i.e., it may "split" into the union of two trajectories

B) The sequence of the gradient flow trajectories in $\mathcal{M}_{J,H}(\gamma_-, \gamma_+)$ may diverge by "bubbling off" a J-holomorphic sphere of degree d. The formal limit of this diverging sequence will be a union of a gradient flow trajectory from $\mathcal{M}(q^d\gamma_-, \gamma_+)$ (which can be thought as a pseudo-holomorphic cylinder in M in the sence which will be explained in the next section) and a J-holomorphic sphere of degree d attached to this cylinder at some point.

In order to have a good intersection theory on manifolds of gradient flow trajectories (which is the main ingredient in the definition of cup-product in Floer cohomology) we should **compactify** them.

The compactification of the manifold $\mathcal{M}(\gamma_-, \gamma_+)$ includes:

A) The loops lying inside the union

$$\mathcal{M}(\gamma_-,\gamma_1)\bigcup\mathcal{M}(\gamma_1,\gamma_2)\bigcup\ldots\bigcup\mathcal{M}(\gamma_{k-1},\gamma_k)\bigcup\mathcal{M}(\gamma_k,\gamma_+)$$

B) Those trajectories in $\mathcal{M}(q^d\gamma_-,\gamma_+)$ which can be obtained by bubbling off from some trajectories in $\mathcal{M}(\gamma_-,\gamma_+)$.

If we considercompactification of the space $\tilde{\mathcal{M}}(\gamma_-,\gamma_+)$ (which has one real dimension lower) then the part A) of the compactification will consist of

$$\bigcup_{\gamma_1,\ldots,\gamma_k} \tilde{\mathcal{M}}(\gamma_-,\gamma_1)\times\tilde{\mathcal{M}}(\gamma_1,\gamma_2)\times\ldots\times\tilde{\mathcal{M}}(\gamma_{k-1},\gamma_k)\times\tilde{\mathcal{M}}(\gamma_k,\gamma_+)$$

Here the union is taken over all intermediate critical points.

The part A) of the compactification is easy to handle. We just add this part to $\mathcal{M}(\gamma_-,\gamma_+)$ to obtain a smooth manifold with corners.

The above constructed manifold with corners is desingularized by a canonical Morse-theoretic procedure of "gluing trajectories" (see [CJS1],[AuBr] for a precise construction) to obtain a smooth manifold with boundary. The boundary of this "desingularized" manifold consists of the gradient flow trajectories going through the intermediate critical points together with "the gluing data" which corresponds to "blowing up" the corners.

The part B) of the compactification is much more complicated object to work with. It was proved by Floer using dimension-counting argument (4.10) that if we bubble off the sphere of degree d such that $< c_1(TM); q^d > \; > 0$ then the corresponding part of the compactification has codimension at least two.

Hofer and Salamon [HS] showed that in the case when M is weakly monotone and the almost-complex structure J_0 on M is "generic" then the spheres of degrees d such that $< c_1(TM); q^d > \; < 0$ cannot be bubbled off. If we bubble off the sphere of degree d such that $< c_1(TM); q^d > \; = 0$ then for "generic" almost-complex structure J_0 on M the corresponding part of the compactification has codimension at least four.

Let us consider the free abelian group $CF_*(M,H)$ genetated by the critical points of the functional $S_{\omega,H}$ on $\widehat{\mathcal{L}M}$. This abelian group has a structure of $Z_{[H_2(M)]}$-module since the group $H_2(M)$ of the covering transformations acts on the set of critical points.

Since the action of the group of covering transformations is free, the module $CF_*(M,H)$ is a free module, generated by the finite set of the critical points of the multivalued functional $S_{\omega,H}$ on the loop-space $\mathcal{L}M$ (before going to the universal cover)

Let us take a completion of this abelian group $CF_*(M,H)$ by allowing certain infinite linear combinations of the critical points of $S_{\omega,H}$ to occur in $CF_*(M,H)$. More precisely, let us tensor our $Z_{[H_2(M)]}$-module $CF_*(M)$ on the Novikov ring Λ_ω over the ring $Z_{[H_2(M)]}$. We will denote this extended abelian group by the same symbol $CF_*(M,H)$ (which is actually an Λ_ω-module) and call it a **Floer chain complex** corresponding to "perturbed symplectic action" $S_{\omega,H}$.

A Floer chain complex $CF_*(M,H)$ has a natural Z-grading **deg** induced from the above-defined grading of the critical points

Let $\{x, y, ...\}$ be some set of critical points of $S_{\omega,H}$ on $\widehat{\mathcal{L}M}$. We assume that this set maps isomorphically onto the set of all critical points of $S_{\omega,H}$ on $\mathcal{L}M$. In other words, we choose one point in the fiber of the universal cover over each critical point.

Now we are ready to define a boundary operator $\delta : CF_*(M,H) \to CF_*(M,H)$ which will:

A) commute with Λ_ω-action (i.e. δ will be Λ_ω-module homomorphism);

B) decrease the Z-grading **deg** by one.

Let us define

$$\delta x = \sum_y \sum_d < \delta x; q^d y > q^d y \tag{4.13}$$

where the sum in the r.h.s. of (3.13) is taken only over such values of y and of d that the critical points x and $q^d y$ have relative Morse index one.

Let $< \delta x; q^d y >$ be the number of connected components of $\mathcal{M}_{J,H}(x, q^d y)$ (all of them are one-dimensional) counted with $\pm$ 1-signs depending on orientations of these components relative to their ends x and $q^d y$

Lemma 4.9. *The boundary operator δ is defined over the Novikov ring Λ_ω*

Let us prove that for any index $i = 1, ..., s$ there exists an integer N_i such that only those values of $(d_1, ..., d_s)$ could contribute to the r.h.s. of (4.13) that $d_i > -N_i$ for all i.

Proof.

By definition of the gradient flow, if the manifold $\mathcal{M}_{J,H}(x, q^d y)$ is non-empty, then $S_{\omega,H}(x) > S_{\omega,H}(q^d y)$ for any J_0-compatible symplectic form ω (and in particular for our basic forms $\{\omega_1, ..., \omega_s\}$) This means that for any positive real number t and for any trajectory $\gamma(\tau, \theta) \in \mathcal{M}_{J,H}(x, q^d y)$ we have

$$S_{\omega_i,H}(x) \quad - \quad S_{\omega_i,H}(q^d y) \quad =$$

$$= \quad \int_{S^1 \times R} \gamma^*(\omega_i) \quad + \quad \int_{S^1} H_\theta(y(\theta))d\theta \quad - \quad \int_{S^1} H_\theta(x(\theta))d\theta \quad > \quad 0 \tag{4.14}$$

Since the values of the integrals $\int_{S^1} H_\theta(y(\theta))d\theta$ and $\int_{S^1} H_\theta(x(\theta))d\theta$ are independent of the symplectic form, and $\int_{S^1 \times R} \gamma^*(\omega_i)$ is a homotopy invariant which depends only on the limit values of γ as $\tau \to \pm\infty$, then we can conclude that in our case $\int_{S^1 \times R} \gamma^*(\omega_i)$ is a homotopy invariant and depends only on the value of d.

It follows directly from the fact that $\{\omega_1, ..., \omega_s\}$ form a basis dual to

$\{q_1, ..., q_s\}$ that if the value of d_i decreases by one then the value of the integral $\int_{S^1 \times R} \gamma^*(\omega_i)$ also decreases by one.

This observation implies that in order the inequality (4.14) to hold the lower bound $-N_i$ on the value of d_i should exist. Since $\omega = \Sigma_i m_i \omega_i$ where the coefficients $\{m_i\}$ are positive, the existence of lower bounds $\{-N_i\}$ imply the statement of the Lemma 4.9.

Theorem 4.10 (Floer, Hofer-Salamon). $\delta^2 = 0$.

The proof of this statement is highly non-trivial and relies heavily on the way how we compactify the manifolds $\{\mathcal{M}(x, y)\}$ of the gradient flow trajectories. This allows one to prove that the contributions to δ^2 "from the boundary" of the appropriate manifold of the gradient flow trajectories will cancel each other.

Lemma 4.11. *Homology $HF_*(M)$ of the Floer chain complex inherit both the N-module structure and the Z-grading* **deg** *from $CF_*(M, H)$.*

Theorem 4.12 (Floer). $HF_*(M) = H_*(M) \otimes \Lambda_\omega$

The idea of the proof of this theorem is the following:

First, Floer proved that the graded module $HF_*(M)$ is well-defined and independent of the choice of "hamiltonian perturbation" H involved in its definition.

He constructed an explicit chain homotopy between Floer chain complexes $CF_*(M, H_1)$ and $CF_*(M, H_2)$ constructed from two different hamiltonians H_1 and H_2 (which are functions from $S^1 \times M$ to R)

Second, if we consider "θ-independent Hamiltonian" $H : M \to R$ which is small in C^2-norm, then all the critical points of "perturbed symplectic action functional" $S_{\omega,H}$ on $\widehat{\mathcal{L}M}$ can be obtained from the critical points of H on M by covering transformations. Here M is embedded in $\widehat{\mathcal{L}M}$ as a submanifold of constant loops as specified above.

Saying the same thing in another words, only constant loops can be critical points of $S_{\omega,H}$. These "critical loops" can take values in the critical points of H on the manifold M and only in those points.

The gradient flow trajectories joining these critical points can be of two types:

A) Lying inside submanifold $M \subset \mathcal{L}M$ of constant loops

B) Not lying inside any submanifold of constant loops

The trajectories of type B) cannot be isolated due to non-triviality of S^1-action (which rotates the loop) on the space of those trajectories.

Thus, only trajectories of type A) can contribute to the Floer boundary operator δ. But the chain complex generated by these trajectories is exactly the Morse complex of M.

Thus, the homology of the Floer complex will be the same as homology of M (tensored by the appropriate coefficient ring due to the action of the group of covering transformations)

Before starting to explain cup-product structure, let us define **Floer cohomology** $HF^*(M)$ and **Floer cochain complex** $CF^*(M,H)$ for both perturbed and unperturbed symplectic action. To define those objects we define:

A) Floer cochain complex $CF^*(M,H) \doteqdot Hom_{\Lambda_\omega}(CF_*(M,H),\Lambda_\omega)$

B) Coboundary operator δ^* in the Floer cochain complex as a conjugate to the biundary operator δ in $CF_*(M,H)$

C) Floer cohomology $HF^*(M)$ as homology of the complex $(CF^*(M,H);\delta^*)$

Remark 1. *The Floer cochain complex $(CF^*(M,H);\delta^*)$, defined above, is a Morse complex for the same perturbed symplectic action functional $S_{\omega,H}$ but with* **reversed time**

Lemma 4.13. *The following statements hold:*

A) $HF^*(M) = Hom_{\Lambda_\omega}(HF_*(M),\Lambda_\omega)$

B) $HF^*(M) = H^*(M)\otimes\Lambda_\omega = H^*(M,\Lambda_\omega)$ i.e. Floer cohomology are isomorphic to ordinary cohomology with the appropriate coefficient ring.

Remark 2. *The Floer cochain complex of the "perturbed symplectic action functional" $S_{\omega,H}$ has a canonical basis corresponding to the critical points $\{q^{d^1}x, q^{d^2}y, ...\}$*

of $S_{\omega,H}$. This basis is dual to the basis of the critical points $\{q^{-d^1}x, q^{-d^2}y, ...\}$ in the Floer chain complex $CF_*(M,H)$. In this basis the "Floer coboundary oberator" δ^* (defined as a conjugate operator to the "Floer boundary operator" δ) **is exactly** *the boundary operator in Morse complex for the functional $S_{\omega,H}$ with reverse time.*

Proceeding as above, we can develop the Morse-Bott-Witten theory for the Morse-Bott functional S_ω on the universal cover of the loop-space $\widehat{\mathcal{L}M}$ in the same way as Floer developed his theory for Morse functional $S_{\omega,H}$ on the same space.

The main ingredient of such a theory is a **Floer chain complex** corresponding to the "unperturbed symplectic action" S_ω . Algebraically this chain complex is defined as $H_*(M) \otimes \Lambda_\omega$.

Geometrically, this Floer chain complex is generated (as an abelian group) by the pseudo-cycles inside the critical submanifolds $\{q^d M\}$ of the symplectic action functional.

Here, as above, we allow certail infinite linear combinations to occur. The occurrence of these infinite linear combinations stands for the fact that we are working over the Novikov ring Λ_ω.

This new Floer chain complex (we will denote it $CF_*(M,0)$) also has a Λ_ω-module structure and the Z-grading **deg**. The latter is defined as follows:

$$deg[q^d \widehat{A}] \;=\; deg[\widehat{A}] \;-\; \sum_{i=1}^{s} d_i \; deg[q_i] \tag{4.15}$$

where $\widehat{A}$ be some homology class of degree $deg[\widehat{A}]$.

Here, as in chapter 3, $A \to \widehat{A}$ stands for Poincare duality isomorphism between cohomology class $A \in H^{2n-deg[\widehat{A}]}(M)$ and homology class $\widehat{A} \in H_{deg[\widehat{A}]}(M)$

Let us fix some (homogenous) basis $\{A_1, A_2, ...\}$ in the cohomology of M , let $\{\eta(A_1), \eta(A_2), ...\}$ be the dual to $\{\eta(A_1), \eta(A_2), ...\}$ basis in homology of M and let $\{\widehat{A_1}, \widehat{A_2}, ...\}$ be the corresponding Poincare dual basis in homology of M . We will always assume that each element in our bases $\{\widehat{A_1}, \widehat{A_2}, ...\}$ and $\{\eta(A_1), \eta(A_2), ...\}$ is represented by some pseudo-cycle in M (which we will denote by the same symbol).

Let $q^{d^1}\widehat{A}$ and $q^{d^2}\widehat{B}$ be two (bihomogenous) elements of the Floer chain complex $CF_*(M,0)$ (represented by pseudo-cycles).

Proceeding as above, we can define:

A) The topological space $\mathcal{M}(q^{d^1}\widehat{A}, q^{d^2}\eta(B))$ of L^2-bounded gradient flow trajectories of S_ω which flow from the cycle $q^{d^1}\widehat{A}$ in $q^{d^1}M$ as $\tau \to -\infty$ to the cycle $q^{d^2}\eta(B)$ in $q^{d^2}M$ as $\tau \to +\infty$. This topological space should not necessarily be a manifold.

We compactify this space by the gradient flow trajectories passing through the intermediate critical submanifolds and by trajectories in $\mathcal{M}(q^{d^1+d}\widehat{A}, q^{d^2}\eta(B))$ obtained by bubbling off.

(Here $\widehat{B} \to \eta(B)$ stands for Poincare duality **in homology** of M which makes sence once we have chosen a basis in $H^*(M)$).

B) Relative Morse index of $q^{d^1}\widehat{A}$ and $q^{d^2}\widehat{B}$ as the virtual dimension of $\mathcal{M}(q^{d^1}\widehat{A}, q^{d^2}\eta(B))$ defined as $deg[q^{d^2}\widehat{B}] - deg[q^{d^1}\widehat{A}]$

C) Z-grading **deg** on the Floer chain complex $CF_*(M)$ (defined by the formula (4.15)) such that the relative Morse index of $q^{d^1}\widehat{A}$ and $q^{d^2}\widehat{B}$ is equal to the difference of their degrees

D) Floer boundary operator $\delta : CF_*(M,0) \to CF_*(M,0)$ which commutes with Λ_ω-action and decreases the Z-grading **deg** by one.

This Floer boundary operator is defined as

$$\delta\widehat{A} = \sum_{\widehat{B}}\sum_{d} < \delta\widehat{A}; q^d\widehat{B} > q^d\widehat{B} \tag{4.16}$$

Here the first sum $\sum_{\widehat{B}}$ in (4.16) is taken over our (fixed) set of pseudo-cycles representing the basis in $H_*(M)$, and $< \delta\widehat{A}; q^d\widehat{B} >$ counts the number (weighted with ±1-signs depending on orientation) of isolated gradient flow trajectoties inside the manifold $\mathcal{M}_d(\widehat{A}, \eta(B))$ defined as

$$\mathcal{M}_d(\widehat{A}, \eta(B)) = \mathcal{M}(\widehat{A}, q^d \eta(B)) \tag{4.17}$$

Here the r.h.s. of (4.17) gives the definition to its l.h.s.

Lemma 4.14 (Givental). *The Floer boundary operator is identically zero in this case.*

The proof of this lemma relies on the fact that any Morse-Bott function which is a hamiltonian of an S^1-action has this property.

Thus, we have

Lemma 4.15. *Floer homology $HF_*(M)$ coincides as an abelian group with the Floer chain complex $CF_*(M,0)$ of the unperturbrd symplectic action functional.*

Let us define **Floer cochain complex** $CF^*(M,0)$ as dual to the Floer chain complex $CF_*(M,0)$.

The Floer cochain complex $CF^*(M,0)$ also has a canonical basis $\{q^{d^1}A_1, q^{d^2}A_2, ...\}$ This basis is dual to the basis $\{q^{-d^1}\widehat{A_1}, q^{-d^2}(\widehat{A_2}), ...\}$ in the Floer chain complex $CF_*(M,0)$.

Later on we will use these two bases in these two Floer cochain complexes when we will work with Floer cohomology instead of Floer homology.

Chapter 5

Cup-Products in Floer Cohomology

Original Floer's motivation for introducing the object which is now known as "symplectic Floer cohomology" was to give an interpretation of fixed points of the symplectomorphism of M in terms of Morse theory.

In order to have such an interpretation, one has to develop some Morse theory on the loop space $\mathcal{L}M$ instead of the usual Morse theory on M. By identifying the fixed points of our symplectomorphism (constructed canonically from "the periodic time-dependent Hamiltonian" $H_\theta : S^1 \times M \to R$) with the critical points of Floer's "perturbed symplectic action functional" on the loop-space, we have such a Morse-theoretic interpretation.

If we assume all the fixed points of our symplectomorphism to be non-degenerate (which is the case only if "the Hamiltonian" H is "generic" in the sense of Lemma 4.4), and use the fact that homology of our Morse-Floer complex $CF_*(M)$ are isomorphic to the classical homology of M, then the lower bound on the number of the fixed points of our symplectomorphism will be given by usual Morse inequalities. This was one part of the Arnold's Conjecture which Floer proved.

The other part of Arnold's Conjecture was: what if we drop the non-degeneracy assumption on the Jacobian at the fixed points? Classical Morse theory gives us the lower bound on the number of (not necessarily non-degenerate) critical points of the function H on the compact manifold M in terms of the so-called cohomological length of M.

Definition. *The cohomological length of the topological space* M *is an integer* $k \in Z_+$ *such that:*

A) There exist $k-1$ cohomology classes $\alpha_1, ..., \alpha_{k-1}$ on M of positive degrees such that $\alpha_1 \wedge ... \wedge \alpha_{k-1} \neq 0$ in $H^*(M)$ and

B) There are no k cohomology classes on M with this property.

Thus we see that in order to try to prove this part of the Arnold's Conjecture in the framework of Floer's Morse theory, **one needs to invent some multiplicative structure in Floer cohomology**. A kind of such a multiplicative structure was also constructed by Floer [F1] and successfully applied to this part of Arnold's Conjecture in another Floer's paper [F2].

However, Hofer [Ho2] have found a proof of this part of Arnold's Conjecture without using Floer homology.

Using the (nontrivial) fact that Floer cohomology $HF^*(M)$ are canonically isomorphic (as Λ_ω-module) tp the ordinary cohomology $H^*(M) \otimes \Lambda_\omega$ the following five statements are equivalent:

A) we have a multiplication in Floer cohomology

$$HF^*(M) \otimes HF^*(M) \rightarrow HF^*(M) \tag{5.1A}$$

which is Λ_ω-module homomorphism and which preserves the Z-grading;

B) we have an action

$$HF^*(M) \rightarrow End(HF^*(M)) \tag{5.1B}$$

of Floer cohomology on itself (by left multiplication) which is Λ_ω-module homomorphism and which preserves the Z-grading;

C) we have an action

$$H^*(M) \to End(HF^*(M)) \tag{5.1C}$$

of classical cohomology of the manifold M on its Floer cohomology which preserves the Z-grading;

D) we have an action

$$H_*(M) \to End(HF^*(M)) \tag{5.1D}$$

of classical homology of the manifold M (related by Poincare duality with the cohomology of M) on its Floer cohomology which preserves the Z-grading;

E) we have an action

$$\Omega_*(M) \to End(CF^*(M, H)) \tag{5.1E}$$

of the space of singular chains in M which can be realized by pseudo-cycles on the Floer cochain complex of M. This action commutes with the boundary operator and preserves the Z-grading.

F) we have an action

$$\Omega_*(M) \to Hom(CF^*(M, H) \otimes CF_*(M, H), Z) \tag{5.1F}$$

Later on we will denote all these four maps $(5.1A) - (5.1F)$ by the same symbol m_F and call them **the Floer multiplication.**

Remark 1. *$End(CF^*(M, H))$ is isomopphic to $Hom(CF^*(M, H) \otimes CF_*(M, H), Z)$*

Remark 2. *The identity map from $CF^*(M, H)$ to $CF_*(M, H)$, which maps each critical point of $S_{\omega,H}$ to itself, gives a canonical isomorphism between $CF^l(M, H)$ $CF_{2n-l}(M, H)$ for every integer l. This map* **is not a chain map** *in the sense of homological algebra.*

In order to define the Floer multiplication m_F in the form $(5.1E)$ (or in the equivalent form $(5.1F)$) it is enough to define its matrix elements $< y|m_F(\widehat{C})|x >$ where

$x \in CF^m(M,H), y \in CF^l(M,H) = CF_{2n-l}(M,H)$, $\widehat{C} \in \Omega_{2n-k}(M)$ and then put

$$m_F(\widehat{C})(x) \quad = \quad \sum_{y,d} < q^d y|m_F(\widehat{C})|x > q^d y \tag{5.2}$$

where the sum in the r.h.s. of (5.2) is taken over the basis in the Λ_ω-module $CF^*(M,H)$ and only over those terms y, d such that $deg[q^d y] - deg[x] = k$.

Let x , y be two "basic" critical points of the functional $S_{\omega,H}$ on $\widehat{\mathcal{L}M}$, and let $q^{d^1}x$ and $q^{d^2}y$ be the corresponding elements of the Floer cochain complex $CF^*(M)$ with the index difference $l - m = k$. Then let us put

$$< q^{d^2}y|m_F(\widehat{C})|q^{d^1}x > \quad = \chi[\mathcal{M}_{J,H}(q^{d^2}y; q^{d^1}x) \bigcap \tilde{ev}_1^{-1}(\widehat{C})] \tag{5.3}$$

Here χ is Euler characteristic of the (zero-dimensional) oriented manifold; $\tilde{ev}$: $S^1 \times \widehat{\mathcal{L}M} \to M$ is the standard "evaluation map" where the circle S^1 is assumed to be embedded as a unit circle $|z| = 1$ in the complex plane C. The map $\tilde{ev}_1$ means evaluation of the loop at the point $z = 1$

Theorem 5.1. *The following two statements hold:*

A) The action m_F of $\Omega_*(M)$ on $CF^*(M,H)$ defined by (5.2) and (5.3) descends to the action m_F of $H_*(M)$ on $HF^*(M)$;

B) The induced action $m_F : H_*(M) \to End(HF^*(M))$ does not depend on the choice of "the Hamiltonian" H assuming that this Hamiltonian is "generic" in the sense of Lemma 4.4.

For the case when M is a **positive** almost-Kahler manifold the Theorem 5.1 was proved by Floer [F1].

We will reproduce this proof (with the appropriate modifications) for the general semi-positive case. In the next section the techniques which is used in this proof will be applied to prove equivalence of Floer's and quantum multiplication.

The main idea behind this proof is to consider "τ-dependent Hamiltonian perturbation" of the equation (4.5). More precisely, let H be some smooth function on $R \times S^1 \times M$. Here, as above, the real line R is equipped with the parameter τ, varying from minus infinity to plus infinity, and the circle S^1 is equipped with the arclength parameter θ.

Let us restrict ourselves to the C^{10n-1}-smooth functions on $R \times S^1 \times M$ which are τ-independent in the region $-\infty < \tau < -1$ and in the region $1 < \tau < +\infty$. This condition means that there exist two functions H_- and H_+ on $S^1 \times M$ such that

$$H(\tau; \theta) = H_-(\theta) \quad if \quad \tau \leq -1 \tag{5.4A}$$

$$H(\tau; \theta) = H_+(\theta) \quad if \quad \tau \geq 1 \tag{5.4B}$$

Let us denote the space of all such functions $\mathcal{G}_{H_-,H_+}$

We would like to stress that the Banach manifold $\mathcal{G}_{H_-,H_+}$ is defined for any choice of "boundary value" Hamiltonians H_+ and H_- (not necessarily "generic"). In the next section we will be interested in situation when either H_+, or H_-, or both, are identically zero.

Then we can study the space of solutions of the following PDE

$$\frac{\partial \gamma_\tau(\theta)}{\partial \tau} = J\frac{\partial \gamma_\tau(\theta)}{\partial \theta} - grad \; H(\tau, \theta) \tag{5.5}$$

which are L^2-bounded in the sense of (4.6)

For any L^2-bounded solution of (5.5) there exist a critical point x of S_{ω,H_+} and a critical point $q^d y$ of S_{ω,H_-} such that

$$\gamma_\tau(\theta) \to x(\theta) \qquad \tau \to +\infty \tag{5.6A}$$

and

$$\gamma_\tau(\theta) \to q^d y(\theta) \qquad \tau \to -\infty \tag{5.6B}$$

The proof reduces to to τ-independent case by cutting the cylinder into pieces.

Following the logic of the section 4, we can define the space $\mathcal{M}_H(q^d y, x)$ of L^2-bounded solutions of (5.5) which we will call "the moduli space of τ-dependent gradient flow trajectories".

The "virtual dimension" of $\mathcal{M}_H(q^d y, x)$ is again given by the spectral flow of the appropriate family of operators on the circle and coincides with the actual dimension for "generic" J and H. The moduli space $\mathcal{M}_H(q^d y, x)$ again carries a natural orientation (following the logic of [FH]).

The moduli spaces $\{\mathcal{M}_H(q^d y, x)\}$ of finite-energy- solutions of (5.5) are compactified by adding gradient flow trajectories obtained by "splitting" and by "bubbling off". The compactified moduli spaces $\{\bar{\mathcal{M}}_H(x, q^d y)\}$ have the structure of stratified spaces such as:

A) The strata are labelled by "degeneration patterns" labelled by almost the same data (labelled trees) as "degeneration patterns" of J-holomorphic spheres, described in the section 2. To avoid confusion, we will spell it out explicitely.

B) Each stratum (including the top stratum) is a smooth manifold **with boundary**.

Definition. *Degeneration patern for the gradient flow trajectory in* $\{\bar{\mathcal{M}}_H(q^d y, x)\}$ *is following set of data:*

DP1) The class $d^0 \in H_2(M)$, the set $\{d^1; ...; d^k\} \subset C \subset H_2(M)$ of non-zero effective classes , and the set $\{a_1; ...; a_k\}$ of positive integers, such that the following identity holds: $d = d^0 + \sum_{i=1}^k a_i d^i$

DP2) The set $\{I_1; ...; I_t\}$ of subsets in the set $\{d^0; d^1; ...; d^k\}$. We do not allow one of $\{I_1; ...; I_t\}$ to be the proper subset of another.

Using the set of data $\{d^0; d^1; ...; d^k; I_1; ...; I_t\}$, we can construct a graph T with $k+1+t$ vertices $\{d^0; d^1; ...; d^k;\ I_1; ...; I_t\}$ as follows:

If the class d^i lies in the set I_j then we join the vertices d^i and I_j by an edge.

DP3) The graph T obtained by above prescription is a tree.

Let $D_d = \{ \{d^0; d^1; ...; d^k\};\ \{a_1; ...; a_k\};\ \{I_1; ...; I_t\};\ T \}$ be some degeneration pattern. Then let us define $\mathcal{M}_H^{D_d}(q^d y, x)$ as a topological subspace in

$\mathcal{M}_H(q^{d^0} y, x) \times \Pi_{i=1}^k [\mathcal{M}^*_{J_0,d^i} \,/\, PSL(2,C)]$ as follows:

An element φ in $\mathcal{M}_H^{D_d}(q^d y, x)$ consists of one gradient trajectory C_0 of $S_{\omega,H}$ which flows from x as $\tau \to +\infty$ to $q^{d^0} y$ as $\tau \to -\infty$ and k unparametrized J_0-holomorphic spheres $\{C_i \in [\mathcal{M}^*_{J_0,d^i}/PSL(2,C)]\ \}$. We require that for any subset $I_j = \{d^{j_1}; ...; d^{j_{n_j}}\}$ from $\{I_1; ...; I_t\}$ the curves $\{C_{j_1}; ...; C_{j_{n_j}}\}$ have a common intersection point. We do not allow this intersection point to lie on any other curve $C_i \subset M$ in our collection[1]

Remark 3. *C_0 will be a cylinder and $\{C_i\}$ for $i > 0$ will be spheres.*

Lemma 5.2. *"The compactification divisor" $\bar{\mathcal{M}}_H(q^d y, x) - \mathcal{M}_H(q^d y, x)$ has codimension at least two.*

Proof. *K-semi-positivity implies that $(c_1(TM); C_i) \geq 0$, i.e., only spheres with non-negative first Chern number can bubble off. If at least one J-holomorphic sphere with positive Chern number bubbles off than codimension of the corresponding component of the compactification divisoris at least twice the Chern number of that sphere (which follows from explicit description of the corresponding component and the formula for its dimension). If only J-holomorphic spheres with Chern number zero bubble off then the last formula in section 2 of [HS] states that each such "bubbling off" decreases the dimension by four. This proves the lemma.*

Remark 4. *The proof given above is essentially contained in [HS].*

[1]If $C_{j_1} \bigcap ... \bigcap C_{j_{n_j}} \bigcap C_i \neq \emptyset$ then our "degenerate trajectory" would lie in the other stratum governed by another "degeneration pattern".

Remark 5. *Each stratum (including the top stratum) is a smooth manifold with boundary (and the boundary has codimension one in the corresponding stratum).*

Remark 6. *The boundary of the stratum* $\mathcal{M}_H^{D_d}(q^d y, x)$ *consists of a union of two components* $\mathcal{M}_H^{D_d}(q^d y, \delta^* x)$ *and* $\mathcal{M}_H^{D_d}(q^d \delta y, x)$

Remark 5. *Since all the spaces we are considering are topological subspaces in* $\widehat{\mathcal{L}M}$ *(or, saying it in another way, wee are working with the image of the universal cover of the space* $Map(S^1 \times R, M)$ *in* $\widehat{\mathcal{L}M}$ *then the boundary of the stratum* $\mathcal{M}_H(q^d y, x)$ *does not contain pieces* $\bigcup_z \mathcal{M}_H(q^d y, z) \bigcup \mathcal{M}_H(z, x)$ *where* z *runs over critical points of index difference greater than one. This holds since the pieces of the form of "broken trajectories" come as "unions of trajectories" and not as "products of trajectories" (as would be the case if we mod out* $\mathcal{M}_H(x, q^d y)$ *by* $\mathbf{R}$*–action).*

For the case of **positive** symplectic manifold the statement of the lemma 5.2. was proved in [F1]. In the semi-positive case the proof was given in [HS] when H was τ-independent. The general case was worked out in [PSS] and [RT2].

The formulas (4.9) – (4.11) have their analogues for the moduli spaces of τ-dependent gradient flow trajectories. This implies that we can fix the additive constant ambiguities in the gradings of the critical points of $\{S_{\omega,H}\}$
for all Hamiltonians simultaneously such that

$$vdim\, \mathcal{M}_H(q^d y, x) \;=\; deg_{CF^*(M,H_+)}(q^d y) \;-\; deg_{CF^*(M,H_-)}(x) \tag{5.7}$$

Theorem 5.3. *For any two "*τ*-dependent Hamiltonians"* $H^{(0)}$ *and* $H^{(1)}$ *lying in the space* $\mathcal{G}_{H_-,H_+}$ *the manifolds of trajectories* $\bar{\mathcal{M}}_{H^{(0)}}(q^d y, x)$ *and* $\bar{\mathcal{M}}_{H^{(1)}}(q^d y, x)$ *are cobordant to each other as stratified spaces with boundary.*

More precisely, there exists a path $\{H^{(t)}\}$ $(0 \leq t \leq 1)$ in $\mathcal{G}_{H_-,H_+}$ joining $H^{(0)}$ and $H^{(1)}$ such that $\bigcup_{0 \leq t \leq 1} \bar{\mathcal{M}}_{H^{(t)}}(q^d y, x)$ gives us the desired cobordism.

Proof. *Let* K *be a positive number greater than* $S_{\omega,H_+}(x) - S_{\omega,H_-}(q^d y)$*,*
$\mathcal{J}_+(M, \omega, K)$ *and* $Map(x, q^d y)$ *defined as above (recall that this is the space of all*

$W^{1,p}_{loc}$*-Sobolev maps from* $S^1 \times R$ *to* M *which tend to* x *and* $q^d y$ *in* $W^{1,p}$*-norm as* $\tau \to \pm\infty)$.

As was mentioned in section 4, the above choice of boundary conditions at $\tau \to \pm\infty$ guarantees exponential decay [APS].

Let us consider the infinite-dimensional Banach bundle $\mathcal{H}$ over $Map(x, q^d y) \times \mathcal{J}_+(M, \omega, K)$. The fiber of the bundle $\mathcal{H}$ over the point $(\gamma; J)$ in $Map(x, q^d y) \times \mathcal{J}_+(M, \omega, K)$ will be the space of all locally L^p-integrable $(0,1)$-forms on $R \times S^1 = C^*$ with the coefficients in $\gamma^*(TM)$ which exponentially tend to zero as $\tau \to \pm\infty$.

Let us consider the pull-back of this Banach bundle to $Map(q^d y, x) \times \mathcal{J}_+(M, \omega, K) \times \mathcal{G}_{H_-,H_+}$ and construct a (canonical) section Φ as follows:

$$\Phi(\gamma) = \frac{\partial \gamma}{\partial \tau} - J\frac{\partial \gamma}{\partial \theta} - grad\ H(\tau, \theta)\ d\bar{z} \tag{5.8}$$

Here $d\bar{z}$ is a canonical $(0,1)$-form on $R \times S^1 = C^*$. The identification between $R \times S^1$ and C^* is given by the map (4.8).

The arguments of McDuff [McD] show that if the function H does not admit any holomorphic symmetries with respect to parameters on $R \times S^1$ then the section Φ is regular over $Map(x, q^d y) \times \mathcal{J}_M \times \{H\}$.

Since the space of functions $\{H\}$ with this property is open and dense in $\mathcal{G}_{H_-,H_+}$ this means that the section Φ is regular over the dense open set in $Map((q^d y, x) \times \mathcal{J}_M \times \mathcal{G}_{H_-,H_+}$. Since the zero-set of the section Φ over $Map(q^d y, x) \times \{J\} \times \{H\}$ is our friend $\mathcal{M}_{J,H}(q^d y, x)$ then we can apply Lemmas 2.4 and 2.6 to prove existence of the above cobordism.

Since the Floer coboundary operator δ^* acts (in general) nontrivially on x and the Floer boundary operator δ acts nontrivially on y then the manifolds $\{\mathcal{M}_{H^{(t)}}(q^d y, x)\}$ are **manifolds with boundary** and the cobordism $\bigcup_{0 \le t \le 1} \bar{\mathcal{M}}_{H^{(t)}}(q^d y, x)$ will have **two extra boundary components**

$\bigcup_{0\le t\le 1}\mathcal{M}_{H^{(t)}}(q^d\delta y,x)$ and $\bigcup_{0\le t\le 1}\mathcal{M}_{H^{(t)}}(q^d y,\delta^* x)$

These extra components appear due to the presence of intermediate critical points of S_{ω,H_-} and S_{ω,H_+} .

Lemma 2.6 implies that such a path $\{H^{(t)}\}$ $(0\le t\le 1)$ in $\mathcal{G}_{H_-,H_+}$ joining $H^{(0)}$ and $H^{(1)}$ exists. The extension of Lemma 5.2. implies that the corresponding smooth cobordism can be compactified as a stratified space and "the compactification divisor" (defined as $\bigcup_{D_d}^{nontrivial}\bigcup_{0\le t\le 1}\mathcal{M}^{D_d}_{H^{(t)}}(q^d y,x)$) will have codimension at least two inside this stratified space.

This fact together with a standard transversality argument of [McD S] implies that the intersection of "the compactification divisor" with (any of) four "boundary components" of the total space of our cobordism will have codimension at least two in this "boundary component" (considered as a stratified space).

The theorem 5.3. is proved.

Now let us temporarily return to the case when "the Hamiltonian" H is τ-independent and let us remember that the manifolds $\{\mathcal{M}_H(q^{d^2}y;q^{d^1}x)\}$ of gradient flow trajectories can be thought either as submanifolds in the loop-space or as submanifolds in the space $Map(q^{d^2}y;q^{d^1}x)$ of maps from the cylinder $R\times S^1$ to M with the fixed "boundary values" at $\tau\to\pm\infty$

Thus, we have a commutative diagram

$$\begin{array}{ccc}\mathcal{M}_H(q^{d^2}y;q^{d^1}x) & \stackrel{ev_{\tau=0}}{\longrightarrow} & \mathcal{L}M\\ \downarrow & & \downarrow \tilde{ev}_1\\ Map(R\times S^1;M) & \stackrel{ev_{(0;1)}}{\longrightarrow} & M\end{array}$$

Having this commutative diagram in mind, we can rewrite the definition (5.3) for the matrix element of the Floer multiplication as:

$$< q^{d^2}y|m_F(\widehat{C})|q^{d^1}x> \quad = \quad \chi[\mathcal{M}_H(q^{d^2}y;q^{d^1}x)\bigcap ev^{-1}_{(0;1)}(\widehat{C})] \tag{5.9}$$

Here $ev_{0,1}$ is the "evaluation at the point (0;1)" map from $Map(R\times S^1;M)$ to M, and $ev_{\tau=0}$ is the "evaluation at the point 0 in τ-direction" map from $Map(R \times S^1; M)$ to $\mathcal{L}M$.

The formula (5.9) for the matrix element $< q^{d^2} y | m_F(\widehat{C}) | q^{d^1} x >$ of the Floer multiplication admits the following generalization:

Let H_+, H_-, x, y and H be defined as above. Then let us put

$$< q^{d^2} y | m_F(\widehat{C}) | q^{d^1} x > \;=\; \chi[\mathcal{M}_H(q^{d^2} y; q^{d^1} x) \bigcap ev^{-1}_{(0;1)}(\widehat{C})] \qquad (5.10)$$

where the r.h.s., as usual, means the Euler characteristic of the (zero-dimensional) intersection, or the intersection index.

Any cycle x in the Floer Chain complex $CF^*(M; H_+)$ can be written as a sum $\sum_k n_k x_k$ where x_k are (possibly coinciding) critical points of S_{ω,H_-} and $n_k = \pm 1$. The same is true for the cycle $y = \sum_l m_l y_l$ in $CF_*(M; H_-)$.

We can consider the manifolds

$$\mathcal{M}_H(q^{d^2} y; q^{d^1} x) \;=\; \bigcup_{k,l} n_k m_l \, \mathcal{M}_H(q^{d^2} y_l; q^{d^1} x_k)$$

Here the factor $n_k m_l = \pm 1$ in front means that the component $\mathcal{M}_H(q^{d^2} y_l; q^{d^1} x_k)$ should be taken with the appropriate orientation.

Lemma 5.4. *If we glue all the components of* $\mathcal{M}_H(q^{d^2} y; q^{d^1} x)$ *together, we will obtain a smooth* $deg(q^{d^2} y) - deg(q^{d^1} x)$*-dimensiona l pseudo-manifold* **without boundary** *(or pseudo-cycle).*

Proof. *Since by remarks 4 and 5 for each index k and l we have*

$$\delta\mathcal{M}_H(q^{d^2} y_l; q^{d^1} x_k) \;=\; \mathcal{M}_H(q^{d^2} \delta y_l; q^{d^1} x_k) \cup \mathcal{M}_H(q^{d^2} y_l; q^{d^1} \delta^* x_k)$$

then by definition of $\delta\mathcal{M}_H(q^{d^2} y; q^{d^1} x)$ we have:

$$\delta\mathcal{M}_H(q^{d^2} y; q^{d^1} x) \;=\; \bigcup_{k,l} n_k m_l \, \delta\mathcal{M}_H(q^{d^2} y_l; q^{d^1} x_k) =$$

$$= \bigcup_l m_l [\bigcup_k n_k \mathcal{M}_H(q^{d^2} y_l ; q^{d^1} \delta^* x_k)] \ \bigcup \ \bigcup_k n_k [\bigcup_l m_l \mathcal{M}_H(q^{d^2} \delta y_l ; q^{d^1} x_k)] \qquad (5.11)$$

Now let us look at this formula more closely. Since x is a cycle in $CF^*(M, H_+)$ and y is a cycle in $CF_*(M, H_-)$ we have

$\bigcup_k n_k \delta^* x_k = \emptyset$ (if a point in $CF^*(M, H_+)$ enters $\bigcup_k n_k \delta^* x_k$, it enters it twice: once with a positive orientation and once with a negative orientation)

and

$\bigcup_l m_l \delta y_l = \emptyset$ (if a point in $CF_*(M, H_-)$ enters $\bigcup_l m_l \delta y_l$, it enters it twice: once with a positive orientation and once with a negative orientation)

This implies that for each l

$$\bigcup_k n_k \mathcal{M}_H(q^{d^2} y_l ; q^{d^1} \delta^* x_k) = \emptyset \qquad (5.12)$$

and for each k

$$\bigcup_l m_l \mathcal{M}_H(q^{d^2} \delta y_l ; q^{d^1} x_k) = \emptyset \qquad (5.13)$$

(each component will enter twice: once with a positive orientation and once with a negative orientation)

The equations (5.11)-(5.13) combined imply that

$$\delta \mathcal{M}_H(q^{d^2} y ; q^{d^1} x) \ = \bigcup_l m_l \, \emptyset \ \bigcup \ \bigcup_k n_k \, \emptyset$$

from which the lemma 5.4. follows.

Theorem 5.5. *For any two τ-dependent Hamiltonians $H^{(0)}$ and $H^{(1)}$ from $\mathcal{G}_{H_-,H_+}$; for any cycle $x = \sum_k n_k x_k$ in $CF^*(M; H_+)$; for any pseudo-cycle $\widehat{C} \in \Omega_m(M)$ and for any cycle $y = \sum_l m_l y_l$ in $CF_*(M; H_-)$ such that the relative index of x and y is equal to m we have*

$$\chi[\mathcal{M}_{H^{(0)}}(q^{d^2}y;q^{d^1}x)\bigcap ev^{-1}_{(0;1)}(\widehat{C})] \;=\; \chi[\mathcal{M}_{H^{(1)}}(q^{d^2}y;q^{d^1}x)\bigcap ev^{-1}_{(0;1)}(\widehat{C})] \qquad (5.14)$$

Proof. *The Theorem 5.3 provides us with a cobordism* $\mathcal{M}^t$ *between*

$\mathcal{M}_{H^{(0)}}(q^{d^2}y;q^{d^1}x)$ and $\mathcal{M}_{H^{(1)}}(q^{d^2}y;q^{d^1}x)$. The fact that both x and y are cycles in the Floer complexes $CF^*(M,H_+)$ and $CF_*(M,H_-)$ respectively, means that the cobordism $\mathcal{M}^t$ between them does not have other boundary components. (All the "extra boundary components" of cobordisms $\{\bigcup_{0\le t\le 1}\mathcal{M}_{H^{(t)}}(q^{d^2}y_l;q^{d^1}x_k)\}$ for different k and l will cancel each other out in pairs after we glue them together).

The theorem 2.1 of [McD S] which claims that the map $ev_{(0;1)}$ from $\mathcal{M}_{H^{(t)}}(q^{d^2}y_l;q^{d^1}x_k)\times\mathcal{J}_M\times\mathcal{G}_{H_-,H_+}$ to M is surjective, allows us to apply the Lemma 2.5. to the evaluation map $ev_{(0;0)}$ taken as a "projection operator." By applying this lemma we have that the cobordism $\mathcal{M}^t$ intersects transversally with $ev^{-1}_{(0;1)}(\widehat{C})$ and the corresponding intersection gives us a smooth one-dimensional submanifold (with boundary).

This submanifold does not intersect the "compactification divisor" $\bar{\mathcal{M}}^t-\mathcal{M}^t$ since by Lemma 5.2. the latter has codimension ≥ 2 and we have the freedom of putting everything "in general position".

Thus, $\mathcal{M}^t\bigcap ev^{-1}_{0,1}(\widehat{C})$ gives us the desired compact one-dimensional cobordism between $\mathcal{M}_{H^{(0)}}(q^{d^2}y;q^{d^1}x)\bigcap ev^{-1}_{0,1}(\widehat{C})$ and $\mathcal{M}_{H^{(1)}}(q^{d^21}y;q^{d^1}x)\bigcap ev^{-1}_{0,1}(\widehat{C})$ The statement of Theorem 5.5. follows.

The same cobordism and transversality arguments together with the crucial analytic Lemma 5.2. prove the following

Lemma 5.6. *If* $\widehat{C}_0$ *and* $\widehat{C}_1$ *be two pseudo-cycles in* M *homologous to each other (which implies that they are actually cobordant to each other in the category of pseudo-manifolds); if* y *is a cycle in* $CF_*(M;H_-)$ *and* x *is a cycle in* $CF^*(M;H_+)$*; and if* (τ_0,θ_0) *and* (τ_1,θ_1) *are any two points on the cylinder* $R\times S^1$ *then*

$$\chi[\mathcal{M}_{H^{(0)}}(q^{d^2}y;q^{d^1}x)\bigcap ev^{-1}_{\tau_0,\theta_0}(\widehat{C}_0)] = \chi[\mathcal{M}_{H^{(1)}}(q^{d^2}y;q^{d^1}x)\bigcap ev^{-1}_{\tau_1,\theta_1}(\widehat{C}_1)] \qquad (5.15)$$

Up to this point we were working with cycles (x and y) in Floer (co)-chain complexes and not with their homology classes. So, we will have to prove that the above-defined matrix elements of Floer multiplication will not change if we change the Floer cycles (x and y) on homologous Floer cycles (in appropriate complexes). For this we will need the following two lemmas:

Lemma 5.7. *Let $x = \bigcup_k n_k x_k$ and $x' = \bigcup_k n'_k x'_k$ be homologous elements in $CF^*(M, H_+)$. Then*

$$< q^{d^2}y|m_F(\widehat{C})|q^{d^1}x > \;=\; < q^{d^2}y|m_F(\widehat{C})|q^{d^1}x' > \qquad (5.16)$$

Proof. *The fact that x and x' are homologous in $CF^*(M, H_+)$ means that there exists an element $z = \bigcup_s n_s z_s \in CF^{*-1}(M, H_+) = CF_{2n+1-*}(M, H_+)$ such that*
$\delta^* z = x - x'$

z is a union (taken with appropriate signs) of a critical points of S_{ω,H_-} of index one higher than the index of x

Then we can consider a space of trajectories

$$\mathcal{M}_H(q^{d^2}y, q^{d^1}z) \;=\; \bigcup_s n_s \mathcal{M}_H(q^{d^2}y; q^{d^1}z_s)$$

by Remark 6 earlier in this chapter

$$\delta\mathcal{M}_H(q^{d^2}y;q^{d^1}z) \;=\; \mathcal{M}_H(q^{d^2}y;q^{d^1}x)\bigcup -\mathcal{M}_H(q^{d^2}y;q^{d^1}x') \qquad (5.17)$$

here the minus sign in front of $\mathcal{M}_H(q^{d^2}y;q^{d^1}x')$ means that this component is taken with the negative orientation.

Now (5.17) implies that

$$< q^{d^2} y | m_F(\widehat{C}) | q^{d^1} x > \quad - \quad < q^{d^2} y | m_F(\widehat{C}) | q^{d^1} x' > =$$

$$= \chi[\mathcal{M}_{H^{(0)}}(q^{d^2} y; q^{d^1} x) \bigcap ev^{-1}_{(0;1)}(\widehat{C})] \; - \; \chi[\mathcal{M}_{H^{(0)}}(q^{d^2} y; q^{d^1} x') \bigcap ev^{-1}_{(0;1)}(\widehat{C})] \; =$$

$$= \quad \chi[\delta[\mathcal{M}_{H^{(0)}}(q^{d^2} y; q^{d^1} z) \bigcap ev^{-1}_{(0;1)}(\widehat{C})]\,] \tag{5.18}$$

Since the expression in (5.18) is algebraic number of points in the boundary of a one-dimensional manifold (in this case, $\mathcal{M}_H(q^{d^2} y; q^{d^1} z) \bigcap ev^{-1}_{(0;1)}(\widehat{C})$) which is always zero. This proves the lemma.

Lemma 5.8. *Let $y = \bigcup_l m_l y_l$ and $y' = \bigcup_l m'_l y'_l$ be homologous elements in $CF_*(M, H_-)$. Then*

$$< q^{d^2} y | m_F(\widehat{C}) | q^{d^1} x > \quad = \quad < q^{d^2} y' | m_F(\widehat{C}) | q^{d^1} x > \tag{5.19}$$

Proof (almost identical to the proof of the Lemma 5.7). *The fact that y and y' are homologous in $CF_*(M, H_-)$ means that there exists an element $z = \bigcup_p m_p z_p \in CF_{*-1}(M, H_+)$ such that $\delta z = y - y'$*

z is a union (taken with appropriate signs) of a critical points of S_{ω, H_-} of index one lower than the index of y

Then we can consider a space of trajectories

$$\mathcal{M}_H(q^{d^2} z; q^{d^1} x) \quad = \quad \bigcup_p m_p \mathcal{M}_H(q^{d^2} z_p; q^{d^1} x)$$

by Remark 6 earlier in this chapter

$$\delta \mathcal{M}_H(q^{d^2} z; q^{d^1} x) \quad = \quad \mathcal{M}_H(q^{d^2} y; q^{d^1} x) \bigcup - \mathcal{M}_H(q^{d^2} y'; q^{d^1} x) \tag{5.20}$$

here the minus sign in front of $\mathcal{M}_H(q^{d^2} y'; q^{d^1} x)$ means that this component is taken with the negative orientation.

Now (5.20) implies that

$$< q^{d^2} y | m_F(\widehat{C}) | q^{d^1} x > \quad - \quad < q^{d^2} y' | m_F(\widehat{C}) | q^{d^1} x > \quad =$$

$$= \chi[\mathcal{M}_H(q^{d^2} y; q^{d^1} x) \bigcap ev^{-1}_{(0;1)}(\widehat{C})] \; - \; \chi[\mathcal{M}_H(q^{d^2} y'; q^{d^1} x) \bigcap ev^{-1}_{(0;1)}(\widehat{C})] \; =$$

$$= \quad \chi[\delta[\mathcal{M}_H(q^{d^2} z; q^{d^1} x) \bigcap ev^{-1}_{(0;1)}(\widehat{C})]\,] \tag{5.21}$$

Since the expression in (5.21) is algebraic number of points in the boundary of a one-dimensional manifold (in this case, $\mathcal{M}_H(q^{d^2} z; q^{d^1} x) \bigcap ev^{-1}_{(0;1)}(\widehat{C})$) which is always zero. This proves the lemma.

Now we are ready to prove the Theorem 5.1. In order to prove it, we should (following Floer):

A) Construct a chain homotopy $h_H : CF^*(M, H_-) \to CF^*(M, H_+)$ (which depends on the choice of the function $H \in \mathcal{G}_{H_-,H_+}$.

B) Prove that the chain homotopy h_H gives a well-defined homomorphism $h_{H_-,H_+} : HF^*(M, H_-) \to HF^*(M, H_+)$ on the level of homology, and this homomorphism is independent of the choice of H.

C) Prove that $h_{H_1,H_3} = h_{H_2,H_3} h_{H_1,H_2}$ for any triple of Hamiltonians H_1, H_2, H_3 defined as functions from $S^1 \times M$ to R

D) Prove that for any singular homology class $\widehat{C}$ in M

$$h_{H_-,H_+}(m_F^{H-}(\widehat{C})) \quad = \quad m_F^{H+}(\widehat{C}) h_{H_-,H_+} \tag{5.22}$$

as operators acting from $HF^*(M, H_-)$ to $HF^{*+deg(c)}(M, H_+)$ Here m_F^{H-} and m_F^{H+} are operators of the action of $H_*(M)$ on the Floer cohomology $HF^*(M, H_-)$ and $HF^*(M, H_+)$ respectively.

Let $\{x_1, x_2, ...\}$ and $\{y_1, y_2, ...\}$ be the bases (over $Z_{H_2(M)}$) of critical points of S_{ω,H_+} and S_{ω,H_-} respectively. Let $x = \sum_k n_k x_k$ be a cycle in $CF^*(M; H_+)$ and $y = \sum_l m_l y_l$ be a cycle in $CF_*(M; H_-)$.

Then the matrix element $< q^{d^2} y | h_H | q^{d^1} x >$ of the desired chain homotopy h_H is by definition the number of zero-dimensional components of $\mathcal{M}_H(q^{d^2} y; q^{d^1} x)$ taken with appropriate orientation. This number is non-zero only if $deg(q^{d^1} x) = deg(q^{d^2} y)$. By our convention, the difference $deg(q^{d^2} y) - deg(q^{d^1} x)$ is given by the spectral flow.

Theorem 5.5 and Lemma 5.6 imply that h_H defined above **on the chain level** is well-defined and independent of the choice of $H \in \mathcal{G}_{H_-,H_+}$ such that the statement A) holds.

Lemma 5,7 and Lemma 5.8 imply that above defined h_H is well-defined **on the homology level** such that the statements B) holds.

The statement D) above is equivalent to the fact that

$$\chi[\mathcal{M}_H(q^{d^2} y; q^{d^1} x) \bigcap ev^{-1}_{(2;0)}(\widehat{C})] = \chi[\mathcal{M}_H(q^{d^2} y; q^{d^1} x) \bigcap ev^{-1}_{(-2;0)}(\widehat{C})] \qquad (5.23)$$

The l.h.s. of (5.23) coincides with the matrix element $< q^{d^2} y | m_F^{H_+}(\widehat{C}) h_{H_-,H_+} | q^{d^1} x >$ of the l.h.s. of (5.22) because of $H(2,\theta) = H_+(\theta)$. The r.h.s. of (5.23) coincides with the matrix element $< q^{d^2} y | h_{H_-,H_+}(m_F^{H_-}(\widehat{C})) | q^{d^1} x >$ of the r.h.s. of (5.22) because $H(-2,\theta) = H_-(\theta)$. Thus, we have reduced the statement D) to the special case of the Lemma 5.6.

Actually, since the operators $m_F^{H_+}(\widehat{C})$ and $m_F^{H_-}(\widehat{C})$ are given only through their matrix elements, completely careful proof of the statement D) would require "gluing theorem C)" which we will prove later. We will repeat this argument in more details in section six.

Statement C) above is a consequence of the procedure of "gluing trajectories". A finite-dimensional Morse-theoretic version of this procedure (where there is no "bubbling-off" phenomenon) is due to Austin and Braam [AuBr]. The analysis of the effects of "bubbling-off" in the infinite-dimensional J-holomorphic curve version (relevant for our purposes) was worked out in [HS] following earlier work [F1] and [McD].

Namely, let us glue two half-cylinders $S^1 \times (-\infty; T]$ and $S^1 \times [-T; +\infty)$ along their boundaries, i.e., we glue the circle $\tau = T$ on the first half-cylinder with the circle $\tau = -T$ on the second half-cylinder. Since we have a "τ-dependent Hamiltonian" H_{12} on the first half-cylinder and a "τ-dependent Hamiltonian" H_{23} on the second half-cylinder such that

$$H_{12}(\tau;\theta) = H_1(\theta) \quad if \quad \tau \leq -1 \qquad\qquad H_{12}(\tau;\theta) = H_2(\theta) \quad if \quad \tau \geq +1$$

$$H_{23}(\tau;\theta) = H_2(\theta) \quad if \quad \tau \leq -1 \qquad\qquad H_{23}(\tau;\theta) = H_3(\theta) if au \geq +1$$

we can glue them together to obtain a new τ-dependent Hamiltonian H_{13}^T which is defined as

$$H_{13}^T(\tau;\theta) = H_{12}(\tau + T;\theta) \quad if \quad \tau \leq 0$$

$$H_{13}^T(\tau;\theta) = H_{23}(\tau - T;\theta) \quad if \quad \tau \geq 0$$

If x_1 and x_3 are any two cycles in $CF_*(M, H_1)$ and in $CF^*(M, H_3)$ respectively of relative Morse index zero then Lemma 5.6. implies that

$$< x_1 | h_{H_1,H_3} | x_3 > \quad = \quad \chi(\mathcal{M}_{H_{13}^T}(x_1; x_3)) \tag{5.24}$$

for **any** finite value of the gluing parameter T. Here χ means the Euler characteristic of the zero-dimensional oriented manifold.

Gluing Lemma 5.9. *For sufficiently large value of T the moduli space $\mathcal{M}_{H^T 13}(x_1, x_3)$ is* **diffeomorphic** *to $\bigcup_y \mathcal{M}_{H_{12}}(x_1, y) \times \mathcal{M}_{H_{12}}(y, x_3)$ where the union is taken over all critical points $\{y\}$ of H_2.*

This theorem was essentially proved by Austin-Braam and by Taubes (in a slightly more general setting of Morse theory on Banach manifolds).

The proof of this Gluing lemma (in a more general setting) will be given in the next section.

This observation implies that C) holds which proves the Theorem 5.1.

Thus, we have a well-defined map

$$m_F : H^*(M) \otimes HF^*(M) \to HF^*(M)$$

Since the Floer cohomology $HF^*(M)$ is isomorphic to the classical cohomology $H^*(M) \otimes \Lambda_\omega$, this "Floer multiplication" gives us some bilinear operation

$$m_F : H^*(M) \otimes H^*(M) \to H^*(M) \otimes \Lambda_\omega$$

in classical cohomology.

In order to calculate this bilinear operation and prove that it coincides with the quantum cup-product, we should examine more closely how the isomorphism between $HF^*(M)$ and $H^*(M) \otimes \Lambda_\omega$ is constructed. In the next section we will construct another isomorphism between $HF^*(M)$ and $H^*(M) \otimes \Lambda_\omega$ which will be used to prove our main theorem .

Chapter 6

The Proof of the Main Theorem

For each cohomology class $C \in H^*(M)$ two linear operators

$$m_Q(C) : H^*(M) \to End(H^*(M)) \otimes \Lambda_\omega \qquad (6.1)$$

and

$$m_F(C) : H^*(M) \to End(H^*(M)) \otimes \Lambda_\omega \qquad (6.2)$$

were defined in the previous three sections. The map $m_Q(C)$ was called **quantum multiplication** (from the left) on the cohomology class C . The map $m_F(C)$ was called **Floer multiplication** (from the left) on the cohomology class C.

The Main Theorem 6.1. *Quantum multiplication coincides with Floer multiplication.*

To prove that the homomorphisms (6.1) and (6.2) are in fact equal, it is sufficient to prove that all their (Λ_ω-valued) matrix elements

$$< B|m_Q(C)|A > \;=\; \sum_d q^d \;< q^d B|m_Q(C)|A > \qquad (6.3)$$

and

$$< B|m_F(C)|A > \;=\; \sum_d q^d \;< q^d B|m_F(C)|A > \qquad (6.4)$$

are the same. (Here A and B run over our once chosen homogenous basis in the total cohomology of M) .

Let H be a C^{10n}- function on $S^1 \times R \times M$ vanishing in the region $||\tau|| > 1$ (which means that $H \in \mathcal{G}_{0,0}$). We assume that H is "generic", i.e., not invariant under any holomorphic automorphism of $S^1 \times R$ (which is identified with C^* by the map (4.9)).

Following the logic of the previous section, let us consider the space of L^2-bounded trajectories $\mathcal{M}_H(\widehat{A}; q^d\eta(B))$ and compactify it as a stratified space. (Here, as in the previous section, we fix pseudo-cycle representatives of $\widehat{A}$ and $\eta(B)$).

Theorem 6.2.

$$\bar{\mathcal{M}}_H(\widehat{A}; q^d\eta(B)) \;=\; \bar{\mathcal{M}}_{J_{gradHd\bar{z}},d} \bigcap ev_0^{-1}(\widehat{A}) \bigcap ev_\infty^{-1}(\eta(B)) \tag{6.5}$$

Let $\gamma = \gamma(\tau,\theta)$ be any L^2-bounded solution of (5.5) with $H=0$ in the region $||\tau|| > 1$. Then γ (considered as a map from the cylinder $S^1 \times R$ to M) can be continiously extended from the cylinder $S^1 \times R$ to the 2-sphere S^2 since the limit value of γ at $\tau \to \pm\infty$ should be constant loops. By Theorem 3.6. of Parker and Wolfson [PW] (removable singularity theorem for J-holomorphic maps), this extension is smooth and $J_{gradHd\bar{z}}$-holomorphic.

Now the statement of the Theorem 6.2. follows directly from the definitions of the l.h.s. and the r.h.s. of (6.5).

The fact that (6.5) is an isomorphism at the level of **compactifications** (as stratified spaces) can be observed by comparing the explicit description of these compactifications that we have.

To justify our considerations, we need the following lemma:

Lemma 6.3. *"The compactification divisor" $\bar{\mathcal{M}}_H(\widehat{A}; q^d\eta(\widehat{B})) - \mathcal{M}_H(\widehat{A}; q^d\eta(\widehat{B}))$ has codimension at least two.*

Remark. *Since the Floer coboundary operator in $CF^*(M,0)$ is identically zero,* *there is no "codimension-one-boundary" (unlike in the case when the Hamiltonian H is "generic").*

Proof. *The inequalities in the lemma 2.9 impliey that "the compactification divisor" in $\bar{\mathcal{M}}_{J_{gradHd\bar{z}},d}$ has codimension at least two. Then our Lemma 6.3. will follow from the following transversality result:*

Lemma 6.4. *D_d The map $ev_0 \times ev_\infty : \bigcup_{H \in \mathcal{G}_{0,0}} \bar{\mathcal{M}}_{J_{gradHd\bar{z}},D_d} \to M \times M$ is surjective for any "degeneration pattern" D_d .*

A stronger statement about surjectivity of p-fold evaluation map (of which our lemma 6.4 is a special case for $p = 2$) was proved in [McD S]. It follows from theorems 5.3.1, 5.4.1 and 6.1.1 of [McD S].

The Theorem 6.2. means that the matrix element of quantum multiplication can be written as

$$< B|m_Q(C)|A > \quad = \quad \sum_d q^d \ \chi[\mathcal{M}_H(\hat{A}; q^d\eta(B)) \bigcap ev_{0,1}^{-1}(\hat{C})] \tag{6.6}$$

where the number in the r.h.s., as usual, is understood as intersection index.

Remark. *The r.h.s. of (6.6) can be thought as a definition of Floer multiplication operation using the Floer cochain complex $CF^*(M,0)$ (which by Lemma 4.14. has identically zero coboundary operator).*

This remark implies that in order to prove the Main Theorem, it is enough to generalize the program implemented in the previous section as follows:

A) To construct the chain homotopies $\{h_{H_{i0}}\}(i = 1;2 \ , H_{i0} \in \mathcal{G}_{H_i,0})$ from $\{CF^*(M, H_i)\}$ to $CF^*(M,0)$ and to construct the chain homotopies $\{h_{H_{0i}}\} : CF^*(M,0) \to CF^*(M, H_i)$ $(H_{0i} \in \mathcal{G}_{0,H_i})$ going in the opposite direction.

B) To generalize to the case $H = 0$ the points B),C) and D) of the program, accomplished in the previous section for H "generic". To do this, we need to prove several lemmas:

Let $\{x_1^i, x_2^i, ...\}$ $(i = 1;2)$ be the bases (over $Z_{H_2(M)}$) in the Floer cochain complexes $\{CF^*(M,H_i)\}$ corresponding to critical points of the Morse functionals S_{ω,H_i} (here the hamiltonians $\{H_i\}$ are taken to be "generic",

and let $\{A_1, A_2, ...\}$ be our fixed homogenous basis in the cohomology of M (which is also a basis (over $Z_{H_2(M)}$) in the Floer cochain complex $CF^*(M,0)$) with zero hamiltonian and with zero coboundary operator).

Then the matrix element $< q^{d^2}A|h_{H_{i0}}|q^{d^1}x^i >$ of the desired chain homotopy h_H is by definition the number of zero-dimensional components of $\mathcal{M}_{H_i}(q^{d^1}x^i; q^{d^2}A)$ taken with appropriate orientation. This number is non-zero only if $deg(q^{d^1}x^i) = deg(q^{d^2}A)$. By our convention, the difference
$deg(q^{d^2}A) - deg(q^{d^1}x^i)$ is given by the spectral flow.

Lemma 6.5. *For any two "generic" τ-dependent Hamiltonians $H_{i0}^{(0)}$ and $H_{i0}^{(1)}$ from $\mathcal{G}_{H_i,0}$ the manifolds of trajectories $\bar{\mathcal{M}}_{H_{i0}^{(0)}}(x_i, \hat{A})$ and $\bar{\mathcal{M}}_{H_{i0}^{(1)}}(x_i, \hat{A})$ are cobordant to each other as stratified spaces with boundary. Also, for any two τ-dependent Hamiltonians $H_{0i}^{(0)}$ and $H_{0i}^{(1)}$ from $\mathcal{G}_{0,H_i}$ the manifolds of trajectories $\bar{\mathcal{M}}_{H_{0i}^{(0)}}(\hat{A}, x_i)$ and $\bar{\mathcal{M}}_{H_{0i}^{(1)}}(\hat{A}, x_i)$ are cobordant to each other as stratified spaces with boundary.*
More precisely, there exists a path $\{H_{i0}^{(t)}\}$ $(0 \leq t \leq 1)$ in $\mathcal{G}_{H_{i0},0}$ joining $H_{i0}^{(0)}$ and $H_{i0}^{(1)}$ such that $\bigcup_{0\leq t\leq 1} \bar{\mathcal{M}}_{H_{i0}^{(t)}}(x_i, \hat{A})$ gives us the desired cobordism. (The same statement is also true if we interchange the indices i and 0).

Proof. *(We are going to prove existence only one of the two cobordisms. The other one is constructed the same way).*

Let us consider the space $Map(x_i, \hat{A})$ of all $W^{1,p}_{loc}$-Sobolev maps from $R \times S^1$ to M which tend (in C^1-norm) to the map $x_i : S^1 \to M$ as $\tau \to -\infty$ and to the constant loops lying in the pseudo-cycle $\hat{A}$ as $\tau \to +\infty$ Let us also consider the infinite-dimensional Banach bundle $\mathcal{H}$ over
$Map(x_i, \hat{A}) \times \mathcal{J}_+(M, \omega, K)$ where K is sufficiently large. The fiber of the bundle $\mathcal{H}$ over the point $(\gamma; J)$ in $Map(x_i, \hat{A}) \times \mathcal{J}_+(M, \omega, K)$ will be the space of all L^p-integrable $(0,1)$-forms on $R\times S^1 \bigcup +\infty = C^* \bigcup \infty$ with the coefficients in $\gamma^*(TM)$ which exponentially tend to zero as $\tau \to -\infty$ (this condition will be automatically satisfied [APS]) and which are tangent to $\hat{A}$ as $\tau \to +\infty$. The last condition means

that if we consider an evaluation map $ev_\infty : \mathcal{H}(\gamma) \to TM(\gamma(\infty))$ then $ev_\infty(\gamma)$ should lie in $T\widehat{A}(\gamma(\infty))$.

Let us consider the pull-back of this Banach bundle to $Map(x_i, \widehat{A}) \times \mathcal{J}_+(M, \omega, K) \times \mathcal{G}_{H_i,0}$ and construct a (canonical) section Φ as follows:

$$\Phi(\gamma) = \frac{\partial \gamma}{\partial \tau} - J\frac{\partial \gamma}{\partial \theta} - grad\ H(\tau, \theta)\ d\bar{z} \tag{6.7}$$

Here $d\bar{z}$ is a canonical $(0,1)$-form on $R \times S^1 = C^*$. The identification between $R \times S^1$ and C^* is given by the map (4.8).

The arguments of McDuff [McD] show that if the function H does not admit any holomorphic symmetries with respect to parameters on $R \times S^1 \bigcup +\infty$ then the section Φ is regular over $Map(x_i, \widehat{A}) \times \mathcal{J}_+(M, \omega, K) \times \{H\}$.

Since the space of the functions $\{H\}$ with this property is open and dense in $\mathcal{G}_{H_i,0}$ this means that the section Φ is regular over the dense open set in $Map(x_i, \widehat{A}) \times \mathcal{J}_+(M, \omega, K) \times \mathcal{G}_{H_i,0}$. Since the zero-set of the section Φ over $Map(x_i, \widehat{A}) \times \{J\} \times \{H\}$ is our friend $\mathcal{M}_H(x_i, \widehat{A})$ then we can apply Lemmas 2.4 and 2.6 to prove existence of the above cobordism.

Since Floer boundary operator δ acts (in general) nontrivially on x_i then the manifolds $\{\mathcal{M}_{H^{(t)}}(x_i, \widehat{A})\}$ are **the manifolds with boundary** and the cobordism $\bigcup_{0 \le t \le 1} \bar{\mathcal{M}}_{H^{(t)}}(x_i, \widehat{A})$ will have **an extra boundary component** $\bigcup_{0 \le t \le 1} \bar{\mathcal{M}}_{H^{(t)}}(\delta x_i, \widehat{A})$

This extra component appear due to the presence of intermediate critical points of S_{ω, H_i}.

The lemma 2.6 implies that such a path $\{H^{(t)}\}$ $(0 \le t \le 1)$ in $\mathcal{G}_{H_i,0}$ joining $H_{i0}^{(0)}$ and $H_{i0}^{(1)}$ exists. The extension of the Lemma 6.2. implies that the corresponding smooth cobordism(which has three boundary components) can be compactified as a stratified space and "the compactification divisor" will have a codimension of at least two inside this stratified space.

This fact together with a transversality argument of sections 6.3. and 6.5. of the book [McD S] implies that the intersection of "the compactification divisor" with (any of) three "boundary components" of the total space of our cobordism will have codimension at least two in this "boundary component" (considered as a stratified space).

The lemma 6.5. is proved.

Lemma 6.6. *For any two "generic" τ-dependent Hamiltonians $H_{i0}^{(0)}$ and $H_{i0}^{(1)}$ from $\mathcal{G}_{H_i,0}$; for any two "generic" τ-dependent Hamiltonians $H_{0i}^{(0)}$ and $H_{0i}^{(1)}$ from $\mathcal{G}_{0,H_i}$; for any cycle $x^i = \sum_k n_k x_k^i$ in $CF_*(M;H_i)$; for any pseudo-cycle $\widehat{A} \in CF^*(M;0)$ and for any pseudo-cycle $\widehat{C} \in \Omega_*(M)$ we have*

$$\mathcal{M}_{H_{i0}^{(0)}}(q^{d^1}x^i;q^{d^2}\widehat{A}) \bigcap ev_{(0;1)}^{-1}(\widehat{C}) = \mathcal{M}_{H_{i0}^{(1)}}(q^{d^1}x^i;q^{d^2}\widehat{A}) \bigcap ev_{(0;1)}^{-1}(\widehat{C}) \tag{6.8}$$

$$\mathcal{M}_{H_{0i}^{(0)}}(q^{d^1}\widehat{A};q^{d^2}x^i) \bigcap ev_{(0;1)}^{-1}(\widehat{C}) = \mathcal{M}_{H_{0i}^{(1)}}(q^{d^1}\widehat{A};q^{d^2}x^i) \bigcap ev_{(0;1)}^{-1}(\widehat{C}) \tag{6.9}$$

Proof. *(Again, we are going to prove only the formula* (6.8)*. The proof of the other formula (6.9) goes exactly the same way).*

The lemma 6.5. provides us with a cobordism $\mathcal{M}^t$ between $\mathcal{M}_{H_{i0}^{(0)}}(q^{d^1}x;q^{d^2}\widehat{A})$ and $\mathcal{M}_{H_{i0}^{(1)}}(q^{d^1}x;q^{d^2}\widehat{A})$. The fact that x is a cycle in the Floer complex $CF^*(M,H_i)$ means that the cobordism $\mathcal{M}^t$ does not have other boundary components. (All the "extra boundary components" of cobordisms $\{\bigcup_{0\leq t\leq 1} \mathcal{M}_{H^{(t)}}(q^{d^1}x_k;q^{d^2}\widehat{A})\}$ for different k will cancel each other out in pairs after we glue them together).

Comment. *The proof of "cancellation of codimension-one peaces" duplicates the proof of Lemma 5.4.*

The theorem 2.1 of [McD S] which claims that the map $ev_{(0,1)}$ from $\mathcal{M}_{H^{(t)}}(q^{d^1}x_k;q^{d^2}\widehat{A}) \times \mathcal{J}_+(M,\omega,K) \times \mathcal{G}_{H_i,0}$ to M is surjective, allows us to apply the Lemma 2.5. to the evaluation map $ev_{(0;1)}$ taken as "projection operator". By applying this lemma we have that the cobordism $\mathcal{M}^t$ intersects transversally with $ev^{-1}_{(0;1)}(\widehat{C})$ and the corresponding intersection gives us smooth one-dimensional submanifold (with boundary).

This submanifold does not intersect the "compactification divisor" $\bar{\mathcal{M}}^t - \mathcal{M}^t$ since by the Lemma 6.3. the latter has codimension ≥ 2 and we have in our hands the freedom of putting everything "in general position".

Thus, $\mathcal{M}^t \bigcap ev^{-1}_{0,1}(\widehat{C})$ gives us the desired compact one-dimensional cobordism between $\mathcal{M}_{H^{(0)}}(q^{d^1}x; q^{d^2}\widehat{A}) \bigcap ev^{-1}_{0,1}(\widehat{C})$ and $\mathcal{M}_{H^{(1)}}(q^{d^1}x; q^{d^2}\widehat{A}) \bigcap ev^{-1}_{0,1}(\widehat{C})$ The statement of the lemma 6.6. follows.

The same cobordism and transversality arguments prove the following:

Lemma 6.7. *If $\widehat{C}_0$ and $\widehat{C}_1$ be two pseudo-cycles in M homologous to each other (which implies that they are actually cobordant to each other in the category of pseudo-manifolds); if x is a cycle in $CF_*(M; H_i)$; $\widehat{A}$ is a cycle in $CF^*(M; 0)$; and (τ_0, θ_0) and (τ_1, θ_1) be any two points on the cylinder $R \times S^1$ then*

$$\mathcal{M}_{H^{(0)}_{i0}}(q^{d^1}x; q^{d^2}\widehat{A}) \bigcap ev^{-1}_{\tau_0,\theta_0}(\widehat{C}_0) = \mathcal{M}_{H^{(1)}_{i0}}(q^{d^1}x; q^{d^2}\widehat{A}) \bigcap ev^{-1}_{\tau_1,\theta_1}(\widehat{C}_1) \tag{6.10}$$

$$\mathcal{M}_{H^{(0)}_{0i}}(q^{d^1}\widehat{A}; q^{d^2}x) \bigcap ev^{-1}_{\tau_0,\theta_0}(\widehat{C}_0) = \mathcal{M}_{H^{(1)}_{0i}}(q^{d^1}\widehat{A}; q^{d^2}x) \bigcap ev^{-1}_{\tau_1,\theta_1}(\widehat{C}_1) \tag{6.11}$$

The proof of these formulas goes in three steps:

Step 1 is to change H while keeping (τ, θ) and $\widehat{C}$ fixed. This was Lemma 6.6.

Step 2 is to change $\widehat{C}$ while keeping H and (τ, θ) fixed. We have to construct a pseudo-cycle $\widehat{C}_t (0 \leq t \leq 1)$ which gives a cobordism between $\widehat{C}_0$ and $\widehat{C}_1$. Then we have to consider the one-dimensional cobordism $\mathcal{M}_H(q^{d^1}\widehat{A}; q^{d^2}x) \bigcap ev^{-1}_{\tau,\theta}(\widehat{C}_t)$ between $\mathcal{M}_H(q^{d^1}\widehat{A}; q^{d^2}x) \bigcap ev^{-1}_{\tau,\theta}(\widehat{C}_0)$ and $\mathcal{M}_H(q^{d^1}\widehat{A}; q^{d^2}x) \bigcap ev^{-1}_{\tau,\theta}(\widehat{C}_1)$

Step 3 is to change (τ, θ) while keeping $\widehat{C}$ and H fixed. We have to choose a path (τ_t, θ_t) in $S^1 \times R$ joining (τ_0, θ_0) and (τ_1, θ_1) and then consider the one-dimensional cobordism $\mathcal{M}_H(q^{d^1}\widehat{A};q^{d^2}x)\bigcap ev^{-1}_{\tau_t,\theta_t}(\widehat{C})$ between $\mathcal{M}_H(q^{d^1}\widehat{A};q^{d^2}x)\bigcap ev^{-1}_{\tau_0,\theta_0}(\widehat{C})$ and $\mathcal{M}_H(q^{d^1}\widehat{A};q^{d^2}x)\bigcap ev^{-1}_{\tau_1,\theta_1}(\widehat{C})$

Combining these three steps, we get our lemma.

If we put $(\tau_0,\theta_0)=(-2;0)$ and $(\tau_1,\theta_1)=(2;0)$ in the above lemma 6.7., we will obtain the following

Lemma 6.8.

$$< q^{d^1}\widehat{A}|h_{H_10}\circ m_F^{H_i}(\widehat{C})|q^{d^2}x > \;=\; < q^{d^1}\widehat{A}|m_F^0(\widehat{C})\circ h_{H_i,0}|q^{d^2}x > \tag{6.12}$$

and

$$< q^{d^1}x|h_{0,H_1}\circ m_F^0(\widehat{C})|q^{d^2}\widehat{A} > \;=\; < q^{d^1}x|m_F^{H_i}(\widehat{C})\circ h_{0,H_i}|q^{d^2}\widehat{A} > \tag{6.13}$$

i.e., the above-constructed chain homotopies intertwine the Floer multiplication operation.

Remark. *We will reformulate the statement that "chain homotopies intertwine the Floer multiplication" at the end of this section (formulas* (6.37) – (6.42) *).*

Now, the proof of our Main Theorem 6.1. essentially reduces to the following

Theorem 6.9.

For any pair of Hamiltonians H_1 and H_2 considered as functions on $S^1\times M$ we have

$$h_{H_1,0}\circ h_{0,H_1} = Id_{CF^*(M,0)} \tag{6.14}$$

and functoriality property

$$h_{H_1,H_2} = h_{0,H_2}\circ h_{H_1,0} \tag{6.15}$$

Proof of (6.14).

Lemma 6.10. *If we choose "the τ-dependent Hamiltonian" $H \in \mathcal{G}_{0,0}$ to be independent of the loop variable θ then*

$$\chi(\mathcal{M}_H(\widehat{A}_i;\eta(A_j))) = \widehat{A}_i \bigcap \eta(A_j)) \tag{6.16}$$

Proof of the lemma. (6.16) *is a statement of the finite-dimensional Morse-Bott theory on M, which proves the lemma.*

Lemma 6.11. *For any two pseudo-cycles $\widehat{A}_i$ and $\widehat{A}_j$ of the same dimension, the number $\chi(\mathcal{M}_H(\widehat{A}_i;\eta(A_j)))$ is independent of "the τ-dependent Hamiltonian" $H \in \mathcal{G}_{0,0}$*

The proof uses exactly the same transversality and cobordism argument as the proof of lemma 6.4. Namely, there exists a one-dimensional cobordism $\bigcup_{0\le t\le 1} \mathcal{M}_{H^{(t)}}(\widehat{A}_i;\eta(\widehat{A}_j))$ between $\mathcal{M}_{H^{(0)}}(\widehat{A}_i;\eta(A_j))$ and $\mathcal{M}_{H^{(1)}}(\widehat{A}_i;\eta(A_j))$. This cobordism has no "extra boundary components" because the Floer chain complex $CF^*(M,0)$ has zero boundary operator. The existence of this cobordism proves the lemma.

Following the strategy of the previous section, let us glue two half-cylinders $S^1 \times (-\infty;T]$ and $S^1 \times [-T;+\infty)$ along their boundaries, namely, we glue the circle $\tau = T$ on the first half-cylinder with the circle $\tau = -T$ on the second half-cylinder. Since we have a "τ-dependent Hamiltonian" H_{10} on the first half-cylinder and a "τ-dependent Hamiltonian" H_{01} on the second half-cylinder such that

$$H_{10}(\tau;\theta) = 0 \quad if \quad \tau \le -1 \qquad\qquad H_{10}(\tau;\theta) = H_1(\theta) \quad if \quad \tau \ge +1$$

$$H_{01}(\tau;\theta) = H_1(\theta) \quad if \quad \tau \le -1 \qquad\qquad H_{01}(\tau;\theta) = 0 \quad if \quad \tau \ge +1$$

then we can glue them together to obtain a new τ-dependent Hamiltonian H^T_{00} which is defined as

$$H_{00}^{T}(\tau;\theta) = H_{10}(\tau+T;\theta) \;\; if \;\; \tau \leq 0$$

$$H_{00}^{T}(\tau;\theta) = H_{01}(\tau-T;\theta) \;\; if \;\; \tau \geq 0$$

If $\{\widehat{A}_1, \widehat{A}_2, ...\}$ be our (once-chosen) basis of pseudo-cycles in M, then the lemmas 6.10 ana 6.11 imply

$$< A_j|h_{H_{00}^T}|A_i > \;\; = \;\; \delta_{ij} \tag{6.17}$$

Since (6.17) obviously holds if dimensions of pseudo-cycles $\widehat{A}_i$ and $\widehat{A}_j$ are different, and

$$< A_j|h_{H_{00}^T}|A_i > \;\; = \;\; \chi(\mathcal{M}_{H_{00}^T}(\widehat{A}_i;\eta(A_j))) = \delta_{ij} \tag{6.18}$$

for **any finite** value of the gluing parameter T (here χ means the Euler characteristics of the zero-dimensional oriented manifold) then (6.14) reduces to:

$$\chi(\mathcal{M}_{H_{00}^T}(\widehat{A}_i;\eta(A_j))) = \chi(\mathcal{M}_{H_{00}^\infty}(\widehat{A}_i;\eta(A_j))) \tag{6.19}$$

here $\mathcal{M}_{H_{00}^\infty}(\widehat{A}_i;\eta(A_j))$ is, by definition, a moduli space which parametrizes pairs of gradient flow trajectories in $\bigcup_y \mathcal{M}_{H_{10}}(\widehat{A}_i;y) \times \mathcal{M}_{H_{01}}(y;\eta(A_j))$ with a uniform bound on total energy. A priory, the set of "intermediate points" y may be arbitrary points in $\widehat{\mathcal{L}M}$, but L^2-boundedness condition ensures that y runs only through critical points of S_{ω,H_1}.

Now, we are going to to prove (6.19) (more precisely, we will prove the stronger result which will imply it), namely,

Gluing Theorem 6.12. *$\mathcal{M}_{H_{00}^\infty}(\widehat{A}_i;\eta(A_j))$ is* **diffeomorphic** *to $\mathcal{M}_{H_{00}^T}(\widehat{A}_i;\eta(A_j))$ for T finite but sufficiently large.*

Remark. *For the application to the proof of* (6.14) *we can assume that both spaces* *are zero-dimensional. But we will not make this assumption here since later we will need the general case of this gluing theorem.*

Proof of the Gluing Theorem.

The proof uses several facts from Morse theory on Banach manifolds:

Let B be a Banach manifold, H_0 be a Morse or Morse-Bott function on B, H_1 be a Morse function on B, H be a function on $B \times R$ which interpolates between the two:

$$H(\gamma, \tau) = H_0(\gamma) \;\; if \;\; \tau < -1 \tag{6.20}$$

$$H(\gamma, \tau) = H_1(\gamma) \;\; if \;\; \tau > 1 \tag{6.21}$$

here γ is a point on B , τ is the parameter on R.

For any critical point x of H_0 and y of H_1 we will consider submanifolds $U(x)$ and $S(x)$ of gradient flow trajectories of H, going "from" x and "to" x respectively, and submanifolds $U(y)$ and $S(y)$ of "unstable" (resp. "stable" gradient flow trajectories of H_1, going from (resp. to) y. Let $\mathcal{M}_H(x, y)$ be the moduli space of gradient flow trajectories of H flowing from x to y. We assume that we are in Floer-type situation, i.e., the relative indices of critical points are well-defined and all the moduli spaces $\mathcal{M}_H(x, y)$ are finite-dimensional. We also assume that the total number of critcal points (resp. submanifolds) is finite.

Theorem (Smale [Sm]). *The intersection of $U(x)$ with the neighborhood of y is locally isomorphic to $U(y) \times \mathcal{M}_H(x, y)$.*

The above-formulated theorem of Smale has its "dual" version which can be obtain from the original one by "time-reversal."

More presisely, let H_0 and H_1 are as above, x is the the critical point of H_0 , y is the the critical point of H_1, H is a real-valued function on $B \times R$ such that

$$H(\gamma,\tau) = H_1(\gamma) \ \ if \ \ \tau < -1 \tag{6.22}$$

$$H(\gamma,\tau) = H_0(\gamma) \ \ if \ \ \tau > 1 \tag{6.23}$$

and the manifolds $U(x)$ and $S(x)$ gradient flow trajectories of H_1, flowing from (resp. to) x. A "dual" version of the Smale's Theorem will be

Proposition. *The intersection of $S(x)$ with the neighborhood of y is locally isomorphic to $S(y) \times \mathcal{M}_H(y,x)$.*

Besides these Smale's result which we will not priove here, we will state another result due to Austin and Braam [AuBr] and Taubes [Ta2] and include a proof for completeness.

For any critical point y of a Morse function H_1 on B there exists its neighborhood which product of a ball of radius ϵ in $U(y)$ times a a ball of radius ϵ in $S(y)$. Let us denote S_u^ϵ and S_s^ϵ the spheres of radius ϵ bounding these balls. and let $p \in S_s^\epsilon$, $q \in S_u^\epsilon$

Theorem ([AuBr],[Ta]). *For ϵ sufficiently small, $p \in S_s^\epsilon$, $q \in S_u^\epsilon$ and real numbers $T_1 < \tau < T_2$ there is a unique solution $\gamma(\tau)$ of*

$$\frac{d\gamma}{d\tau} = gradH(\gamma(\tau),\tau) \tag{6.24}$$

such that

$$\pi^s(\gamma(T_1)) = p \quad \pi^u(\gamma(T_2)) = q \quad ||x(\tau)|| \leq 2\epsilon$$

The solution depends smoothly on the parameters p, q, T_1 and T_2 and will be denoted $x(\tau, p, q, T_1, T_2)$

Here π^s and π^u are projection operators onto "stable" (resp. "unstable") part of our neighborhood.

The proof (in this special case) is rather easy. We can express

$$\gamma(\tau) = \phi(\tau, T_1)p + \phi(\tau, T_2)q \tag{6.25}$$

where $\phi(\tau, T_1)$ is operator of the gradient flow from time T_1 to time τ and $\phi(\tau, T_2)$ is an operator of the gradient flow from the time T_2 to the time τ (backwards).

Any solution of the ODE (6.24) can be expressed in the form (6.25) since this ODE in the neighborhood of y splits as a product of ODE in B_u^ϵ and an ODE in B_s^ϵ and the operators $\phi(\tau, T_1)$ and $\phi(\tau, T_2)$ are contracting. This implies existence and uniqueness of solution of (6.24).

Now we will proceed with the proof of the gluing theorem (6.12) as follows: we will construct a map ψ_T from $\mathcal{M}_{H_{00}^\infty}(\hat{A}_i; \eta(A_j))$ to $\mathcal{M}_{H_{00}^T}(\hat{A}_i; \eta(\hat{A}_j)$ which will be proved to be a diffeomorphism for T sufficiently large.

Let $\phi \in \mathcal{M}_{H_{00}^\infty}(\hat{A}_i; \eta(A_j)$ which means that there exist:
A) a critical point y of the "perturbed symplectic action functional" S_{ω,H_1} on $\widehat{\mathcal{L}M}$,
B) functions H_{01} and H_{10} on $M \times S^1 \times R$ (τ-dependent Hamiltonials) which "connect" 0 and H_1 respectively. More presisely,

$$H_{10}(\tau;\theta) = 0 \quad if \quad \tau \leq -1 \qquad H_{10}(\tau;\theta) = H_1(\theta) \quad if \quad \tau \geq +1$$

$$H_{01}(\tau;\theta) = H_1(\theta) \quad if \quad \tau \leq -1 \qquad H_{01}(\tau;\theta) = 0 \quad if \quad \tau \geq +1$$

C) a pair $(\phi_- \in \mathcal{M}_{H_{01}}(\hat{A}_i; y)\ ;\ \phi_+ \in \mathcal{M}_{H_{10}}(y; \eta(A_j))$

Comment. *$\mathcal{M}_{H_{00}^\infty}(\hat{A}_i; \eta(A_j) = \bigcup_y \mathcal{M}_{H_{01}}(\hat{A}_i; y) \times \mathcal{M}_{H_{10}}(y; \eta(A_j))$ where the union is taken over all critical points $\{y\}$ of S_{ω,H_1} on $\widehat{\mathcal{L}M}$ such that the virtual dimensions of both $\mathcal{M}_{H_{01}}(\hat{A}_i; y)$ and $\mathcal{M}_{H_{10}}(y; \eta(A_j))$ are non-negative .*

By the theorem of Smale for ϵ sufficiently small for any $p \in S_s^\epsilon \bigcap \mathcal{M}_H(x, y)$, $q \in S_u^\epsilon$ there exist a unique solution to the ODE $\frac{d\gamma}{d\tau} = gradH(\gamma(\tau), \tau)$ such that $\gamma(T_1) = p + q/2$ for T_1 large enough (and specified by ϵ).

By the dual version of the theorem of Smale, for ϵ sufficiently small for any $p \in S_s^\epsilon$,

$q \in S^{\epsilon}_u \bigcap \mathcal{M}_H(y, x^i)$ there exist a unique solution to the ODE $\frac{d\gamma}{d\tau} = gradH(\gamma(\tau), \tau)$ such that $\gamma(-T_2) = p/2 + q$ for T_2 large enough (and specified by ϵ).

By the theorem of Austin-Braam and Taubes, there exists a unique solution $\gamma(\tau)$ to the gradient flow equation (6.24) uch that $\pi_u(\gamma(T_1 - T)) = q/2$ and $\pi_s(\gamma(T_2 - T) = p/2$

On the other hand, by our choice of T_1 and T_2, $\pi_s(\gamma(T_1 - T)) = p$ and $\pi_u(\gamma(T_2 - T) = q$

This implies that by "gluing" the solution from three pieces: the solution of (6.24) in the interval $\tau \leq (T_1 - T)$, the solution of (6.24) in the interval $(T_1 - T) \leq \tau \leq (T_2 - T)$, and the solution of (6.24) in the interval $\tau \geq (T_2 - T)$ we obtain a smooth solution to (6.24) which is uniquely specified by the pair (p, q) and hence, uniquely specified by a pair $(u, v) \in \mathcal{M}_{H^\infty}(x, x^i)$ This completes the proof of the Gluing Theorem 6.12., and also proves the gluing lemma 5.9 from the previous section and proves (6.14).

Proof of (6.15). *The Theorem 5.5. implies that for any pair of cycles* $x = \sum_k n_k x_k$ *in* $CF_*(M; H_1)$ *and* $y = \sum_l m_l y_l$ *in* $CF^*(M; H_2)$ *the number*

$$< q^{d^2} y | h_H | q^{d^1} x > \quad = \quad \chi(\mathcal{M}_H(q^{d^1} x; q^{d^2} y))$$

is independent of the choice of $H \in \mathcal{G}_{H_1, H_2}$

Following the pattern of the proof of.(6.14), we can glue two half-cylinders $S^1 \times (-\infty; T]$ and $S^1 \times [-T; +\infty)$ along their boundaries. Since we have a "τ-dependent Hamiltonian" H_{01} on the first half-cylinder and a "τ-dependent Hamiltonian" H_{20} on the second half-cylinder such that

$$H_{10}(\tau; \theta) = H_1(\theta) \;\; if \;\; \tau \leq -1 \qquad\qquad H_{10}(\tau; \theta) = 0 \;\; if \;\; \tau \geq +1$$

$$H_{02}(\tau; \theta) = 0 \;\; if \;\; \tau \leq -1 \qquad\qquad H_{02}(\tau; \theta) = H_2(\theta) \;\; if \;\; \tau \geq +1$$

then we can glue them together to obtain a new τ-dependent Hamiltonian H_{12}^T which is defined as

$$H_{12}^T(\tau;\theta) = H_{10}(\tau + T;\theta) \quad if \quad \tau \le 0$$

$$H_{12}^T(\tau;\theta) = H_{02}(\tau - T;\theta) \quad if \quad \tau \ge 0$$

If we choose a one-parameter family $\{H_{12}^T\}$ of τ-dependent Hamiltonians defined above and then take the limit $T \to +\infty$ (in the sence specified below) then we claim

Theorem 6.13.

$$< q^{d^2} y | h_{H_{12}} | q^{d^1} x > \quad = \sum_i < q^{d^2} y | h_{H_{02}} | \eta(A_i) >< \widehat{A}_i | h_{H_{10}} | q^{d^1} x > \tag{6.26}$$

which is equivalent to (6.15) and implies the Main Theorem.

Before proving (6.26) let us introduce some notation:

Let $\mathcal{M}_{cyl,H_{10}}(q^{d^1} x_k)$ be a moduli space of $J_{H_{01}d\bar{z}}$-holomorphic maps from $S^1 \times R \bigcup +\infty = CP^1 - \{0\}$ to M which tend to x_k as $\tau \to -\infty$, and let $\mathcal{M}_{cyl,H_{10}}(q^{d^1} x) \;=\; \bigcup_k n_k \mathcal{M}_{cyl,H_{10}}(q^{d^1} x_k)$.

Let $\mathcal{M}_{cyl,H_{02}}(q^{d^2} y_l)$ be a moduli space of $J_{H_{02}d\bar{z}}$-holomorphic maps from $S^1 \times R \bigcup -\infty = C$ to M which tend to $q^{d^2} y_l$ as $\tau \to +\infty$, and let $\mathcal{M}_{cyl,H_{02}}(q^{d^2} y) \;=\; \bigcup_l m_l \mathcal{M}_{cyl,H_{10}}(q^{d^2} y_l)$.

Let us define evaluation maps

$ev_\infty : \; \mathcal{M}_{cyl,H_{10}}(q^{d^1} x) \; \to M$ and $ev_0 : \; \mathcal{M}_{cyl,H_{02}}(q^{d^2} y) \; \to M$

by evaluation at $z = \infty$ and $z = 0$ respectively, and the product map

$$ev = ev_\infty \times ev_0 : \; \mathcal{M}_{cyl,H_{10}}(q^{d^1} x) \; \times \; \mathcal{M}_{cyl,H_{02}}(q^{d^1} y) \; \to M \times M$$

Then let us define

$$\mathcal{M}_{H_{12}^{\infty}}(q^{d^1}x, q^{d^2}y) = ev^{-1}(M_{diag}) \subset \mathcal{M}_{cyl,H_{10}}(q^{d^1}x) \times \mathcal{M}_{cyl,H_{02}}(q^{d^2}y)$$

Lemma 6.14.

$$\chi(\mathcal{M}_{H_{12}^{\infty}}(q^{d^1}x, q^{d^2}y)) = \sum_i < q^{d^2}y|h_{H_{02}}|\eta(A_i) >< \widehat{A}_i|h_{H_{10}}|q^{d^1}x > \tag{6.27}$$

Proof. *Take $D \subset M \times M \times [0;1]$ to be a pseudo-manifold with boundary, such that the boundary ∂D consists of two components: $M_{diag} = D\bigcap[M \times \{0\}]$ and $\bigcup_i \widehat{A}_i \times \eta(A_i) = D\bigcap[M \times \{1\}]$.*

Then $ev^{-1}(D) \subset \mathcal{M}_{cyl,H_{10}}(q^{d^1}x) \times \mathcal{M}_{cyl,H_{02}}(q^{d^2}y)\times[0;1]$ gives us one-dimensional cobordism between $\mathcal{M}_{H_{12}^{\infty}}(q^{d^1}x, q^{d^2}y)$ and $\bigcup_i \mathcal{M}_{H_{10}}(q^{d^1}x, \widehat{A}_i) \times \mathcal{M}_{H_{02}}(\eta(A_i), q^{d^2}y)$.

Since both x and y are cycles in the relevant Floer complexes, the above-constructed cobordism "does not have extra boundary components" which implies the equality of Euler characteristics of its l.h.s. and r.h.s., which is exactly (6.27). The lemma is proved.

The lemma 6.14 reduces the statement of the Theorem 6.12 (and the statement of our Main Theorem 6.1) to the following statement:

$$\chi(\mathcal{M}_{H_{12}^{\infty}}(q^{d^1}x, q^{d^2}y)) = \chi(\mathcal{M}_{H_{12}^{T}}(q^{d^1}x, q^{d^2}y)) \tag{6.28}$$

for sufficiently large (but finite) T.

To prove (6.28), we use a deep analytic result of McDuff and Salamon:

Let $\mathcal{M}_{H_{12}^{\infty},K}(q^{d^1}x, q^{d^2}y)$ be open subset in $\mathcal{M}_{H_{12}^{\infty}}(q^{d^1}x, q^{d^2}y)$ consisting of pairs of maps u, v with the C^0-bound $|du| < K$, $|dv| < K$ on the first derivative.

Theorem A.5.2 of [McD S].

For arbitrary large K the space $\mathcal{M}_{H^{\infty}_{12},K}(q^{d^1}x, q^{d^2}y)$ is **orientation-preserving-diffeomorphic** to the open set in the space $\mathcal{M}_{H^T_{12}}(q^{d^1}x, q^{d^2}y)$ for sufficiently large (but finite) T. In the case when these spaces are zero-dimensional, the spaces themselves (and not just their open subsets) are orientation-preserving-diffeomorphic.

The diffeomorphism between the two open subsets was explicitely constructed in [McD S], Appendix A.

This theorem of [McD S] implies equality of Euler characteristics, the formula (6.28) and the formula (6.15).

For convenience of the reader, we reproduce a proof of the
Theorem A.5.2 of [McD S].

The proof of these formulas goes in three steps:

Step 1 - to construct a gluing map g_T which sends each pair
$u \in \mathcal{M}_{H_{10}}(\eta(A_i); q^{d^1}x)$ and $v \in \mathcal{M}_{H_{02}}(\widehat{A}_i); q^{d^2}y)$ "an approximate H^T_{00}-gradient flow trajectory" w_T which is union of the piece of u when $\tau < T$, the piece of v when $\tau > -T$ and some extra piece (which "smooths" the ends).

Step 2 - to prove the elliptic estimates for these "approximate H^T_{00}-gradient flow trajectories" to be able to apply an implicit function theorem in Banach spaces in the following form:

Let B be a Banach manifold, $E \to B$ be a Banach vector bundle over B,
Φ be a regular Fredholm section of E, $\mathcal{M}$ be a zero-set of Φ (the moduli space in question), $D : T(B) \to E$ be a differential of the section Φ. We assume that the operator field D has a right inverse Q which has a norm uniformly bounded by a constant c. Then there exists a small number ϵ such that the preimage under Φ of the radius-ϵ-ball-bundle D_ϵ in the total space of E will lie in $\mathcal{M} \times B_{c\epsilon}$ where $B_{c\epsilon}$ is a ball in the normal bundle in B to $\mathcal{M}$ of radius $c\epsilon$. Moreover, there exists a unique well-defined projection π from $\Phi^{-1}(D_\epsilon)$ to $\mathcal{M}$

Step 3. By taking the composition of π and g_T above to obtain a desired map from $\mathcal{M}_{H^\infty}$ to $\mathcal{M}_{H^T}$.

Step 4. Prove that this map is a local diffeomorphism.

Step 5. Prove that this map is an actual diffeomorphism provided that $\mathcal{M}_{H^\infty}$ and $\mathcal{M}_{H^T}$ are zero-dimensional.

To make our notation consistent with notation of [McD S], let us make a change of variables $z = exp(\tau + i\theta)$ and put $R = exp(T)$. Using this notation, given any pair

$$(u, v) \in \mathcal{M}_{H^\infty_{12}}(q^{d^1}x, q^{d^2}y) \subset \mathcal{M}_{cyl,H_{10}}(q^{d^1}x) \times \mathcal{M}_{cyl,H_{02}}(q^{d^2}y)$$

(considered as a pair of maps: v from $CP^1 - \{0\}$ to M and u from C to M such that $u(0) = v(\infty)$), the step 1 is to construct "approximate solution" $w_R = u\sharp_R v : C^* \to M$ to the gradient flow equation which satisfies

$$w_R(z) = \begin{cases} v(R^2 z), \ \ if \ \ |z| \leq \frac{\delta}{2R}, \\ u(0) = v(\infty), \ \ if \ \ \frac{\delta}{R} \leq |z| \leq \frac{1}{\delta R}, \\ u(z), \ \ if \ \ |z| \geq \frac{2}{\delta R}. \end{cases}$$

To define the map w_R on the rest of the annulus $\frac{\delta}{2R} \leq |z| \leq \frac{2}{\delta R}$ we fix a cutoff function $\rho : C \to [0; 1]$ such that

$$\rho(z) = \begin{cases} 1, \ \ if \ \ |z| \geq 2 \\ 0. \ if \ \ |z| \leq 1 \end{cases}$$

Let us use the exponential map in the neighborhood of the intersection point $p = u(0) = v(\infty)$. Let $\xi_u(z) \in T_p(M)$ for $|z| < \varepsilon$ and $\xi_v(z) \in T_p(M)$ for $|z| > \frac{1}{\varepsilon}$ be the vector fields such that $u(z) = exp_p(\xi_u(z))$ and $v(z) = exp_p(\xi_v(z))$. Define

$$w_R(z) = exp_p(\rho(\delta R z)\xi_u(z) + \rho(\delta R/z)\xi_v(R^2 z))$$

for $\frac{\delta}{2R} \leq |z| \leq \frac{2}{\delta R}$. This is well-defined if $R > 2/\delta\varepsilon$ and consistent with the above expressions for $\frac{\delta}{2R} \leq |z| \leq \frac{2}{\delta R}$. . Moreover, the number $\varepsilon > 0$ depends on L^∞-bound on du and dv. The map w_R is not $J_{gradHd\bar{z}}$-holomorphic. However, w_R weakly converges to the pair (u, v) and, by the lemma A.3.2 of [McD S], it converges also in $W^{1,p}$-norm.

Moreover, $W^{1,p}$-convergence implies that for any $p > 2$ and $K > 0$ there exist constants $R_0 > 0$ and and $c > 0$ such that

$$||dw_R||_{0,p,R} \leq c \tag{6.29}$$

for $R \geq R_0$ and $(u, v) \in \mathcal{M}_{H_{12}^{\infty},K}(q^{d^1}x, q^{d^2}y)$

Here $||\cdot||_{0,p,R}$ means the L^p-norm on C weighted with the function $(\theta_R(z))^{-2}$ where

$$\theta_R(z) = \begin{cases} R^{-2} + R^2|z|^2, & if\ |z| \leq 1/R \\ 1 + |z|^2, & if\ |z| \geq 1/R \end{cases}$$

The purpose of introducing this norm is to make the formulas symmetric w.r.t. change of u and v.

For our future convenience let us introduce maps

$$u_R(z) = \begin{cases} w_R(z), & if\ \ |z| \geq 1/R \\ u(0), & if\ \ |z| \leq 1/R \end{cases} \quad v_R(z) = \begin{cases} w_R(z/R^2), & if\ \ |z| \leq R \\ v(\infty), & if\ \ |z| \geq R \end{cases}$$

The estimate (6.29), proved in [McD S], plays an important role in applying implicit function theorem, as discussed above (for the gluing theorem in the case of "generic" H_1) and will be discurred below (for the gluing theorem for $H_1 = 0$).

The second ingredient necessary to apply an implicit function theorem, which we use to obtain a true (and not approximate) gradient flow trajectory, is to prove existence of a uniform bound on the rigrt inverse of the $\bar{\partial}$-type operator D_{w_R} (we will denote this right inverse by Q_{w_R}). Using the estimate (6.31) below, [McD S] Constructed a linear operator $Q_R : \mathcal{H}|w_R \to W^{1,p}(w_R^*(TM))$ which satisfies the bound

$$||D_{w_R} \circ Q_R - 1|| < 1/2 \tag{6.30}$$

Unlike (6.29), this is a hard part of [McD S] proof. To obtain this uniform bound, they use several steps:

As an auxiliary tool, a special cut-off function $\beta(z) : C \to R$, which is identically equal to 1 if $|z| \leq \delta$ and is given by the formula $\beta(z) = \frac{log|z|}{log\delta}$, if $0 \leq \delta \leq 1$

The cut-off functions β (dependent on the real parameter δ introduced above) have two properties crucial to prove the desired bound for Q_{w_R}:

1) If $\delta \to 0$ then the sequence of β's tends to zero in $W^{1,2}$-norm but not in L^∞-norm.

2) If we put $\beta_\lambda(z) = \beta(\lambda z)$ then

$$||\nabla\beta_\lambda \cdot \xi||_{L^p} \leq \varepsilon||\xi||_{W^{1,p}} \tag{6.31}$$

where ε and δ are related as $\delta = exp(-2\pi/\epsilon)$

The formula (6.31) is a lemma A.1.2 in [McD S] and was carefully proven on p.167 of that book.

Now we will reproduce the proof of (6.30) and will derive the Gluing Theorem from (6.29) and (6.30).

Following notation of [McD S], let us define the Banach spaces

$$W^{1,p}_{u,v} = \{(\xi_u, \xi_v) \in W^{1,p}(u^*(TM) \times W^{1,p}(v^*(TM)|u(0) = v(\infty)\}$$

and

$$L^p_u = L^p(\Lambda^{0,1}T^*CP^1 \otimes_J u^*(TM)$$

And analogously, for L^p_v.

In our previous notations, $W^{1,p}_{u,v} = T(Map_(x,y))_{|(u,v)}$, $L^p_u = \mathcal{H}_{|u}$, $L^p_v = \mathcal{H}_{|v}$

To prove (6.30), we are going to prove that the $\bar{\partial}$-operator
$D_{u,v} : W^{1,p}_{u,v} \to L^p_u \times L^p_v$ has a uniformly bounded right inverse
$Q_{u,v} : L^p_u \times L^p_v \to W^{1,p}_{u,v}$ such that $||Q_{u,v}|| \leq c_0$ and the constant c_0 depend only on L^∞-boumd K on u and v and does not depend on the particular choice of u and v satisfying these bounds. Since the space $\mathcal{M}_{H^\infty_{12,K}}(x,y)$ is compact (if the map lies in the "compactification divisor", its derivative "blows up" at some point), it is sufficient to prove the estimate $||Q_{u,v}|| \leq c_0$ and the estimate (6.30) locally in the neighborhood

of (u,v) in $\mathcal{M}_{H^{\infty}_{12,K}}(x,y)$. If we do this, we will obtain a uniform bound on the norm of $Q_{w_R} = Q_R(D_{w_R} \circ Q_R)^{-1}$.

Following [Sa2] and [McD S], let us reproduce the proof of the uniform bound on $||Q_{u,v}||$. Since the space $\mathcal{M}_{H^{\infty}_{12,K}}(x,y)$ is compact, it is sufficient to prove that $||Q_{u,v}|| \leq c_0$ locally in the neighborhood of the point $(u,v) \in ||Q_{u,v}||$. By assumption of regularity of almost-complex structure J, the operator $D_{u,v}$ is onto.

First, let us consider the case when the index of $D_{u,v}$ is zero. Then the operator $D_{u,v}$ is invertible and by inverse function theorem its inverse $Q_{u,v}$ is bounded. Moreover, the operators $\{D_{u,v}\}$ depend continuously on (u,v) in the norm topology.

To be precise, the domains and ranges of operators $D_{u,v}$ and D_{u_1,v_1} for the nearby pairs (u,v) and (u_1,v_1) are different. To make the notion of contonuous dependence precise, we have to identify $W^{1,p}_{u,v}$ and $W^{1,p}_{u_1,v_1}$ as well as $L^p_u \times L^p_v$ and $L^p_{u_1} \times L^p_{v_1}$ by the operator $\Phi_\xi = exp_{(u,v)}(D_{u,v}(\xi)$ of parallel transport along the geodesics $\tau \to \tau\xi$ which joins (u,v) and (u_1,v_1) (here ξ is a vector field on (u,v) whose exponential $exp_{(u,v)}(\xi)$ gives (u_1,v_1)) .

Since the map $(u,v) \to D_{u,v}$ is continuous, and the map $D_{u,v} \to Q_{u,v}$ is also continuous (since the operators $\{D_{u,v}\}$ for all $\{(u,v)\}$ are invertible, then the map $(u,v) \to Q_{u,v}$ is also continuous, which proves the existence of the required bound in the index-zero case.

To handle the case of positive index, let us first consider the case when $Ind(D_{u,v}) = 2nl$, i.e., index is divisible by $2n$. Then let us fix l points $z_1, \ldots, z_l$ on u and cut down subspace $W^{1,p}_{u,v}(z_1,\ldots,z_l)$ in $W^{1,p}_{u,v}$ by imposing l conditions $\xi(z_1) = \ldots = \xi(z_l) = 0$ where $\xi \in W^{1,p}_{u,v}$. Then the rerstriction of the operator $D_{u,v}$ on the subspace $W^{1,p}_{u,v}(z_1,\ldots,z_l)$ has index zero and we can repeat the above arguments for the index-zero case to prove the uniform bound on the norm of $Q_{u,v}$.

So, we are left with the case when the index of $D_{u,v} = 2nl + k$ where $0 < k < 2n$. Then, let us fix one more point $z_0 \in u$ and consider the evaluation map $ev = ev_{z_0} \times ev_{z_1} \times \ldots ev_{z_l} : \mathcal{M}_{H^{\infty}_{12,K}}(q^{d^1}x, (q^{d^2}y) \to M^{l+1}$. Since we have chosen our almost-complex structure J to be "regular", and $dim(\mathcal{M}_{H^{\infty}_{12,K}}((q^{d^1}x, (q^{d^2}y)) = 2nl + k$, the ve can choose the points $z_0, z_1, ..., z_2$ to be "generic" such that the "evaluation map" $ev_{z_0} \times ev_{z_1} \times \ldots ev_{z_l}$ is injective. Let N be a subbundle of $T(M^{n+1})|_{ev(\mathcal{M}_{H^{\infty}_{12,K}})((q^{d^1}x,(q^{d^2}y))}$ of dimension $2n-k$, normal to the tangent bundle to $ev(\mathcal{M}_{H^{\infty}_{12,K}}((q^{d^1}x, (q^{d^2}y))$.

Then let us cut down the subspace $W^{1,p}_{u,v}(N)$ in $W^{1,p}_{u,v}$ of codimension $2nl+k$ by imposing conditions $(\xi(z_0), \xi(z_1), ..., \xi(z_l)) \in N$. Then the restriction of $D_{u,v}$ on the subspace $W^{1,p}_{u,v}(N)$ has index zero and we can repeat the above arguments for the index-zero case to prove the uniform bound on the norm of $Q_{u,v}$. This proves the required uniform bound in the remaining case.

We will construct the desired operator Q_R as follows: outside the annulus $1/R \leq |z| \leq R$ we put $Q_R = Q_{u,v}$. Inside the annulus $1/R \leq |z| \leq R$ we will define Q_R in terms of cut-off function β.

More precisely, let us define Q_R by means of a commutative diagram

$$\begin{array}{ccc} L^p_{w_R} & \xrightarrow{Q_R} & W^{1,p}_{w_R} \\ \downarrow & & \uparrow \\ L^p_u \times L^p_v & \xrightarrow{Q_{u,v}} & W^{1,p}_{u,v} \end{array}$$

It is convenient to modify the diagram slightly and replace u and v by the cylindrical-end-curves u_R and v_R defined above. Then u_R converges to u in $W^{1,p}$-norm and similiarly for v_R. Hence the operators D_{u_R,v_R} still have uniformly bounded right inverses Q_{u_R,v_R}

The following commutative diagram should give a better explanation of definition of Q_R.

$$\begin{array}{ccc} \eta & \xrightarrow{Q_R} & \xi \\ \downarrow & & \uparrow \\ (\eta_u, \eta_v) & \xrightarrow{Q_{u_R,v_R}} & (\xi_u, \xi_v) \end{array}$$

The left vertical map is given by cut-off $\eta \in L^P_{w_R}$ along the circle $|z| = 1/R$:

$$\eta_u(z) = \begin{cases} \eta_R(z), & if\ |z| \geq 1/R, \\ 0 & if\ |z| \leq 1/R \end{cases} \quad \eta_v(z) = \begin{cases} R^{-2}\eta_R(R^{-2}z), & if\ |z| \leq 1/R, \\ 0, & if\ |z| \geq 1/R \end{cases}$$

(The discontinuities in η_u and η_v do not cause problems because only their L^p-norms enter the estimates).

The lower horizontal map is given as $(\xi_u, \xi_v) = Q_{u_R, v_R}(\eta_u, \eta_v)$

In particular,

$$\xi_u(0) = \xi_v(\infty) = \xi_0 \in T_p(M)$$

The right vertical map is given in terms of cut-off function β as follows:

$$\xi(z) = \begin{cases} \xi_u(z), & if\ |z| \geq \frac{1}{\delta R}, \\ \xi_u(z) + (1 - \beta(1/Rz))(\xi_v(R^2 z) - \xi_0), & if\ \frac{1}{R} \leq |z| \leq \frac{1}{\delta R}, \\ \xi_v(R^2 z) + (1 - \beta(Rz))(\xi_u(z) - \xi_0), & if\ \frac{\delta}{R} \leq |z| \leq \frac{1}{R}, \\ \xi_v(R^2 z), & if |z| \leq \delta \leq R \end{cases}$$

So, we defined $\xi = Q_R(\eta)$. Now let us make a few remarks to clarify this definition.

Remark 1. *In the annulus $\delta/R \leq |z| \leq 1/R$ the maps u_R, v_R, w_R take the constant value p and the vector $\xi(z) \in T_p(M)$ is simply sum of the vectors $((1 - \beta(Rz))((\xi_u(z) - \xi_0)$ and $(1 - \beta(1/Rz))(\xi_v(R^2 z) - \xi_0)$.*

Remark 2. *In the first term of the sum in the previous remark, the cutoff function $1 - \beta(Rz)$ is non-zero only in the region $|z| \leq 1/R$ where $\xi_u = D_{u_R}\xi_u \equiv 0$. Similiarly, for the second term of the above sum, the cut-off function $1 - \beta(1/Rz)$ is non-zero only in the region $|z| \geq 1/R$ where $\xi_v = D_{v_R}\xi_v \equiv 0$*

Remark 3. *The formula for ξ is invariant with respect to the symmetry $u(z) \to v(1/z)$, $v(z) \to u(1/z)$, $w_R(z) \to w_R(1/R^2 z)$, $\xi_u(z) \to \xi_v(1/z)$, $\xi_v(z) \to \xi_u(1/z)$, $\xi(z) \to \xi(1/z)$*

Proof of (6.30). *We have to prove that*

$$||D_{w_R}\xi - \eta||_{0,p,R} \leq \frac{1}{2}||\eta||_{0,p,R} \tag{6.32}$$

Using the fact that $D_{w_R}\xi(z) = \eta(z)$ outside tha annulus $\frac{\delta}{R} \leq |z| \leq \frac{1}{\delta R}$, and using the symmetry of the remark 3, it is sufficient to make the estimate of the l.h.s. of (6.32) only in the annulus

$$\frac{\delta}{R} \leq |z| \leq \frac{1}{R}$$

In this region, $u_R = v_R = w_R$ is a constant map. Therefore, over this annulus $D_{u_R} = D_{v_R} = D_{w_R} = \bar{\partial}$. Furthermore, in the fact that $|z| \leq 1/R$ implies that $D_{w_R}\xi_v(R^2 z) = \eta(z)$. Hence, with the notation $\beta_R(z) = \beta(Rz)$,

$$D_{w_R}\xi - \eta = D_{u_r}((1-\beta_R)(\xi_u - \xi_0)) = (1-\beta_R)D_{u_R}(\xi_u - \xi_0) + \bar{\partial}\beta_R \otimes (\xi_u - \xi_0) =$$

$$= (\beta_R - 1)D_{u_R}(\xi_0) + \bar{\partial}\beta_R \otimes (\xi_u - \xi_0) \tag{6.33}$$

Here we have used the crucial fact that $D_{u_R}\xi_u = 0$ in the region $|z| \leq 1/R$. Now he have to estimate the norm of the 1-form $D_{w_R}\xi - \eta$ w.r.t. R-dependent metric. The next crucial point is to observe that the weighting function for 1-forms is $\theta_R(z)^{p-2}$. Since $p > 2$ and $\theta_R(z) \leq \theta_1(z) \leq 2$ in the region $|z| \leq 1/R$, it follows that $(0, p, R)$-norm of our 1-form is smaller that twice the ordinary L^p-norm. Hence we obtain the inequality

$$||D_{w_R}\xi - \eta||_{0,p,B_{1/R}} \leq 2||D_{w_R}\xi - \eta||_{L^p(B_{1/R})} \leq$$

$$\leq 2||D_{u_R}\xi_0||_{L^p(B_{1/R})} + 2||\bar{\partial}\beta_R \otimes (\xi_u - \xi_0)||_{L^p(B_{1/R})} \leq \pi R^{\frac{-2}{p}}|\xi_0| + \epsilon||\xi_u - \xi_0||_{W^{1,p}} \tag{6.34}$$

The first term in last inequality in (6.34) follows from the fact that the term $D_{u_R}\xi_0$

can be pointwise estimated by $|\xi_0|$ (since $|\xi_0|$ is a constant vector) and L^p-norm is taken over an area at most π/R^2. The second term in the last inequality in (6.34) follows from the fact that we can choose β such that (6.31) holds.

By choosing ϵ and $1/R$ to be as small as we want, we can make (6.34) less than arbitrary small constant (in particular, 1/2), which proves the required estimate.

Now let us summarize which estimates were proved so far and how to derive the Gluing Theorem A.5.2. of [McD S] from these estimates.

We constructed an embedding $g_R : \mathcal{M}_{H_{12}^{\infty}}(x,y) \to Map(x,y)$,
$(u,v) \to w_R$, $R = exp(T)$ ("approximate J-holomorphic curve") such that the image $\mathcal{M}_{H_{12}^{\tilde{T}}}(x,y)$ of g_R satisfies the following properties:
1) $\bar{\partial}(w_R) \leq cR^{-2/p}$ if $R > R_\delta$. The constants c and R_δ depend only on the pointwise bound K on du and dv. This property was stated as Lemma A.4.3 of [McD S] and proved there.

2) The operator $D_{w_R} = \bar{\partial}(w_R)$ has a right inverse $Q_{w_R} = Q_R(D_{w_R} \circ Q_R)^{-1}$ with the norm bounded by $2||Q_R|| \leq 2c_1$ (since the above-constructed operator Q_R is uniformly (in the image of g_R) bounded by the constant c_1 and (6.30) implies that $||(D_{w_R} \circ Q_R)^{-1}|| \leq 2$

Having the properties 1) and 2) of the map g_R we can apply implicit function theorem to the Banach bundle map $D : T(Map(x,y)) \to \mathcal{H}$ we obtain that for R sufficiently large image of g_R lies inside the disc bundle $\mathcal{M}_{H_{12}^T} \times B_{2c_1cR^{-2/p}}$ (which is an open neighborhood of $\mathcal{M}_{H_{12}^T}(q^{d^1}x, q^{d^2}y)$ in $(Map(q^{d^1}x, q^{d^2}y)))$. Thus, by taking the projection π from $\mathcal{M}_{H_{12}^T} \times B_{2c_1cR^{-2/p}}$ to $\mathcal{M}_{H_{12}^T}(q^{d^1}x, q^{d^2}y)$ we obtain a desired map $\pi \circ g_R$ from $\mathcal{M}_{H_{12}^{\infty}}(q^{d^1}x, q^{d^2}y)$ to $\mathcal{M}_{H_{12}^T}(q^{d^1}x, q^{d^2}y)$. To prove that this map is locally an injection and has degree one

(from which it will follow that it is a local orientation-preserving diffeomorphism since the dimensions are the same) we need another result of [McD S]

Lemma 3.3.4 of [McD S].

The map π is given by a limit of a Newton-type iteration $\xi \to \xi - Q_{w_R}\mathcal{F}(\xi)$ where the map $\mathcal{F} : W^{1,p}(w_R^*(TM)) \to L^p(\Lambda^{01}(C^*) \otimes_J w_R^*(TM))$ is defined by

$$\mathcal{F}(\xi) = \Phi_\xi(\bar{\partial}_J(exp_{w_R}(\xi))) \tag{6.35}$$

where

$$\Phi_\xi : \mathcal{H}_{exp_{w_R}(\xi)} \to \mathcal{H}_{w_R}$$

denotes the parallel transport along the geodesic $\tau \to exp_{w_R}(\tau\xi)$. The proof of convergence of Newton-type iteration follows from implicit function theorem in Banach spaces.

To prove injectivity of the limit of the above Newton-type iteration, we need

Proposition 3.3.5 of [McD S]. *Let $p > 2$ and $1/p + 1/q = 1$. For every constant $c_0 > 0$ there exists a constant $\delta > 0$ such that the following holds.*

Let $w_R : \Sigma \to M$ be a $W^{1,p}$-map and Q_{w_R} be a right inverse of D_{w_R} such that $||Q_{w_R}|| \leq c_0$, $||w_R||_{L^p} \leq c_0$ with respect to the metric on Σ such that $Vol(\Sigma) \leq c_0$. If $v_o = exp_{w_R}(\xi_0)$ and $v_1 = exp_{w_R}(\xi_1)$ are J-holomorphic curves such that $\xi_0, \xi_1 \in W^{1,p}(w_R^*(TM))$ satisfy $||\xi_0||_{W^{1,p}} \leq \delta$, $||\xi_0||_{W^{1,p}} \leq c_0$, and $||\xi_1 - \xi_0||_{L^\infty} \leq \delta$, $\xi_1 - \xi_0 \in ImQ_{w_R}$

Then $v_0 = v_1$.

Proof [McD S]: Choose $\eta = D_{w_R}\hat{\xi} \in L^p(\Lambda^{01}T^*\Sigma \otimes_J w_R^*TM)$ so that

$$\hat{\xi} = \xi_1 - \xi_0 = Q_{w_R}\eta$$

and note that $D_{w_R}\xi = \eta$. Let $\xi \to \mathcal{F}(\xi)$ be the map defined by (6.35).
Then $\mathcal{F}(\xi_0) = \mathcal{F}(\xi_1) = 0$.
Since by Taylor formula the function $\mathcal{F}$ satisfies a quadratic estimate

$$||\mathcal{F}(\xi + \hat{\xi}) - \mathcal{F}(\xi) - d\mathcal{F}(\xi)\hat{\xi}||_{L^p} \leq c_1||\hat{\xi}||_{L^\infty}||\hat{\xi}||_{W^{1,p}} \tag{6.36}$$

we can obtain

$$||\widehat{\xi}||_{W^{1,p}} = ||Q_{w_R}\eta||_{W^{1,p}} \leq c_0||\eta||_{L^p} = c_0||D_{w_R}\widehat{\xi}||_{L^p} = c_0||d\mathcal{F}(0)\widehat{\xi}||_{L^p} =$$

$$= c_0||\mathcal{F}(\xi_1) - \mathcal{F}(\xi_0) - d\mathcal{F}(0)\widehat{\xi}||_{L^p} \leq$$

$$\leq c_0||\mathcal{F}(\xi + \widehat{\xi}) - \mathcal{F}(\xi) - d\mathcal{F}(\xi_0)\widehat{\xi}||_{L^p} + c_0||(d\mathcal{F}(\xi_0) - d\mathcal{F}(0))\widehat{\xi}||_{L^p} \leq$$

$$\leq c_0c_1||\widehat{\xi}||_{L^\infty}||\widehat{\xi}||_{W^{1,p}} + c_2\delta||\widehat{\xi}||_{W^{1,p}} \leq c_3\delta||\widehat{\xi}||_{W^{1,p}}$$

To get the second inequality, we are using (6.35) to estimate the first term, and the estimate on the norm of ξ_0 to estimate the second term. To get the last inequality we are using the estimate on $||L^\infty||$-norm of $\widehat{\xi}$.
Here the constant c_3 depends only on the uniform bounds on dw_R and Q_{w_R}. By choosing δ such that $c_3\delta < 1$ we will come to contradiction to the assumption that $\widehat{\xi} \neq 0$. This proves the proposition 3.5. of [McD S] and proves that the kernel of D_{w_R} gives a good local coordinate on the region of $\mathcal{M}_{H^T_{12}}(q^{d^1}x, q^{d^2}y)$.

This finishes the proof of the first part of the Gluing Theorem of [McD S] (local orientation-preserving diffeomorphism).

To prove (6.28) in the case $\mathcal{M}_{H^T_{12}}(q^{d^1}x, q^{d^2}y)$ and $\mathcal{M}_{H^\infty_{12}}(q^{d^1}x, q^{d^2}y)$ are zero-dimensional, we need to prove that the above-constructed map $f_T = \pi \circ g_T$ is an actual orientation-preserving diffeomorphism.

Moreover, we need to prove that if the spaces $\mathcal{M}_{H^T_{12}}(q^{d^1}x, q^{d^2}y)$ and $\mathcal{M}_{H^\infty_{12}}(q^{d^1}x, q^{d^2}y)$ are of positive dimension k but their subspaces (we will denote them $\mathcal{M}_{H^T}$ and $\mathcal{M}_{H^\infty}$) are cut down from $\mathcal{M}_{H^T_{12}}(q^{d^1}x, q^{d^2}y)$ and $\mathcal{M}_{H^\infty_{12}}(q^{d^1}x, q^{d^2}y)$ by the maps $ev_{z_1}(\widehat{C}_1) \times \cdot \times ev_{z_s}(\widehat{C}_s)$ where total codimension of the pseudo-cycles $\{\widehat{C}_1, \cdot \widehat{C}_s\}$ is k then the zero-dimensional oriented manifolds $\mathcal{M}_{H^T}$ and $\mathcal{M}_{H^\infty}$ are **diffeomorphic**.

These two cases can be considered at once, since the above reproduced proof of the Gluing Theorem A.5.2 of [McD S] is "local" and depends only on the data in the "annulus" $\frac{\delta}{2R} < |z| < \frac{2}{\delta R}$.

Since both manifilds $\mathcal{M}_{H^T}$ and $\mathcal{M}_{H^\infty}$ are compact (and finite) by transversality argument, the part of Gluing theorem A.5.2 of [McD S] that we proved already, implies, that for each point $(u, v) \in \mathcal{M}_{H^\infty}$ there exists a unique point $f_T(u, v)$ in $\mathcal{M}_{H^T}$. We have to prove that for T large enough every point in $\mathcal{M}_{H^T}$ has to be of the form $f_T(u, v)$ for some $(u, v) \in \mathcal{M}_{H^\infty}$.

The proof of the proposition 3.3.5 of [McD S], reproduced here, implies that for any point $(u, v) \in \mathcal{M}_{H^\infty}$ (all these points are isolated and there is finite number of them) there exists only one element in $\mathcal{M}_{H^T}$ in sufficiently small ϵ-neighborhood of (u, v) in C^0-norm for sufficiently small ϵ (this element is precisely $f_T(u, v)$ and is obtained from (u, v) by above-constructed Newton-type iteration.

To prove the desired diffeomorphism, we have to prove that there are no elements in $\mathcal{M}_{H^T}$ **outside** this ϵ-neighborhood of (u, v) in C^0-norm for T sufficiently large.

Let us assume the opposite and then come to the contradiction.

Let us choose K large enough such that all members of $\mathcal{M}_{H^\infty}$ have energy less than $K/2$.

Suppose there exists a sequence $\{T_m \to \infty\}$ and a sequence $\{w_{T_m} \in \mathcal{M}_{H^{T_m},K}\}$ $(m \to \infty)$ such that every member of this sequence lies outside ϵ-neighborhood of $\mathcal{M}_{H^\infty,K}$ in C^0-norm for ϵ sufficiently small. Since the energy of the maps $\{w_{T_m}\}$ is uniformly bounded by K then, by a version of Gromov compactness theorem, which we are not proving here (see [HS] or [RT1] for the proof in the setting we are using) there exists a subsequence $\{T_\nu \to \infty\}$ such that the subsequence $\{w_{T_\nu} \in \mathcal{M}_{H^{T_\nu},K}\}$ $(\nu \to \infty)$ is weakly convergent. The weak limit of this subsequence might be either a "trajectory with bubbled-off J-holomorphic sphere" (which is excluded by "codimension-two versus dimension-one argument") or a "splitted trajectory" (or an element in $\mathcal{M}_{H^\infty,K}$).

In section 6 of [RT1] one can find a proof that weak convergence in the sence of Gromov implies convergence in the norms used here for the gluing theorem. The diffeomorphism follows from this.

This finishes the proof of the Gluing Theorem of [McD S].

Now let us sumarize what we have proved already and what is left to prove the Main Theorem 6.1.

We proved that for any pair of "generic" hamiltonians $\{H_i\} : S^1 \times M \to R$ and for any integer m there exist isomorphisms

$$h^m_{0,H_i} : HF^m(M,0) \to HF^m(M,H_i)$$

$$h^m_{H_i,0} : HF^m(M,H_i) \to HF^m(M,0)$$

$$h^m_{H_1,H_2} : HF^m(M,H_1) \to HF^m(M,H_2)$$

such that

$$h^m_{H_1,H_2} = h^m_{0,H_2} \circ h^m_{H_1,0} : HF^m(M,H_1) \to HF^m(M,H_2)$$

$$Id_{HF^m(M,0)} = h^m_{H_2,0} \circ h^m_{0,H_1} : HF^m(M,0) \to HF^m(M,0)$$

We also proved that for any cohomology class $C \in H^k(M)$ the Floer multiplication operators

$$m^H_F(C) : HF^m(M,H) \to HF^{m+k}(M,H)$$

$$m_Q(C) = m^0_F(C) : HF^m(M,0) \to HF^{m+k}(M,0)$$

are defined such that the following diagrams are commutative:

$$\begin{array}{ccc} HF^m(M,0) & \stackrel{m_Q(C)}{\to} & HF^{m+k}(M,0) \\ \downarrow h^m_{0H} & & \downarrow h^{m+k}_{0H} \\ HF^m(M,H) & \stackrel{m^H_F(C)}{\to} & HF^{m+k}(M,H) \end{array} \tag{6.37}$$

$$\begin{array}{ccc} HF^m(M,H) & \overset{m_F^H(C)}{\rightarrow} & HF^{m+k}(M,H) \\ \downarrow h^m_{H0} & & \downarrow h^{m+k}_{0H} \\ HF^m(M,0) & \overset{m_Q(C)}{\rightarrow} & HF^{m+k}(M,0) \end{array} \tag{6.38}$$

(The commutative diagrams (6.37); (6.38) are the same as the formulas (6.13); (6.12) respectively).

Also, we have established an isomorphism between $HF^m(M,0)$ and a piece in $H^*(M) \otimes \Lambda_\omega$ homogenous of degree m. If it will not lead to a confusion, we will denote $HF^m(M,0)$ by $H^m(M,\Lambda_\omega)$

What we have to prove is

$$m_Q(C) = h^{m+k}_{H,0} \circ m_F^H(C) \circ h^m_{0,H} \tag{6.39}$$

$$M_F^H(C) = h^{m+k}_{0,H} \circ m_Q(C) \circ h^m_{H,0} \tag{6.40}$$

i.e., that the following two diagrams are commutative

$$\begin{array}{ccc} H^m(M,\Lambda_\omega) & \overset{m_Q(C)}{\rightarrow} & H^{m+k}(M,\Lambda_\omega) \\ \downarrow h^m_{0,H} & & \uparrow h^{m+k}_{H,0} \\ HF^m(M,H) & \overset{m_F^H(C)}{\rightarrow} & HF^{m+k}(M,H) \end{array} \tag{6.41}$$

$$\begin{array}{ccc} HF^m(M,H) & \overset{m_F^H(C)}{\rightarrow} & HF^{m+k}(M,H) \\ \downarrow h^m_{H,0} & & \uparrow h^{m+k}_{0,H} \\ H^m(M,\Lambda_\omega) & \overset{m_Q(C)}{\rightarrow} & H^{m+k}(M,\Lambda_\omega) \end{array} \tag{6.42}$$

In order to prove equality of operators, we have to prove equality of their matrix elements in the bases chosen. To prove (6.39) let us do the following manipulations (using (6.13) and the Gluing theorems):

$$< B|m_Q(C)|A >=< B|m_Q(C) \circ h_{H0} \circ h_{0H}|A >=$$

$$= \sum_x < B|m_Q(C) \circ h_{H0}|x >< x|h_{0H}|A >=$$

$$= \sum_x < B|h_{H0} \circ m_Q(C)|x >< x|h_{0H}|A >=$$

by the Morse-theory-Gluing theorem 6.12 for generic H

$$= \sum_{x,y} < B|h_{H0}|y >< y|m_F^H(C)|x >< x|h_{0H}|A >$$

Which gives the r.h.s. of (6.39).

By similiar manipulations (using (6.12) and the Gluing Theorem:

$$< y|m_F^H(C)|x >=< y|h_{0H} \circ h_{H0} \circ m_F^H(C)|x >=$$

$$= \sum_B < y|h_{0H}|B >< \eta(B)|h_{H0} \circ H_F^m(C)|x >=$$

$$= \sum_B < y|h_{0H}|B >< \eta(B)|m_Q(C) \circ h_{H,0}|x >=$$

by Gluing theorem A.5.2. of [McD S]

$$= \sum_{A,B} < y|h_{0H}|B >< \eta(B)|m_Q(C)A >< \eta(A)|h_{H,0}|x >$$

which gives the r.h.s. of (6.40)

Remark. *In the manipulations above we used the formulas* (6.12) *and* (6.13) *respectively.*

So, we have proved the formulas (6.39) and (6.40) which are equivalent to our Main Theorem 6.1. The Main Theorem is proved. Before going to computation in example, let us give one more equivalent formulation of our Main Theorem:

Main Theorem. *For any "generic" Hamiltonian $H : S^1 \times M \to R$ we constructed canonical isomorhpisms*

$$h^m_{0,H_i} : H^m(M, \Lambda_\omega) \to HF^m(M, H_i)$$

$$h^m_{H_i,0} : HF^m(M, h_i) \to H^m(M, \Lambda_\omega)$$

This pair of isomorphisms intertwines operator of quantum multiplication on any comomology class $C \in H^*(M)$ with the operator of Floer multiplication on the same cohomology class, i,e, (6.39) - (6.42) hold.

Chapter 7

Floer Cohomology of complex Grassmanians

As an example of applications of our Main theorem, let us give a rigorous proof of the formula for Floer cohomology ring of the complex Grassmanian $G(k, N)$ ov k-planes in complex N-dimensional vector space V. The formula for the quantum cohomology ring $HQ^*(G(k, N))$ was conjectured long ago by Vafa [Va1]. More detailed analysis of quantum cohomology of Grassmanians was worked out by Intrilligator [I] and recently by Witten [Wi5] in relation with the Verlinde algebra. Witten also mentioned that Floer cohomology ring of the Grassmanian should be given by the same formula.

Now we need to discuss the cohomology of $G(k, N)$. We begin with the classical cohomology. Over $G(k, N)$ there is a "tautological" k-plane bundle E (whose fiber over $x \in G(k, N)$ is the k-plane in V labeled by x) and a complementary bundle F of rank $N - k$:

$$0 \to E \to V^* = C^N \to F \to 0$$

Obvious cohomology classes of $G(k, N)$ come from Chern classes. We set

$$x_i = c_i(E^*)$$

where $*$ denotes the dual. (It is conventional to use E^* rather than E, because the line bundle $\det(E^*)$ is ample.) This is practically where Chern classes come from, as $G(k, N)$ for $N \to \infty$ is the classifying space of the group $U(k)$. It is known that the x_i generate $H^*(G(k, N))$ with certain relations. The relations come naturally from the existence of the complementary bundle F in Let $y_j = c_j(F^*)$, and let
$c_t(\cdot) = 1 + tc_1(\cdot) + t^2c_2(\cdot) + \dots$. Then $H^*(G(k, N))$ is generated by the $\{x_i, y_j\}$ with relations

$$c_t(E^*)c_t(F^*) = 1 \tag{7.1}$$

Since the left hand side of (7.1) is *a priori* a polynomial in t of degree N) the classical relations are of degree $2, 4, \dots, 2N$. The first $N-k$ of these relations (uniquely) express the $\{y_j\}$ in terms of the $\{x_i\}$. This means that the classical cohomology ring of $H^*(G(k, N))$ is generated by the k generators $\{x_i\}$ with k relations of degree $2N - 2k + 2, 2N - 2k + 4, \dots, 2N$.

Let us now work out the quantum cohomology ring $HQ^*(G(k, N))$ of the Grassmannian. We can consider a subring in $HQ^*(G(k, N))$ generated by $\{x_i, y_j\}$

Conjecture (Vafa). *quantum cohomology ring $HQ^*(G(k, N))$ of the Grassmannian is generated by $\{x_i, y_j\}$ with "deformed relations"*

$$c_t(E^*)c_t(F^*) = 1 + q(-1)^{N-k}t^N \tag{7.2}$$

where q is (the unique) Kahler class in $H^2(G(k, N), Z)$.

To prove this Vafa's conjecture it is sufficient to prove that

A) $\{x_i\}$ generate the whole quantum cohomology ring

B) $\{y_j\}$ are expressed in terms of the $\{x_i\}$ by the same formulas as in the classical cohomology ring

C) The relations on $\{x_i\}$ in our quantum cohomology ring form an ideal

D) This ideal of relations is generated by k relations of degrees

$2N-2k+2, 2N-2k+4, \ldots, 2N$ coming from expansion of the l.h.s. of (7.2) in powers of t and taking coefficients of degrees $2N-2k+2, 2N-2k+4, \ldots, 2N$ **without any extra relations**

The fact that $\{y_j\}$ are expressed in terms of the $\{x_i\}$ by the same formulas as in the classical cohomology ring and the fact that these k Vafa's relations indeed take place was proved (rigorously) by Witten [Wi5] by examining the fact that

A) The classical relations of degree $2, 4, \ldots, 2N-2$ cannot deform since $deg[q] = 2N$, and

B) There is a "quantum correction" to the the "top" relation $c_k(E^*)c_{N-k}(F^*) = 0$ of degree $2N$

C) The relations on $\{x_i\}$ in our quantum cohomology ring form an ideal

D) This ideal of relations is generated by k relations of degrees $2N-2k+2, 2N-2k+4, \ldots, 2N$ coming from expansion of the l.h.s. of (7.2) in powers of t and taking coefficients of degrees $2N-2k+2, 2N-2k+4, \ldots, 2N$ **without any extra relations**

$c_k(E^*)c_{N-k}(F^*) = 0$ of degree $2N$ in the classical cohomology. This "deformed relation" has the form $c_k(E^*)c_{N-k}(F^*) = a$ for some number a which can be computed by examining degree-one rational curves in the Grassmannian. The value of this unknown number a was (rigorously) computed by Witten and was shown to be equal to $(-1)^{N-k}$.

The statement C) that the relations on $\{x_i\}$ in our quantum cohomology ring form an ideal will follow from the associativity of the quantum cohomology ring (which was proved rigorously after [Wi5] was finished).

Thus, the only things we need to prove after Witten are:

A) $\{x_i\}$ generate the whole quantum cohomology ring,

and

D) that there are no extra relations (in degree hihger than $2N$) on these generators.

The statement A) can be proved inductively by the degree **deg**. Let us assume that all the elements in $HQ^*(G(k,N))$ of degree less than m can be expressed as polynomials in $\{x_i\}$. Let us prove that this also holds for all the elements in $HQ^*(G(k,N))$ of derree m.

Let $A \in H^m(G(k,N),Z) \subset HQ^*(G(k,N))$ be some homogenous element of degree m. Then we know that in the classical cohomology ring we have

$$A = P_m(x_1,...,x_k)$$

for some polynomial P_m of degree m. The fact that $deg[q] = 2N$ is positive means that in the quantum cohomology ring we have

$$A = P_m(x_1,...,x_k) + \sum_d q^d A_d$$

for some (unknown) cohomology classes $A_d \in H^{m-2Nd}(G(k,N),Z)$ of degree $m - Nd$.

But by our induction hypothesis we know that all $\{A_d\}$ can be expressed as some polynomials in $\{x_i\}$. This simple observation proves the statement A).

To prove the last remaining statement D) let us note that the rank (over the ring $Z_{<q>}$) of the quantum cohomology of the Grassmanian $HQ^*(G(k,N))$ should be equal to the rank (over Z) of the classical cohomology $H^*(G(k,N))$.

If there were some extra relations among the generators $\{x_i\}$ this would mean that the rank (over the ring $Z_{<q>}$) of the free polynomial ring in $\{x_i\}$ moded out by the ideal generated by the coefficients of the l.h.s. of (6.2) would be strictly greater than the rank of $H^*(G(k,N))$.

But we know that any two Z-graded rings generated by k homogenous generators $\{x_1,...,x_k\}$ of degrees $\{2,4,...,2k\}$ with k homogenous relations of degrees $2N-2k+2, 2N-2k+4, \ldots, 2N$ should have the same rank.

This proves the statement D) and the Vafa's conjecture.

The arguments presented here together with the results of Ruan and Tian [RT] who proved "the handle-gluing formula" of Witten [Wi1] give a complete proof to a more refined formula of Intrilligator [I] for the certain intersection numbers (known as Gromov-Witten invariants) on the moduli space of holomorphic maps of **higher genus curves** to the Grassmanian. The proof of this formula was previously known only for the special case ($G(2, N)$) of the Grassmanians of 2-planes and is due to Bertram,Daskaloupulos and Wentworth [BDW],[Be]. Our arguments prove this formula in the full generality.

Chapter 8

Discussion

The Main Theorem 6.1, proved in the present paper can be thought as mathematical implementation of the program of Vafa [Va] of understanding quantum cohomology through geometry of the loop space. The notion of "BRST-quantization on the loop space" considered by string theorists (see [Wi1] for the best treatment), can be put in the mathematically rigorous framework of symplectic Floer cohomology.

If we are studying geometry of Kahler manifold M from the point of view of the string propagating on it, we can extract more algebrogeometrical information on M than is contained in its quantum cohomology ring $HQ^*(M)$

The String Theory on M also provides us with:

A) Deformation of the classical cohomology ring $H^*(M)$ with respect to all (and not just two-dimensional) cohomology classes,

B) Some explicitely constructed cohomology classes of the moduli spaces of punctured curves known as Gromov-Witten classes.

Kontsevich and Manin [KM], [Ko2] formulated the theory of "Gromov-Witten classes" (in this broader sense) algebraically and applied these new invariants to some classical problems in algebraic geometry. [KM] formulated the list of formal properties these "Gromov-Witten classes" should satisfy. It is still an open problem to prove these "formal properties" of [KM].

During the preparation of the present paper there has been some new developments in Floer homology.

Fukaya [Fu1],[Fu2] constructed analogues of the classical Massey products in Floer homology of Lagrangian intersections. In order to construct these "quantized Massey products", Fukaya used the loop space generalization of a finite-dimensional Morse-theoretic construction, which was not known before. Results of [Fu1],[Fu2] together with the work of Cohen-Jones-Segal [CJS2] and Betz-Cohen [BK] give a hope to understand what is "quantum homotopy type" and "Floer homotopy type" of a semi-positive almost-Kahler manifold. (See also [RT2] and [BR] for the further developements).

There is (formally) another cup-product structure in Floer cohomology, defined in [CJS] using "pair of pants". It was conjectured in [CJS],[McD S] anf [Fu1] that this cup-product structure in Floer cohomology is also equivalent to quantum cup-product. The proof of this conjecture is announced in [PSS] and in [RT2].

If our Kahler manifold M is the moduli space of flat $SU(2)$- or $SO(3)$-connections on a two-dimensional surface (which is only a stratified space and not a manifold for the $SU(2)$-case), symplectic Floer homology of this "manifold" is conjectured [A] to be isomorphic to instanton Floer homology of a circle bundle over this surface with even (resp. odd) first Chern class. See [DS],[Y],[Li] for the proof of this conjecture and [Don],[Ta2],[KrM] for further developements.

The multiplicative structure in symplectic Floer homology coresponds under this isomorphism to relative Donaldson invariants of some 4-dimensional manifolds with boundary. Thinking about these relative Donaldson invariants as some matrix elements of quantum multiplication on the moduli space of flat connections we can interpret gluing formulas [BrD] and recursion relations [KrM] for Donaldson invariants as recursion relations coming from associativity of quantum multiplication.

M.Callahan [Cal] was able to to prove using this gauge theory techniques and cup-products in Floer cohomology that there exists a symplectomorphism ϕ which is isotopic to the identity, but not symplectically is isotopic to the identity. In the example of [Cal] M is the moduli space of flat $SO(3)$-connections on a genus-two-surface, ϕ is symplectomorphism of M induced by the Dehn twist around the loop separating the two handles. The way Callahan proves that ϕ not symplectically is isotopic to the identity is that he shows that algtough Floer homology of the identity and Floer homology of ϕ are isomorphic as modules modules over the Novikov ring, the $H^*(M)$-module structure on these two Floer homologies is different. This gives the first application of our results.

Bibliography

[A] M.Atiyah, *New invariants of three- and four-dimensional manifolds.* Proc. Symp. Pure Math. **48** (1988), 38-48.

[AB] M.Atiyah, R.Bott, *The moment map and equivariant cohomology.* Topology **23** (1984), 1–28.

[AuBr] D.Austin, P.Braam, *Morse-Bott Theory and equivariant cohomology.* Preprint, 1993

[AJ] M.Atiyah, L.Jeffrey, *Topological Lagrangians and cohomology.* J. Geom. Phys. **7** (1990), 119–136.

[AM] P.Aspinwal, D.Morrison, *Topological Field Theory and Rational Curves.* Commun. Math. Phys. **151** (1993), 245–262.

[APS] M.Atiyah,V.K.Patodi, I.Singer, *Spectral assymmetry amd Riemannian Geometry.* Math. Proc. Camb. Phil. Soc. **77** (1975), 43–69 ;

[AS1] M.Atiyah, I.Singer, *The index of elliptic operators.* Ann.Math. **87** (1968), 546–604 ; **93** (1971), 119–138.

[AS] A.Astashkevich, V.Sadov, *Quantum Cohomology of Partial Flag Manifolds* $F_{n_1 \dots n_k}$. Commun. Math. Phys **170** (1995), 503-528

[BS] L.Baulieu, I.Singer ,*Topologucal sigma-models.* Commun. Math. Phys **125** (1989), 227–237

[Ba] V.Batyrev, *Quantum cohomology ring of Toric Manifolds.* alg-geom 9310004

[BDW] A.Bertram, G.Daskalopoulos, R. Wentworth, *Gromov Invariants for Holomorphic Maps from Riemann Surfaces to Grassmannians.*, alg-geom 9306005

[Be] A.Bertram, *Towards a Schubert calculus for maps from a Riemann surface to a Grassmannian.* alg-geom 9403007

[BC] M.Betz, R.Cohen, *Graph Moduli Spaces and Cohomology Operations.* Preprint, 1993.

[BR] M.Betz, J.Rade, *Products and Relations in Symplectic Floer Homology.* Preprint, 1994.

[BrD] P.Braam, S.Donaldson, *Fukaya-Floer Homology and Gluing Formulas for Polynomial Invariants.* Preprint, 1992.

[BF] J-M. Bismut, D.Freed, *Analysis of elliptic families.* Commun. Math. Phys **106** (1986), 159–176 and **107** (1986), 103–163

[BR] M.Betz and J.Rade, *Products and relations in symplectic Floer Homology.* Preprint, 1994.

[Cal] M.Callahan, PhD thesis, Oxford, in preparation.

[COGP] P.Candelas, X.de la Ossa, P.Green, L.Parkes, *A Pair of Calabi-Yau manifolds as an exactly soluble superconformal field theory.* Nucl. Phys. **359** (1991) 21–74

[CJS1] R.Cohen, J.D.S.Jones, G.Segal *Morse Theory and Classifying Spaces.* Preprint 1992.

[CJS2] R.Cohen, J.D.S.Jones, G.Segal *Floer's Infinite-Dimensional Morse Theory and Homotopy Theory.* Preprint 1993.

[CKM] H.Clemens, J.Kollar, S.Mori, *Higher dimensional complex geometry.* Asterusque **166** (1988).

[De] P.Deligne, . Letters to Yu.I.Manin and M.Kontsevich, Princeton, 1994.

[DK] S.Donaldson, P.Kronheimer, *The geometry of Four-Manifolds.* (Oxford Math. Monogr.) Oxford, Clarendon, 1990.

[Don] S.Donaldson, *Floer Homology and Algebraic Geometry.* Preprint 1994.

[DoSa] S.Dostoglou, D.Salamon, *Self-Dual Instantons and Holomorphic Curves.* Preprint, 1992

[DSWW] M.Dine, N.Seiberg, X-S.Wen, E.Witten, *Non-perturbative effects on string worldsheet.* Nucl. Phys. **278** (1986) 769–789 and **289** (1987) 319–363

[F1] A.Floer,*Symplectic fixed points and holomorphic spheres.* Commun.Math.Phys. **120** (1989), 575–611.

[F2] A.Floer, *Cuplength estimates on lagrangian intersections.* Commun. Pure. Appl. Math. **42** (1989), 335–356.

[F3] A.Floer, *Morse theory and lagrangian intersections.* J. Diff. Geom. **28** (1988), 513–547.

[F4] A.Floer, *Witten's complex and Infinite-dimensional Morse theory.* J. Diff. Geom. **30** (1989), 207–221.

[F5] A.Floer, *A relative Morse index for the symplectic action.* Commun. Pure. Appl. Math. **41** (1988), 393–407.

[F6] A.Floer, *The unfegularized gradient flow for the symplectic action.* Commun. Pure. Appl. Math. **41** (1988), 775–813.

[F7] A.Floer, *A refinement of the Conley index and the application to the stability of hyperbolic invariant sets.* Ergod. Theory and Dynam. Systems **7** (1987), 93–103.

[F8] A.Floer, *A proof of the Arnold Assumption for surfaces and generalizations to certain Kahler manifolds.* Duke Math. J. **53** (1986), 1–32.

[FH] A.Floer, H.Hofer, *Coherent orientations for periodic orbit problems in symplectic geometry.* Mathematische Zeichrift **212: 1** (1993), 13–38.

[FHS] A.Floer, H.Hofer, and D.Salamon, *Transversality in elliptic Morse theory for the symplectic action.* to appear in *Duke Math. Journal.*

[Fu1] K.Fukaya, *Morse homotopy, A_∞-category and Floer homologies.* Preprint, 1993

[Fu2] K.Fukaya, *Morse homotopy and its quantization.* Preprint, 1994

[FU] D.Freed, K.Uhlenbeck, *Instantons and 4-Manifolds.* Second Edition, New York, Springer-Verlag (1991)

[Fu] W.Fulton, *Intersection Theory.* Berlin, New York, Springer-Verlag (1984) , 470 p.

[GFP] S.Gelfand, S.Fomin and A.Postnikov, *Quantum Schubert Polynomials.* Preprint AMSPS 19605-14-08.

[GK] A.Givental, B.Kim, *Quantum Cohomology of Flag Manifolds and Toda Lattices.* hep-th 9312096.

[Gr1] M.Gromov, *Pseudo-holomorphic curves in almost complex manifolds.* Invent. Math. **82:2** (1985), 307–347.

[Gr2] M.Gromov, Private communication.

[GP] V.Guillemin, A.Pollack, *Differential Topology.* Prentice Hall, 1974.

[Ha] R.Hartshorne, *Algebraic Geometry.* Springer-Verlag, 1977.

[HS] H.Hofer, D.Salamon, *Floer homology and Novikov rings.* Preprint, 1992.

[Ho] H.Hofer, Annales de l'Institut Poincare **5:5** (1988), 495–499.

[I] K.Intrilligator *Fusion Residues.* Mod. Phys. Let. **A6** (1987), 3543–

[K] M.Kontsevich, *A^∞-algebras in mirror symmetry.* Preprint, 1993.

[K2] M.Kontsevich, *Enumeration of Rational curves via Torus actions.* hep-th 9405035

[K3] M.Kontsevich, *Unpublished.* .

[KM] M.Kontsevich, Yu.I.Manin, *Gromov-Witten Classes, Quantum Cohomology and Enumerative Geometry.* hep-th 9402147

[KrM] P.Kronheimer, T.Mrowka, *Recurrence Relations and Assymptotics for 4-manifold invariants.* Bull.AMS. **30** (1994), 215-221. *Embedded Surfaces and the Structure of Donaldson's Polynomial Invariants for 4-manifold invariants.* Preprint, 1994.

[Ku1] M.Kuranishi *New Proof for the existence of locally complete families of complex structures.* in Proceedings of the Conference on Complex Analysis (ed. A.Aeppli et. al.), Minneapolis (1964), New-York, Springer-Verlag.

[LVW] W.Lerche, C.Vafa, N.Warner *Chiral rings in N=2 Superconformal Theory.* Nucl. Phys. **B324** (1989) 427-474.

[Li] W.Li Lecture given at MIT and to appear.

[Liu] G.Liu. *On the equivalence of Quantum cohomology and Floer cohomology.* Preprint, 1995.

[McD1] D.McDuff, *Examples of symplectic structures.* Invent. Math. **89:1** (1987), 13-36 ; *Elliptic methods in symplectic geometry.* Bull.AMS. **23** (1990), 311-358.

[McD S] D.McDuff, D.Salamon, *J-holomorphic curves and Quantum Cohomology.* American Mathematical Society, University Lecture Series **6**, 1994.

[Mi] J.Milnor, *Topology from the differential viewpoint.* The Univ. Press of Virginia ,1965

[Q] D.Quillen *Determinants of Cauchy-Riemann operators over a Riemann surface.* Funct. Anal. Appl. **19** (1987), 31–34

[No] S.P.Novikov, *The Hamiltonian formalism and a multi-valued analogue of Morse Theory.* Russ. Math. Surv. **37:5** (1982), 1–56

[ONO] K.Ono, *The Arnold conjecture for weakly monotone symplectic manifolds.* to appear *Inv. Math.*

[Pid] V.Pidstrigach, *Deformations of instanton surfaces.* Math. USSR Izvestiya, **38** (1992), p.313-331.

[PSS] S.Piunikhin, D.Salamon and M.Schwarz, *Symplectic Floer-Donaldson Theory and Quantum Cohomology.* To appear ii Proceedings of Symplectic Geometry Workshop. Isaak Newton Inst., Cambridge, 1994.

[PW] T.Parker and J.Wolfson, *A compactmess theorem for Gromov's moduli space.* Michigan State Univ. Preprint, 1991.

[Ru1] Y.Ruan, *Topological sigma model and Donaldson type invariants in Gromov theory.* Preprint, 1993

[Ru2] Y.Ruan, *Symplectic Topology and Extremal Rays.* Geometric And Funct. Anal. **3:4** (1993), 395–430

[RT1] Y.Ruan, G.Tian *Mathematical theory of Quantum cohomology.* J. Diff. Geom. **42:1** (1995), 259-367

[RT2] Y.Ruan, G.Tian. *Bott-type symplectic Floer cohomology and its multiplication structures.* Math. research letters, **2:2** (1995), 203-219.

Preprint, 1994. and private communication.

[RRW] M.S.Ravi, J. Rosenthal, X.Wang, *Degree of the generalized Plücker embedding of a Quot scheme and Quantum cohomology.* alg-geom/9402011

[S] V.Sadov, *On equivalence of Floer's and quantum cohomology.* Commun. Math. Phys **173** (1995), 77-100

[Sa] D.Salamon, *Morse Theory, the Conley Index and Floer homology.* Bulletin of London Math. Soc. **22** (1990) 113 - 140

[Sa2] D.Salamon, personal communication.

[SeSi] R.Seeley, I.Singer, *Extending $\bar{\partial}$ to Singular Riemann Surfaces.* J. Geom. Phys. **5:1** (1988)

[SU] J.Sacks, K.Ulenbeck, *The existence of minimal immersions of two-spheres.* Ann. Math. **13** (1981), 1–24

[SaZ] D.Salamon, E.Zendler *Morse Theory for periodic solutions of Hamiltonian Systems and the Maslov Index.* Comm. Pure Appl. Math. **45** (1992) 1303 - 1360

[Sc1] M.Schwarz, *Morse Homology,* Birkhäuser Verlag, 1993.

[Sm] S.Smale. *On gradient Dynamical Systems,* Ann. Math., bf 74 (1961), 199-206.

[ST] B.Siebert, G.Tian, *On Quantum Cohomology Rings of Fano Manifolds and a Formula of Vafa and Intriligator.* alg-geom/9403010

[Ta] C.Taubes, *Self-Dual Connections on 4-manifolds with Indefinite Intersection Matrix.* Journal of Diff. Geom, **19:2** (1984), 517–560

[Ta2] C.Taubes, preprint.

[Ta3] C.Taubes, personal communication.

[Va1] C.Vafa, *Topological mirrors and quantum rings.* in: S.-T. Yau (*Ed.*), *Essays on mirror manifolds.* International Press Co., Hong Kong, 1992.

[Vt] C.Viterbo, *The cup-product on the Thom–Smale–Witten complex, and Floer cohomology.* To appear in: Progress in Math., v. 93, Birkhauser, Basel.

[Wi1] E.Witten, *Topologucal sigma-models.* Commun. Math. Phys **118** (1988), 353–386

[Wi2] E.Witten, *Two-dimensional gravity and intersection theory on moduli space.* Surveys in Diff. Geom. **1** (1991), 243–310.

[Wi3] E.Witten,*Mirror manifilds and topological field theories.* in: S.-T. Yau (*Ed.*), *Essays on mirror manifolds.* International Press Co., Hong Kong, 1992.

[Wi4] E.Witten, *The N-matrix Model and gauged WZW Models.* Nucl. Phys. **B371** (1992), 191–245

[Wi5] E.Witten, *The Verlinde algebra and the cohomology of the grassmanian.* hep-th 9312104

[Wi6] E.Witten, *On the structure of the topological phase of two-dimensional gravity.* Nucl. Phys. **B340** (1990) 281–332

[Wi7] E.Witten, *Supersymmetry and Morse theory.* J. Diff. Geom. 17 (1982), 661–692.

[Y] T.Yoshida, *Floer homology and Holomorphic Curves - the Atiyah Conjecture.* Preprint, 1993.