Plasma Wave Induced Chaos in a Magnetic Field

by

Felicisimo Galicia

Submitted to the Department of Physics
and the Department of Electrical Engineering and Computer Science
in Partial Fulfillment of the Requirements for the Degrees of

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Signature of Author....

Department of Electrical Engineering and Computer Science

May 9, 1996

Certified by ......

Abraham Bers
Professor of Electrical Engineering
Thesis Supervisor

Accepted by .......

June Matthews
Department of Physics Undergraduate Thesis Coordinator

Accepted by ...........

F. R. Morgenthaler
Chairman, Department Committee on Graduate Theses

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Abstract

The dynamics of a charged particle in the presence of a constant magnetic field and a perpendicularly propagating plasma wave are known to be stochastic for finite wave amplitudes. The region of phase space that is stochastic depends on the wave amplitude, plasma wave frequency and the magnetic field strength and this stochasticity can provide a mechanism for heating plasmas. The region of phase space that exhibits stochasticity has been determined for particle energies normalized to their energy at the phase velocity of the plasma wave much greater than the ratio of the plasma wave frequency to the cyclotron frequency. The results are based on theoretical analyses supplemented with numerical simulations and they are valid for both the on and off-resonance cases. The bounds on phase space are robust and extend nicely to lower energies. Furthermore, they show great improvement numerically over existing bounds.

Thesis Supervisor: Abraham Bers
Title: Professor of Electrical Engineering
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Chapter 1

Introduction

1.0 Problem Description

The classical motion of a charged particle in a constant magnetic field is a simple problem studied in freshman physics. The Lorentz force causes the particle to traverse circular orbits of a constant radius with a constant frequency. If one perturbs the system with a longitudinal electrostatic wave, the motion becomes much more complex. Within a plasma, such situations arise. Internal plasma waves can take the form of longitudinal electrostatic waves. These conditions might be found in a tokamak as well as in the earth's ionosphere.

The motion of a charged particle in a constant magnetic field in the presence of a wave can become stochastic for sufficiently large amplitudes of the perturbing plasma wave. The phase space region of stochasticity depends on the ratio of the frequency of
the wave to the cyclotron frequency as well as the particle energy normalized to its energy at the phase velocity of the wave. This stochasticity provides a method for heating the plasma. In a stochastic phase space, particles with energies near the lower energy bound of the stochastic region can diffuse to the higher energy bound of the region. In this manner, the plasma wave can impart energy and momentum to the plasma particles.

Determining which region of phase space is stochastic for a given frequency ratio and wave amplitude is a long-standing problem in plasma physics and non-linear dynamics. The case when the plasma wave propagates at an oblique angle to the magnetic field has been studied and solved by Smith and Kaufman.\(^1\) However, this solution does not extend to the case of perpendicular propagation. In perpendicular propagation, the problem separates into two different cases. The on-resonance case occurs if the frequency ratio takes the form of an integer. In this case, dynamic phase space resonances can be found in the system for finite wave amplitudes and this problem has been studied and solved by Fukuyama \textit{et al.}^2 for the special case of normalized energies much greater than the frequency ratio. The off-resonance case occurs if the frequency ratio does not take the form of an integer. In this case, for small wave amplitudes, phase space resonances do not exist. For this reason, the standard techniques for studying non-linear conservative systems are not easily applied to this system. In this region, the work of Karney\(^3,4\) is the most notable and his result is based on theoretical analyses supplemented by numerical simulations. The work of Karney is also valid only for normalized energies much greater than the frequency ratio.

In this thesis, we focus on the case of perpendicular propagation. We will also derive an upper bound in energy for stochasticity based on analytical work and numerical simulations as a function of the frequency ratio and wave amplitude. Again, this bound will be valid for normalized energies much greater than the frequency ratio. However, the bound we will derive is valid for both the on and off-resonance case and numerically, it extends nicely to all normalized energies. In addition, this upper bound will be better motivated both analytically and numerically than that of Karney.
1.1 Organization of Thesis

In Chapter 2, we introduce some of the techniques for studying the dynamics of physical systems by examining the unperturbed system, a charged particle in a constant magnetic field. In Section 2.3, we introduce some of the vocabulary associated with the phase space of a system with resonances and we also introduce the notion of a fixed point and in Section 2.4 we introduce the notion of action-angle coordinates.

In Chapter 3, we examine the case of oblique propagation. This chapter uses the techniques described in Lichtenberg and Lieberman\textsuperscript{5} to determine the stochastic regime. The Hamiltonian is derived in Section 3.1, a fixed point analysis is performed in Section 3.4, and the Chirikov condition is introduced and applied in Section 3.5.

In Chapter 4, we examine the limit of perpendicular propagation. In Section 4.0 we show that the solution of Smith and Kaufman does not apply to this case. In Section 4.1 we derive the Hamiltonian for this system and define the normalizations used in our treatment. The fixed point analysis for the general system is carried out in Section 4.2. The special case of on or near-resonance is treated briefly in Section 4.3 where we derive the resonance Hamiltonian, and in Section 4.4 where the fixed point analysis is repeated. In Section 4.5 we relate the physics of the system to the island structure in phase space and argue physically for the conditions of island formation. Finally, in Section 4.6 we discuss the onset of stochasticity in the case of perpendicular propagation.

In Chapter 5, the lower and upper bounds of Karney are derived. Section 5.1 is a derivation for the lower bound, and Section 5.3 is a derivation for the upper bound. This is followed by our derivation for the upper bound in Section 6.1. Using numerical calculations, the results of Karney's bound and that derived in Chapter 6 are compared in Section 8.1. A description of how these numerical results are obtained is found in Section 7.1.

Also of note is Appendix A. In this appendix, we use the standard methods described by Lichtenberg and Lieberman\textsuperscript{5} to treat the problem of perpendicular propagation when all harmonics of the plasma wave are included. This problem is studied with arbitrary wave amplitudes and the Chirikov condition is determined to second order. Appendix C contains a complete collection of all the numerical data compared to the
bounds of Karney and those of Chapter 6. Finally, Appendix D contains the computer code used to generate numerical data.
Chapter 2
A Charged Particle in a Constant Magnetic Field as a Linear Harmonic Oscillator

2.0 Introduction

In this chapter, we begin investigating the dynamics of the unperturbed system, \textit{i.e.} in the absence of the plasma wave. Thus, the unperturbed system consists of a charged mass in a constant magnetic field. This is a simple problem and its analysis will allow us to make clear some of the notation and conventions of this thesis as well as to introduce some general techniques for analyzing physical systems.
2.1 Deriving the Hamiltonian

We will begin by deriving the exact Hamiltonian for such a system. We start with the non-relativistic equations of motion:

\[ \frac{d\vec{r}}{dt} = \vec{v} \]  
\[ \frac{d\vec{v}}{dt} = \frac{e}{m} \vec{v} \times \vec{B} \]  
\[ \vec{B} = B_0 \hat{z} \]

where we take the magnetic field to be constant in the z-direction and (2.2) is the Lorenz equation. Rewriting these equations component by component gives:

\[ \frac{dv_x}{dt} = \frac{eB_0}{m} v_y \]  
\[ \frac{dv_y}{dt} = -\frac{eB_0}{m} v_x \]  
\[ \frac{dv_z}{dt} = 0 \]

The motion in the z-direction is trivial and we focus on the motion in the x-direction. Eliminating \( v_y \), defining the momentum \( p_x = mv_x \), and taking \( \omega_0 = \frac{eB_0}{m} \), the cyclotron frequency gives:

\[ \frac{dp_x}{dt} = -m\omega_0^2 x \]  
\[ \frac{dx}{dt} = \frac{p_x}{m} \]

These equations of motion are the familiar equations for the simple harmonic oscillator. The Hamiltonian for this system takes the form:

\[ H = \frac{p_x^2}{2m} + \frac{1}{2} m\omega_0^2 x^2 \]  

Notice that applying Hamilton's equations:

\[ \frac{dp_x}{dt} = -\frac{\partial H}{\partial x} = -m\omega_0^2 x \]
\[
\frac{dx}{dt} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \tag{2.11}
\]

\[
\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0 \tag{2.12}
\]

gives the desired equations, (2.7) and (2.8). Furthermore, (2.12) implies that \( H = E_0 \) is a constant of the motion and from (2.9), we expect the trajectories of the system in phase space to be ellipses.

### 2.2 Trajectories of the Harmonic Oscillator

For the case of the simple harmonic oscillator, it is possible to solve exactly the trajectories of the system. Combining equations (2.7) and (2.8) gives a second order differential equation:

\[
\frac{d^2x}{dt^2} = -\omega_0^2 x \tag{2.13}
\]

and the solution to this equation is:

\[
x = A \sin(\omega_0 t + \phi) \tag{2.14}
\]

This implies that:

\[
p_x = A m \omega_0 \cos(\omega_0 t + \phi) \tag{2.15}
\]

using (2.8). Furthermore, because of (2.12), the motion is restricted to a constant energy, \( E \). Hence, the constant \( A \) is defined using equation (2.9):

\[
A = \sqrt{\frac{2E}{m \omega_0^2}} \tag{2.16}
\]

Equations (2.14) and (2.15) show that at constant energy, trajectories in phase space for the harmonic oscillator are ellipses. This agrees with what we expect because the energy is conserved. Figure 2.1 demonstrates one such trajectory for an energy \( E_0 \).

### 2.3 Fixed Point Analysis

A technique that can provide insight into the dynamics of a physical system is a fixed point analysis. A fixed point is a location in phase space that does not change with time. For a two dimensional conservative system, such as those we will examine in this
Figure 2.1: The phase space trajectories of a charged particle in the x-direction of the unperturbed system for a couple of different energies. The x and p_x intercepts are defined as $A_1 = \sqrt{2E_0/m}$ and $A_2 = \sqrt[4]{\frac{2E_0}{m\omega_0^2}}$. 
thesis, there are two types of fixed points, stable fixed points and unstable fixed points. Near a stable fixed point, small perturbations keep trajectories in the same neighborhood of the fixed point. However, near an unstable fixed point, small perturbations can take trajectories very far from the original fixed point. The stable fixed points are called elliptic fixed points, and the unstable fixed points are either hyperbolic fixed points, or parabolic fixed points. In our case, we are concerned only with the elliptic and hyperbolic fixed points. These fixed points are so named because the trajectories in the neighborhood of an elliptic fixed point take the shape of ellipses, and those near a hyperbolic fixed point take the form of hyperbolas. The trajectories near an elliptic fixed point are referred to as islands, and the trajectory through a hyperbolic fixed point is called a separatrix. Figure 2.2 illustrates these definitions.

Consider a general two dimensional system:

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix}
f(x, y) \\
g(x, y)
\end{bmatrix}
\]  

(2.17)

where \( f \) and \( g \) are functions describing the time derivatives of \( x \) and \( y \). The fixed points exist at the points where the time derivative of both variables is zero. In such a case, these points do not vary with time. For the general system of (2.17), the fixed points occur when:

\[
f(x, y) = g(x, y) = 0
\]  

(2.18)

It is possible to classify the stability of the fixed points by examining the dynamics of the linearized system near these phase space points. Assume the existence of a fixed point at the origin. The linearized system then takes the form:

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = J \begin{bmatrix} x \\ y \end{bmatrix}
\]  

(2.19)

where:

\[
J = \begin{bmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{bmatrix}
\]  

(2.20)

\[J(0,0)\]
Figure 2.2: Illustration of the fixed points and their structure for a conservative system.
is the Jacobian matrix. A fixed point is stable (elliptic) if the square of the trace of $J$ is less than 4 times the determinant of $J$, and it is unstable (hyperbolic or parabolic) otherwise.

It is now possible to determine and classify the fixed points for the harmonic oscillator. The equations of motion of the harmonic oscillator are already linear, and they take the form:

$$\begin{bmatrix}
\dot{x} \\
\dot{p}_x
\end{bmatrix} = \begin{bmatrix}
0 & \frac{1}{m} \\
-m\omega_0^2 & 0
\end{bmatrix} \begin{bmatrix}
x \\
p_x
\end{bmatrix} \quad (2.21)$$

It is easy to see that the only fixed point for the harmonic oscillator occurs at the origin. Further, the trace of the Jacobian matrix is zero. This means that the origin is an elliptic fixed point. Comparing Figures 2.1 and 2.2, this makes sense.

### 2.4 Action-Angle Coordinates

Another useful method for describing the dynamics of a conservative periodic system is in terms of action-angle coordinates. Action-angle coordinates allow us to easily determine conserved quantities of the system. If it is possible to determine a canonical transformation from the original coordinates to a set of coordinates where one of the variables is conserved, the motion of the system becomes simple to analyze. For time independent systems, the energy represents one such conserved variable. However, for time dependent systems, this is not always the case. The action-angle coordinates are such a set of variables for both time dependent and time independent systems. There is a standard technique for determining the action-angle coordinates for a periodic system although this method is not always easy to carry out.7

A transformation to action-angle coordinates is only one example of a canonical transformation. Equations (2.10) and (2.11) use Hamilton's equations to determine the time derivatives of the position and momentum. These equations are only valid when the Hamiltonian of the system is written as a function of a pair of canonical coordinates. Canonical transformations are the set of transformations that preserve Hamilton's equations.7 A canonical transformation from one set of canonical variables always leads to another set of canonical variables. In general, any canonical transformation can be
represented by a generating function. Furthermore, any transformation that can be represented by a generating function is canonical.

In the case of the simple harmonic oscillator, the action-angle coordinates are easy to find. The action is defined as:

$$J = \frac{1}{2\pi} \oint pdq = \frac{1}{2\pi} \pi A_1 A_2 = \frac{E_0}{\omega_0} = \frac{H}{\omega_0}$$  \hspace{1cm} (2.22)

and the angle is:

$$\theta = \tan^{-1} \left( \frac{m\omega_0 x}{p_x} \right)$$  \hspace{1cm} (2.23)

Since the system is time independent, the action is just a multiple of the energy. These variables give a much simpler form of the Hamiltonian:

$$H = \omega_0 J$$  \hspace{1cm} (2.24)

The original coordinates in terms of the new action angle coordinates are:

$$x = \sqrt{\frac{2J}{m\omega_0}} \sin \theta$$  \hspace{1cm} (2.25)

$$p_x = \sqrt{2m\omega_0 J} \cos \theta$$  \hspace{1cm} (2.26)

From (2.12), it is clear that the action is a conserved quantity. The dynamics of the system are now simple. For a fixed energy, the trajectories in action-angle phase space are straight lines.

As we earlier stated, any canonical transformation can be represented by generating function. In this case, the generating function takes the form:

$$F = \frac{1}{2} m\omega_0 x^2 \cot \theta$$  \hspace{1cm} (2.27)

Then, the transformation is calculated as:

$$p_x = \frac{\partial F}{\partial x} = m\omega_0 x \cot \theta$$  \hspace{1cm} (2.28)

$$J = \frac{\partial F}{\partial \theta} = \frac{1}{2} m\omega_0 x^2 \csc^2 \theta$$  \hspace{1cm} (2.29)
and equation (2.28) implies that:

\[ \theta = \tan^{-1}\left( \frac{m\omega_0 x}{p_x} \right) \]  

(2.30)

so that:

\[ p_x = A m \omega_0 \cos \theta, \quad x = A \sin \theta \]  

(2.31)

and by conservation of energy, \( A \) is again defined by (2.16) so that:

\[ J = \frac{1}{2} m \omega_0 x^2 \frac{A^2}{x^2} = \frac{E}{\omega_0} = \frac{H}{\omega_0} \]  

(2.32)

It is clear that both methods give the same canonical transformation. Equations (2.30) and (2.32) are equivalent to equations (2.23) and (2.22).
Chapter 3
A Charged Particle in an Obliquely Propagating Electrostatic Wave

3.0 Introduction

In Chapter 2, a Hamiltonian for the unperturbed system of a charged particle in a constant magnetic field was derived and we studied its dynamics. In this chapter we will examine the system when we include an electrostatic plasma wave that propagates across the system at an angle less than $90^\circ$ to the magnetic field. This problem has been treated by Smith and Kaufman\(^1\) and the work here follows the techniques described by Lichtenberg and Lieberman.\(^5\) It turns out that by including a non-zero parallel wavenumber, determining bounds for stochasticity is a much easier task.
3.1 Deriving the Hamiltonian

Having gained some experience with Hamiltonians in Chapter 2, we begin this section with the general Hamiltonian in the presence of a magnetic field:

\[ H_0 = \frac{1}{2m} |\vec{p} - e\vec{A}|^2 = \frac{1}{2} m\vec{v} \cdot \vec{v} \]  

(3.1)

where \( \vec{p} = m\vec{v} + e\vec{A} \) is now the generalized momentum, \( \vec{A} = -B_0 y\hat{\imath} \) is the vector potential associated with the assumed constant magnetic field \( \vec{B} = B_0 \hat{\imath} \) and \( \vec{v} \) is the velocity. This is the unperturbed Hamiltonian and it is the same as that in equation (2.9), except that all three dimensions are included. For the case of oblique propagation, it is not possible to ignore the motion in the z direction. The generalized momentum may be calculated as:

\[ p_x = mv_x + e(-B_0 y) = m\left(v_x - \frac{eB_0}{m} y\right) = m(v_x - \omega_0 y) \]  

(3.2)

\[ p_y = mv_y \]  

(3.3)

\[ p_z = mv_z \]  

(3.4)

where as in Chapter 2, \( \omega_0 = \frac{eB_0}{m} \). The unperturbed Hamiltonian is perturbed by a plasma wave propagating at an oblique angle to the magnetic field. The plasma wave for oblique propagation takes the form:

\[ E(y, z) = -\Phi_0 k_\perp \cos(k_z z + k_y y - \omega t)\hat{\jmath} - \Phi_0 k_z \cos(k_z z + k_y y - \omega t)\hat{\imath} \]  

(3.5)

where we take the plasma wave in the y-z plane. The potential associated with this wave is:

\[ \Phi = \Phi_0 \sin(k_z z + k_y y - \omega t) \Rightarrow \vec{E} = -\vec{\nabla}\Phi \]  

(3.6)

Hence, the Hamiltonian for the perturbation takes the form:

\[ H_1 = e\Phi = e\Phi_0 \sin(k_z z + k_y y - \omega t) \]  

(3.7)

and the full system takes the form:

\[ H = H_0 + \varepsilon H_1 \]  

(3.8)

where \( \varepsilon = 1 \) is an ordering parameter to remind us that the perturbation is on a much smaller amplitude than the unperturbed system. Notice, it make sense to treat the effects
of the plasma wave as a perturbation only if the amplitude of this wave, \( \Phi_0 \ll B_0 \) the strength of the magnetic field.

### 3.2 Action-Angle Coordinates of the Unperturbed System

As we have already mentioned in Chapter 2, the dynamics of the system are easier to analyze in action-angle coordinates. In Chapter 2, we showed that the action-angle coordinates could be found using a generating function. In this section, we use a generating function equivalent to that in equation (2.27) to again perform the desired canonical transformation to action-angle coordinates of the unperturbed system. This generating function has the form:

\[
F_i = m\omega_0 \left[ \frac{1}{2} (y - Y)^2 \cot \phi - xY \right]
\]  
(3.9)

The relationship between the new and old coordinates, \((\phi, Y)\) and \((x, y)\), is then:

\[
\begin{align*}
p_x &= mv_x - eB_0 y = \frac{\partial F_i}{\partial x} = -m\omega_0 Y \\
p_y &= mv_y = \frac{\partial F_i}{\partial y} = m\omega_0 (y - Y) \cot \phi \\
P_x &= \frac{\partial F_i}{\partial y} = m\omega_0 (y - Y) \cot \phi - m\omega_0 x \\
P_\phi &= \frac{\partial F_i}{\partial \phi} = \frac{1}{2} m\omega_0 (y - Y)^2 \csc^2 \phi
\end{align*}
\]  
(3.10)-(3.13)

where the expressions for the generalized momentum have been inserted from equations (3.2)-(3.4). If we define \( v_\perp^2 = v_x^2 + v_y^2 \) and \( \rho = \frac{v_\perp}{\omega_0} \) as the perpendicular velocity and the Larmor radius respectively, then equations (3.10)-(3.13) imply that:

\[
\begin{align*}
\tan \phi &= \frac{v_x}{v_y} \Rightarrow v_x = -v_\perp \sin \phi, \quad v_y = -v_\perp \cos \phi \\
y &= Y - \rho \sin \phi \\
P_\phi &= \frac{m}{2\omega_0} v_\perp^2
\end{align*}
\]  
(3.14)-(3.16)
It is then possible to rewrite the unperturbed Hamiltonian of (3.1) as:

$$H_0 = \frac{P_z^2}{2m} + \omega_0 P_\phi \quad (3.17)$$

In terms of these coordinates, the perturbation takes the form:

$$H_1 = e\Phi \sin(k_1 z + k_2 (Y - \rho \sin \phi) - \omega t) \quad (3.18)$$

Notice that the unperturbed Hamiltonian is a function only of the momenta and it is independent of both $z$ and $\phi$. Hamilton's equations in the absence of the perturbation give:

$$\frac{\partial H_0}{\partial z} = 0 = \dot{P}_z, \quad \frac{\partial H_0}{\partial \phi} = 0 = \dot{P}_\phi \quad (3.19)$$

and this implies that both momenta are conserved quantities for the unperturbed system. This result is no different than that in Chapter 2, where this time $P_\phi$ is a measure of the perpendicular momentum which is conserved in the presence of a constant magnetic field. However, it is not true that the entire Hamiltonian is independent of $z$ and $\phi$. This is obvious from (3.18). This is the general strategy of finding conserved quantities in a perturbation scheme. We first find the conserved quantities of the unperturbed system. These quantities are not necessarily conserved for the entire system. We next examine the lowest order effects of the perturbation on trajectories of the system. We find conserved quantities of the unperturbed system plus the lowest order perturbation. These quantities are again not necessarily conserved for the entire system, so we include the next lowest order effects of the perturbation and continue. In this manner, we find conserved quantities of the system to any order. Although in this chapter it is sufficient to carry out this process only once, in Appendix A we treat a problem where second order effects must be taken into account.

If the effects of the perturbation were only significant up to a finite order, the technique we just described would lead to quantities which are conserved for all practical purposes. In such a case, the dynamics of the system are again simple. For this reason, it is most interesting to study the system at the places where this perturbation scheme breaks down. The scheme breaks down at places called resonances. At a resonance, the effects
of the perturbation at higher orders play an important role in the trajectories of the system. At a resonance, these higher order terms are amplified by denominators that go to zero.

3.3 Methods of Reducing the Hamiltonian

In this section, we try to further reduce the Hamiltonian to a simpler form. In a conserved system, perturbation theory breaks down at resonances. A common technique used when a system has a resonance is to perform an averaging so that the effects of the resonance may be isolated. This is the goal of this section. We begin this section by performing a canonical transformation from the old coordinates, \((z, P_z)\), to the new coordinates, \((\psi, P_{\psi})\). This transformation will simplify the argument of the sinusoid in (3.18) and it is generated by:

\[
F_2 = (k_z z + k_\perp Y - \omega t)P_{\psi}
\]

(3.20)

Notice that \(Y\) is a constant since its conjugate momentum does not appear anywhere in the entire Hamiltonian. The transformation generated is:

\[
P_z = \frac{\partial F_2}{\partial z} = k_z P_{\psi}
\]

(3.21)

\[
\psi = \frac{\partial F_2}{\partial P_{\psi}} = k_z z + k_\perp Y - \omega t
\]

(3.22)

In the new coordinates, the unperturbed Hamiltonian takes the form:

\[
\tilde{H}_0 = H_0 + \frac{\partial F_2}{\partial t} = \frac{k_z^2 P_{\psi}^2}{2m} - \omega P_{\psi} + \omega_0 P_{\phi}
\]

(3.23)

and the perturbed Hamiltonian:

\[
\tilde{H}_1 = e\Phi_0 \sin(\psi - k_\perp \rho \sin \phi)
\]

(3.24)

At this point, we decompose the perturbation into harmonics of the angle coordinate. In this way, it is possible to isolate resonance terms. This is accomplished by invoking the Bessel function identity:

\[
\exp i(a + b \sin c) = \sum_{n=-\infty}^{\infty} J_n(b) \exp i(nc + a)
\]

(3.25)
where $J_n$ is the $n^{th}$ order Bessel function of the first kind. Applying this identity to (3.24) gives:

$$
\tilde{H}_1 = e\Phi_0 \sum_n J_\sigma(k_z \rho) \sin(\psi - n\phi)
$$

It is clear from this formulation that a resonance occurs at stationary phase for some $n = n_0$ in the perturbation. 

$$
\frac{d(\psi - n_0\phi)}{dt} = \psi - n_0\dot{\phi} = 0
$$

The resonance condition may be approximated to lowest order in $\varepsilon$ by examining the frequencies of the unperturbed system. These are found using Hamilton's equations:

$$
\omega_\phi = \dot{\phi} = \frac{\partial H_0}{\partial P_\phi} = \omega_0
$$

$$
\omega_\psi = \dot{\psi} = \frac{\partial H_0}{\partial P_\psi} = \frac{k_z^2}{m} P_\psi - \omega
$$

The resonance condition may be restated as:

$$
P_\psi = \frac{m}{k_z^2} (\omega + n_0\omega_0)
$$

where $n_0$ is any integer. As we have already mentioned, we want to try to isolate the resonance. We will isolate a resonance at $n_0 = l$. To do this we perform another canonical transformation into a set of coordinates where one of the frequencies is approximately zero at a resonance. We can then perform a time average and isolate the resonance. We use the generating function:

$$
F_2 = (\psi - l\phi)\tilde{P}_\psi + \phi \tilde{P}_\psi
$$

where $l$ is some integer to change variables to the tilde frame. The only two variables effected by this transformation are:

$$
P_\phi = \frac{\partial F_2}{\partial \phi} = \tilde{P}_\phi - l\tilde{P}_\psi, \quad \tilde{\psi} = \frac{\partial F_2}{\partial \tilde{P}_\psi} = \psi - l\phi
$$

Notice, however, that near a resonance:

$$
\dot{\psi} = \psi - l\phi = 0
$$
This is the reason the transformation was performed. The Hamiltonian in the new coordinates takes the form:

\[
\tilde{H}_0 = \frac{k_z^2}{2m} \tilde{P}_\psi^2 + \omega_0 (\tilde{P}_\phi - l \tilde{P}_\psi) - \omega \tilde{P}_\psi \tag{3.34}
\]

\[
\tilde{H}_1 = e \Phi_0 \sum_n J_n(k_\perp \rho) \sin(\tilde{\psi} - (n - l) \tilde{\phi}) \tag{3.35}
\]

A time average of (3.35) would leave only the terms in the summation near stationary phase. Because of (3.33), it is reasonable to assume that \( \dot{\psi} << \dot{\phi} \) so that only the term with \( n = l \) would survive such an average. Furthermore, Hamilton's equations to lowest order in \( \epsilon \) imply that both momenta are conserved quantities and hence almost independent of time. Hence, time averaging the system would isolate the phase space resonance. It isolates the term with stationary phase. The averaged system takes the form:

\[
\bar{H}_0 = \frac{1}{2\pi} \int_0^{2\pi} H_0 d\tau = \frac{k_z^2}{2m} \tilde{P}_\psi^2 + \omega_0 (\tilde{P}_\phi - l \tilde{P}_\psi) - \omega \tilde{P}_\psi \tag{3.36}
\]

where the time dependence of the momenta is ignored. This is the same as (3.34). However, the time averaged perturbation takes the form:

\[
\bar{H}_1 = e \Phi_0 \sum_n \frac{1}{2\pi} \int_0^{2\pi} J_n(k_\perp \rho) \sin(\tilde{\psi} - (n - l) \tilde{\phi}) d\tau = e \Phi_0 J_l(k_\perp \rho) \sin \tilde{\psi} \tag{3.37}
\]

We have thus successfully reduced the Hamiltonian. Notice that this reduction can be performed whenever a resonance exists. Near a resonance, the terms that are averaged out of the perturbation act at a higher order on the system. The complete Hamiltonian takes the form:

\[
\tilde{H} = \bar{H}_0 + e \bar{H}_1 + \epsilon^2 \tilde{H}_2 \tag{3.38}
\]

where:

\[
\tilde{H}_2 = e \Phi_0 \sum_{n \neq l} J_n(k_\perp \rho) \sin(\tilde{\psi} - (n - l) \tilde{\phi}) \tag{3.39}
\]

and the \( \epsilon^2 \) reminds us that this is a higher order effect.
3.4 Fixed Point Analysis

We will now perform a fixed point analysis on the reduced Hamiltonian described by (3.36) and (3.37). The fixed points occur when the time derivatives of all variables are simultaneously zero. The time derivatives are calculated using Hamilton's equations:

\[ \dot{P}_\psi = 0 = \varepsilon e \Phi_0 J_i (k_\perp \rho) \cos \tilde{\psi} \]  
(3.40)

\[ \dot{\psi} = 0 = \frac{k_\parallel^2}{2m} \tilde{P}_\psi - l\omega_o - \omega + \varepsilon e \Phi_0 \sin \tilde{\psi} \frac{\partial J_i (k_\perp \rho)}{\partial \tilde{P}_\psi} \]  
(3.41)

\[ \dot{\phi} = 0 = \omega_o + \varepsilon e \Phi_0 \sin \tilde{\psi} \frac{\partial J_i (k_\perp \rho)}{\partial \tilde{P}_\phi} \]  
(3.42)

where equation (3.42) is satisfied always so that \( \tilde{P}_\phi = \tilde{P}_{\phi,0} \) is some constant and in terms of the tilde variables:

\[ \rho = \sqrt{\left( \frac{2}{m\omega_o} \right) (\tilde{P}_\phi - \tilde{P}_\psi)} \]  
(3.44)

using (3.16) and (3.32). The interesting fixed points are the elliptic points and these occur when:

\[ \tilde{\psi} = \tilde{\psi}_o = \pm \frac{\pi}{2} \]  
(3.45)

and,

\[ \frac{k_\parallel^2}{m} \tilde{P}_\psi - l\omega_o - \omega = \pm \varepsilon e \Phi_0 \frac{\partial J_i (k_\perp \rho)}{\partial \tilde{P}_\psi} \]  
(3.46)

The solutions to (3.46) are the values of \( \tilde{P}_\psi \) at the fixed points which we denote as \( \{ \tilde{P}_{\psi,0} \} \). For simplicity, we perform a translation on the angle so that there is a fixed point at \( \tilde{\psi} = 0 \):

\[ \tilde{\psi} \rightarrow \tilde{\psi} + \frac{\pi}{2} \]  
(3.47)
The fixed points in (3.45) are now:
\[ \Psi = \Psi_0 = 0, \pi \]  
(3.48)
Also, although the unperturbed Hamiltonian is unaffected by this translation, the averaged perturbation now takes the form:
\[ \overline{H}_1 = e^{i\Phi_0}J_1(k_1p)\cos\Psi \]  
(3.49)
We will not proceed any further with the fixed point analysis.

3.5 The Chirikov Condition for Stochasticity

In Section 3.2, we explained that the dynamics of the system are interesting near resonances. If we are far away from a resonance, it is easy to find conserved quantities and the motion cannot become stochastic. The motion near a resonance is that of a nonlinear oscillator. Hence, trajectories near the elliptic fixed points are well-behaved and deterministic. This is the island structure. It is this motion that dominates the system near a resonance. However, each island structure has a finite width. The motion becomes complex when neighboring islands overlap. Particles can then jump from island to island and the motion is no longer predictable. Trajectories begin to stochastically fill areas of phase space. This concept is captured by the Chirikov condition. Islands in phase space are separated in both action and angle. If the width in action or angle due to the perturbation becomes larger than the separation of the islands, the Chirikov condition is satisfied and the system becomes stochastic. In this section, we are concerned with the Chirikov condition in action; this is the standard application. Figure 3.1 illustrates the notion of the Chirikov condition.

At this point, we are ready to develop a stochastic threshold. We begin by expanding the unperturbed Hamiltonian of (3.36) about the elliptic fixed points, \( \{ \tilde{P}_{\psi,0}, \tilde{\psi} \} \):
\[ \overline{H}_0(\Delta \tilde{P}_\psi) = A_0(\tilde{P}_{\psi,0}, \tilde{P}_{\phi,0}) + C_0(\tilde{P}_{\psi,0})\Delta \tilde{P}_\psi + G(\Delta \tilde{P}_\psi)^2 \]  
(3.50)
Figure 3.1: The definitions of island separation and island width as well as the notion of island overlap. In a), the island separation is greater than the width so that island overlap doesn’t occur. In b), the island separation is less than the width and island overlap occurs. In this case, the Chirikov condition is satisfied.
where \( \tilde{P}_\psi = \tilde{P}_{\psi,0} + \Delta \tilde{P}_\psi \) and the constants are defined as:

\[
\tilde{H}_0(\tilde{P}_{\psi,0}) = A_0 = \frac{k_z^2}{2m} \tilde{P}_{\psi,0}^2 + \omega_0 (\tilde{P}_{\psi,0} - l\tilde{P}_{\phi,0}) - \tilde{P}_{\psi,0} \tag{3.51}
\]

\[
\frac{\partial \tilde{H}_0}{\partial \tilde{P}_{\psi,0}}|_{\tilde{P}_{\psi,0}} = C_0 = \frac{k_z^2}{m} \tilde{P}_{\psi,0} - l\omega_0 - \omega \tag{3.52}
\]

\[
\frac{\partial^2 \tilde{H}_0}{\partial \tilde{P}_{\psi,0}^2} = G = \frac{k_z^2}{m} \tag{3.53}
\]

If we recall the fixed point condition in (3.46), the constant \( C_0 \) is small:

\[
C_0 = \pm \varepsilon \Phi_0 \frac{\partial J_i(k_{\perp} \rho)}{\partial \tilde{P}_\psi} \bigg|_{\tilde{P}_{\psi,0}} = O(\varepsilon) \tag{3.54}
\]

so that its contribution to the Hamiltonian may be ignored. Further, the contribution due to \( A_0 \) is always constant and can be removed by a translation on \( \tilde{P}_{\phi,0} \). Hence, the remaining unperturbed Hamiltonian takes the form:

\[
\tilde{H}_0 = G(\Delta \tilde{P}_\psi)^2 \tag{3.55}
\]

We also rewrite the perturbation as:

\[
\tilde{H}_1 = -F \sin \tilde{\psi} \tag{3.56}
\]

where,

\[
F = -e\Phi_0 J_i(k_{\perp} \rho) \tag{3.57}
\]

and remind ourselves that \( \varepsilon = 1 \) was an ordering parameter, and then the entire system may be described as:

\[
\tilde{H} = G(\Delta \tilde{P}_\psi)^2 - F \sin \tilde{\psi} \tag{3.58}
\]

This is the Hamiltonian for the non-linear pendulum, a carefully studied problem. We can now quote the results for the island widths in action:\(^5,^{10}\)

\[
\Delta \tilde{P}_\psi = 4 \sqrt{\frac{F}{G}} = \frac{4}{k_z} \sqrt{m e \Phi_0 J_i(k_{\perp} \rho)} \tag{3.59}
\]
and to first order, the separation between islands is determined by (3.30) so that:

$$\delta P_v = \frac{m\omega_0}{k_z^2}$$  \hspace{1cm} (3.60)

If we take $S$ as the ratio of the island widths to the island separation, the system becomes stochastic when $S > 1$ where:

$$S = \frac{\Delta P_v}{\delta P_v} = \frac{4k_z}{\omega_0} \sqrt{\frac{e\Phi_0 f_1(k_1 \rho)}{m}}$$  \hspace{1cm} (3.61)

The study of oblique propagation is now complete. Notice that $S$ depends on the normalized Larmor radius, the strength of the electric field and the parallel wave number, $k_z$. The limit of $k_z$ goes to zero will be discussed in the next chapter.
Chapter 4

A Charged Particle in a Perpendicularly Propagating Electrostatic Wave

4.0 Introduction

In Chapter 3, we determined the stochasticity parameter for the case of oblique propagation. The techniques applied were standard methods for handling dynamic phase space resonances. The case of perpendicular propagation is more difficult to analyze. For perpendicular propagation, phase space resonances do not exist for arbitrary values of the non-linearity parameter and frequency ratio. For this reason, the techniques described in Chapter 3 are not easy to apply to this system. In Appendix A, we examine another system that always contains phase space resonances, a system with perpendicular propagation and harmonics of the plasma wave. In this chapter, we study the case of perpendicular propagation with only one plasma wave. It would be convenient if it were
possible to use the stochasticity parameter of Chapter 3 in the limit of perpendicular propagation. Recall equation (3.61):

\[ S = \frac{4k}{\omega_0} \sqrt{\frac{e\Phi_0 J_0(k_1 \rho)}{m}} \]  

(4.1)

In the limit when \( k_z \) goes to zero, the stochasticity parameter goes to zero linearly. This means that the techniques of Chapter 3 predict that under perpendicular propagation, the system never becomes stochastic. However, this is not the case. In the following sections, we begin to analyze the case of perpendicular propagation.

### 4.1 Deriving the Hamiltonian

In deriving the Hamiltonian for perpendicular propagation, we take advantage of the work we have already presented in Chapter 2. In this case, the motion of interest may again be restricted to one dimension. The unperturbed Hamiltonian is taken from (2.9):

\[ H_0 = \frac{p_z^2}{2m} + \frac{1}{2} m \omega_0^2 x^2 \]  

(4.2)

In perpendicular propagation, we model the electric field as:

\[ \vec{E} = \hat{z}E_0 \sin(kx - \omega t), \quad \Phi = \frac{E_0}{k} \cos(kx - \omega t) \]  

(4.3)

and hence the perturbation takes the form:

\[ H_1 = e\Phi = \frac{eE_0}{k} \cos(kx - \omega t) \]  

(4.4)

At this point, we normalize our equations. This is a standard procedure in non-linear dynamics. We perform the following scaling transformation:

\[ \omega t \rightarrow \tau \]  

(4.5)

\[ kx \rightarrow q \]  

(4.6)

\[ \frac{kp_z}{m\omega} \rightarrow p \]  

(4.7)

We must also normalize the Hamiltonian:

\[ \frac{k^2}{m\omega^2}(H_0 + H_1) \rightarrow H \]  

(4.8)
The new Hamiltonian takes the form:
\[
H = \frac{k^2}{2m^2 \omega^2} \mathbf{p}^2 + \frac{k^2 \omega_0^2}{2 \omega^2} x^2 + \frac{eE_0k}{m\omega^2} \cos(kx - \omega t) 
\] (4.9)
and replacing the old coordinates with the normalized coordinates:
\[
H = \frac{1}{2} \mathbf{p}^2 + \frac{1}{2} \frac{\omega_0^2}{\omega^2} q^2 + \frac{eE_0k}{m\omega^2} \cos(q - \tau) 
\] (4.10)
Further, if we define \( T = \frac{\omega}{\omega_0} \), the ratio between the frequency of the plasma wave and the cyclotron frequency, and \( \epsilon = \frac{eE_0k}{m\omega_0^2} \), the non-linearity parameter which is the square of the ratio of the bounce frequency to the frequency of the plasma wave, the new form of the perturbation and the unperturbed system is:
\[
H_0 = \frac{1}{2} \mathbf{p}^2 + \frac{1}{2} \frac{\omega_0^2}{T^2} q^2 \quad (4.11)
\]
\[
H_1 = \epsilon \cos(q - \tau) \quad (4.12)
\]
As in Chapter 2, we again find the action-angle coordinates for the unperturbed system:
\[
I = \frac{1}{2\pi} \oint pdq = TH_0 
\] (4.13)
\[
\psi = \tan^{-1}\left(T \frac{q}{p}\right) \quad (4.14)
\]
and the original coordinates are related to the action-angle coordinates as:
\[
q = \sqrt{2TI} \sin \psi \quad (4.15)
\]
\[
p = \sqrt{\frac{2I}{T}} \cos \psi \quad (4.16)
\]
If we define \( \rho = \sqrt{2TI} \), the normalized Larmor radius, the Hamiltonian may now be written in terms of the action angle coordinates as:
\[
H_0 = \frac{1}{T} I 
\] (4.17)
\[
H_1 = \epsilon \cos(\rho \sin \psi - \tau) = \epsilon \sum_m J_m(\rho) \cos(m\psi - \tau) 
\] (4.18)
where the Bessel function identity of (3.25) has again been used to decompose the perturbing Hamiltonian in terms of harmonics of the angle coordinate. As in Chapter 3, this allows us to isolate any phase space resonances which occur at stationary phase in the sinusoids of (4.18).

4.2 General Fixed Point Analysis

The fixed points are found by simultaneously setting the time derivatives of the action and angle to zero. We do this first for the general system. Hamilton's equations give:

\[ \dot{\psi} = \frac{\partial H}{\partial I} = \frac{1}{T} + \varepsilon \sum m J_n'(\rho) \cos(m\psi - \tau) \tag{4.19} \]

\[ \dot{I} = -\frac{\partial H}{\partial \psi} = -\varepsilon \sum m J_n(\rho) \sin(m\psi - \tau) \tag{4.20} \]

These equations are very difficult to analyze. In particular, they are time dependent and they contain an infinite summation over Bessel functions.

4.3 The Hamiltonian for the On or Near Resonance Case

As in Chapter 3, a phase space resonance occurs if a stationary phase condition is satisfied:

\[ \frac{d(n\psi - \tau)}{d\tau} = n\dot{\psi} - 1 = 0 \Rightarrow \dot{\psi} = \frac{1}{n} \tag{4.21} \]

To lowest order in \( \varepsilon \), combining equations (4.19) and (4.21), the phase space resonance condition is equivalent to:

\[ n = T \tag{4.22} \]

or that the wave frequency is some integer multiple of the cyclotron frequency. When the frequency ratio is exactly an integer, the system is on-resonance. Let us assume that we are near a resonance so that \( T \approx n \). We then perform a canonical transformation using the generating function:

\[ F_2 = (n\psi - \tau)\tilde{I} \tag{4.23} \]
which gives the transformation to the \((\tilde{I}, \phi, \tilde{H})\) coordinates:

\[
I = \frac{\partial F_2}{\partial \psi} = n\tilde{I} \tag{4.24}
\]

\[
\phi = \frac{\partial F_2}{\partial \tilde{I}} = n\psi - \tau \tag{4.25}
\]

\[
\tilde{H} = H + \frac{\partial F_2}{\partial \tau} = H - \tilde{I} \tag{4.26}
\]

This transformation is performed because in these coordinates, \(\dot{\phi} = n\psi - 1 = 0\) by equation (4.21). As in Section 3.3, this will allow us to isolate the phase space resonance. The Hamiltonian for this system looks like:

\[
\tilde{H} = \left(\frac{n}{T} - 1\right)\tilde{I} + \varepsilon \sum_m J_m(\rho) \cos \left(\frac{m}{n} \phi - \left(1 - \frac{m}{n}\right)\tau\right) \tag{4.27}
\]

Notice that in these coordinates:

\[
\rho = \sqrt{2\tilde{T}\tilde{I}} = \sqrt{2n\tilde{I}} \tag{4.28}
\]

Performing a time average of this system as in Section 3.3, only the term with stationary phase survives so that:

\[
\bar{H} = \left(\frac{n}{T} - 1\right)\tilde{I} + \varepsilon J_n(\rho) \cos \phi \tag{4.29}
\]

This is the resonance Hamiltonian.

### 4.4 The On or Near Resonance Fixed Point Analysis

We now determine the conditions for the existence of fixed points in the system for the near or on-resonance case. Using (4.29), Hamilton's equations give:

\[
\frac{\partial \bar{H}}{\partial \phi} = \varepsilon J_n(\rho) \sin \phi = 0 \tag{4.30}
\]

\[
\frac{\partial \bar{H}}{\partial \tilde{I}} = \left(\frac{n}{T} - 1\right) + \varepsilon \frac{nT}{\rho} J'_n(\rho) \cos \phi = 0 \tag{4.31}
\]

The elliptic fixed points are given by the condition that \(\sin \phi = 0\) and the hyperbolic
fixed points are given by $\sin \phi \neq 0$. This means that for the elliptic fixed points we have:

$$\phi = m\pi, \text{ or } \psi = \frac{m}{n}\pi$$  \hspace{5em} (4.32)

$$J'_n(\rho) = \mp \frac{1}{\varepsilon} \left( \frac{n}{T} - 1 \right) \frac{\rho}{nT}$$  \hspace{5em} (4.33)

where the top sign is given for $m$ even so that $\cos m\pi = 1$. Notice that for very small $\varepsilon$ and $T \neq n$ there are no elliptic fixed points. The methods from Chapter 3 cannot be applied when elliptic fixed points do not exist. This means that these methods are not easily to apply to the near or off-resonance case.

We now restrict our attention the case when $T = n$. If we repeat the fixed point analysis for the elliptic fixed points we find the conditions:

$$\phi = m\pi, \text{ or } \psi = \frac{m}{n}\pi$$  \hspace{5em} (4.34)

$$J'_n(\rho) = 0$$  \hspace{5em} (4.35)

In this case, the action coordinates of the elliptic fixed points are given by the zeroes of the derivative of the $n^{th}$ Bessel function. We now determine expressions for the hyperbolic fixed points. We know these occur when $\sin \phi \neq 0$. This leaves:

$$J_n(\rho) = 0$$  \hspace{5em} (4.36)

$$\phi = \pm \cos^{-1} \left( \frac{1}{\varepsilon J'_n(\rho)} \cdot \frac{nT - n^2}{\rho} \right)$$  \hspace{5em} (4.37)

Again, repeating the analysis for hyperbolic fixed points for the case where $T = n$ gives:

$$J_n(\rho) = 0$$  \hspace{5em} (4.38)

$$\phi = (2m+1)\frac{\pi}{2}, \text{ or } \psi = \frac{2m+1}{n} \cdot \frac{\pi}{2}$$  \hspace{5em} (4.39)

### 4.5 Phase Space Plots and the Physical System

A code has been written to numerically integrate the equations of motion for the case of perpendicular propagation. This code, including graphics display routines, is listed in Appendix D. The output of the code is a phase space plot. The phase space plots generated by this code are the tool we use to explore this system. In this section, we will
try to develop a relationship between the phase space plots that we numerically generate with the physics of the system. In particular, we will focus on the relationship between island formation in the phase space plots and phase space resonances in the physical picture.

Notice the Hamiltonian for the system, (4.17) and (4.18) is invariant if $\tau \rightarrow \tau + 2\pi$ or $\phi \rightarrow \phi + 2\pi$. Because of this symmetry, we may model the trajectories of the system as orbits on a torus. The phase space plots represent the intersection of these trajectories with the plane representing $\tau = 2\pi m$, where $m$ is an integer. In the phase space plots, it is possible for one initial condition to generate a series of islands. Such a trajectory is closed in the motion on the whole torus and travels about the torus with some period in $\tau, \Gamma$. Let us represent the state of the system by a vector, $\vec{x}(\tau) = (x(\tau), \dot{x}(\tau))$. Then periodic motion implies that $\vec{x}(\tau) = \vec{x}(\tau + \Gamma)$. The points on the phase space plots are samples of this motion at intervals of $2\pi$. We represent the points on the phase space plots as $\vec{x}[n] = \vec{x}(2\pi n)$, but again, periodic motion implies that $\vec{x}[n] = \vec{x}(2\pi n \mod \Gamma)$.

Let us briefly describe the physics of our system. A charged particle moves in circular orbits in the x-y plane with a frequency $\omega_o$, the cyclotron frequency, due to a constant magnetic field in the z-direction. This motion is perturbed by an electrostatic plasma wave with a frequency $\omega$ and wavenumber $k$, which propagates across the magnetic field in the x-direction. The wave-particle is in resonance when the stationary phase condition, (4.21), is met. To lowest order in $\epsilon$, this condition requires that the time it takes the particle to complete one cyclotron orbit is equivalent to the time it takes for $m_0$ integer wavelengths of the plasma wave to pass any fixed position in space. More generally, to $n_0^{th}$ order in $\epsilon$, this condition requires that the time it takes the particle to complete $n_0$ orbits is equivalent to the time it takes for $m_0$ wavelengths to pass any fixed position in space. In the presence of the plasma wave, the cyclotron frequency may be altered. We define $\Omega$ as the perturbed cyclotron frequency without any normalization. The phase space resonance condition as we have just described to $n_0^{th}$ order in $\epsilon$ takes the
where \( \langle \Omega \rangle_{n_0} \) is the time average of the perturbed frequency over \( n_0 \) orbits. Using simple arithmetic, (4.40) is equivalent to:

\[
\frac{\langle \Omega \rangle_{n_0}}{\omega} = \frac{n_0}{m_0}
\] (4.41)

However, under the normalization we have adopted, (4.5), frequency is normalized to \( \omega \). Hence:

\[
\frac{\langle \Omega \rangle_{n_0}}{\omega} = \langle \psi \rangle_{n_0} = \frac{n_0}{m_0}
\] (4.42)

where \( \langle \psi \rangle_{n_0} \) is the normalized perturbed frequency described by (4.19).

We will now relate this phase space resonance to an \( m_0^{th} \) order island, one in which \( m_0 \) closed orbits are traced by the intersection of a single trajectory with the plane \( \tau = 2\pi m \). Let us move into the rotating frame of the particle. The particle sees a perturbation of the form:

\[
E(\tau) = E_0 \sin(x(\tau) - \tau)
\] (4.43)

where \( x(\tau) \) describes the position of the particle at \( \tau \). Under the phase space resonance condition,

\[
x \left( \tau + \frac{2\pi n_0}{\langle \psi \rangle_{n_0}} \right) = x(\tau + 2\pi m_0) = x(\tau)
\] (4.44)

and also,

\[
E \left( \tau + \frac{2\pi n_0}{\langle \psi \rangle_{n_0}} \right) = E(\tau + 2\pi m_0) = E_0 \sin(x(\tau + 2\pi m_0) - \tau - 2\pi m_0) = E_0 \sin(x(\tau) - \tau)
\] (4.45)

So, every \( n_0 \) cyclotron orbits, the particle sees the same electric field. This analytically justifies the wave-particle resonance condition. But let us take this analysis further. In the time it takes for the particle to traverse \( n_0 \) cyclotron orbits, the electric field at each fixed
position will have oscillated $m_0$ times. The plasma wave is continuous in both space and time. This means that in the frame of the charged particle, in this same time interval, $m_0$ wavelengths of the electric field will have passed. This explains an $m_0^{th}$ order island. In one period of this system, a charged particle, initially at rest is exposed to $m_0$ wavelengths of a plane wave. We expect its energy to fluctuate $m_0$ times. In particular, in the case of a primary first order island, which we define as $n_0=1$ and $m_0=m$, every cyclotron orbit of the charged particle is exposed to $m$ wavelengths of the plasma wave. We expect to see $m$ islands in phase space. This is exactly what happens for the on-resonance case.

Now let us imagine what happens as we increase the amplitude of the plasma wave. As the amplitude of the plasma wave is increased, the wave-particle interaction may alter the cyclotron frequency of the charged particle. Let us assume that the system initially met the wave-particle or phase space resonance condition, but as the amplitude of the plasma wave is increased, the cyclotron frequency is slightly perturbed by $\Delta$ so that:

$$\langle \psi \rangle_{n_0} = \frac{n_0}{m_0} + \Delta \quad (4.46)$$

and we further impose that $\Delta = \frac{n_1}{m_1}$, $n_1, m_1 \in \mathbb{Z}$. Under these conditions, according to our analysis, the system meets a new resonance condition:

$$\langle \psi \rangle_{n_0} = \frac{n_0}{m_0} + \Delta = \frac{n_0}{m_0} + \frac{n_1}{m_1} = \frac{n_0 m_1 + n_1 m_0}{m_0 m_1} \quad (4.47)$$

The charged particle must now complete $n_0 m_1 + n_1 m_0$ cyclotron orbits before completing a full period of its motion. Furthermore, in this full period, $m_0 m_1$ wavelengths of the plasma wave will have passed. Hence we expect to see $m_0 m_1$ islands in each island chain. Numerically, this indeed is the case. As the amplitude of the perturbation increases, the islands in the original chain multiply to form higher order island chains.

An interesting question arises from this description of phase space. For any finite value of the non-linearity parameter, an infinite spectrum of rational numbers may be realized by the perturbed frequency. Which rational values are realized, or which order islands form and how are they chosen? This problem has not been dealt with in this thesis.
4.6 The Onset of Stochasticity

In this section, a qualitative description of the onset of stochasticity is presented. We deal separately with the on-resonance case and the off-resonance case and first describe the on-resonance case. The on-resonance case is special because fixed points exist for any finite value of the non-linearity parameter. This is clear from (4.34), (4.35), (4.38) and (4.39). For any finite value of the non-linearity parameter, phase space is filled with primary islands. These are the first order islands predicted in Section 4.5 and they exist because the phase space resonance condition is satisfied for non-zero values of the non-linearity parameter. Figure 4.1 demonstrates the almost regular structure of these islands also predicted by the fixed point analysis. If the simulations were run for a long period of time, all trajectories would close. As the non-linearity parameter is increased, the perturbation acts to alter the frequency of the particle. As a result, as we explained in Section 4.5, some of the orbits in an island structure break up into a set of higher order islands. As we continue to increase the non-linearity parameter, groups of these higher order islands within the same island structure overlap and small regions of stochasticity contained within an island structure appear. Figure 4.2 illustrates a system in which higher order islands have appeared and small regions of stochasticity are found. Eventually, the first order islands neighboring in action overlap and stochasticity in large regions of phase space occurs. This is illustrated by Figure 4.3.

The off-resonance case is similar to the on-resonance case, but different in an important way. The general and near-resonance fixed point analysis in Section 4.2 and 4.4 show that elliptic fixed points do not exist for all values of the non-linearity parameter. This means that for small values of the non-linearity parameter, phase space is uninteresting, i.e. there is no island structure and it is clear that conserved quantities are easily found. This is illustrated by Figure 4.4. As the non-linearity parameter is increased, the frequency is perturbed and island structures are formed. This is illustrated in Figure 4.5. Notice, these islands need not be first order islands and they do not correspond to the islands that form for small values of the non-linearity parameter in the on-resonance case. Again, as the parameter continues to increase, these islands begin to overlap and stochasticity again dominates regions of phase space. This is illustrated by Figure 4.6.
Figure 4.1: The on-resonance case for a small non-linearity parameter. In this simulation, the frequency ratio is 5.0 and the non-linearity parameter is $10^3$. Phase space is dominated by the primary island structure.
Figure 4.2: The on-resonance case for a larger non-linearity parameter. In this simulation, the frequency ratio is 5.0 and the non-linearity parameter is 0.05. Higher order islands have appeared and small regions of stochasticity are found within an island structure.
Figure 4.3: The on-resonance case for a large non-linearity parameter. In this simulation, the frequency ratio is 5.0 and the non-linearity parameter is 0.15. Stochasticity dominates the low action regime of phase space.
Figure 4.4: The off-resonance case for a small non-linearity parameter. In this simulation, the frequency ratio is 5.3 and the non-linearity parameter is 0.0015. No fixed points exist and the trajectories are straight lines.
Figure 4.5: The off-resonance case for a larger non-linearity parameter. In this simulation, the frequency ratio is 5.3 and the non-linearity parameter is 0.027. Higher order islands have appeared in phase space and they do not correspond to the islands in Figure 4.1, the on-resonance case.
Figure 4.6: The off-resonance case for a large non-linearity parameter. In this simulation, the frequency ratio is 5.3 and the non-linearity parameter is 0.12. Stochasticity dominates phase space in the regime of low action.
Chapter 5
Extending Island Formation to Stochasticity

5.0 Introduction

In this section, we will bound the regions of phase space that are stochastic using two arguments. The arguments are analogous to those presented by Karney.\textsuperscript{3,4} A lower bound is achieved using a trapping argument. This lower bound is valid for all cases. It is valid on and off-resonance, for low and high action and for low and high frequency ratios. An upper bound is achieved by first determining a condition for the existence of primary islands. This condition is then extended in an empirical manner to an upper bound for stochasticity. The condition for primary islands is valid only for values of normalized action much greater than the frequency ratio and also only for the off-resonance case.
5.1 A Lower Bound for Stochasticity

The effects of the electrostatic wave are significant only under the conditions of stationary phase. If the electrostatic wave meets the charged particle with different phases, it will not have any net effect. The system becomes stochastic due to the effects of the perturbation. This means that if the perturbation cannot interact with the particles, the system will never be stochastic. The phase velocity of the wave is:

\[ v_w = \frac{\omega}{k} \]  \hspace{1cm} (5.1)

The wave interacts with the particle when its phase velocity is equal to the particle velocity. In this case, the particle sees a constant phase of the wave. For this reason, a simple lower bound on stochasticity then takes the form:

\[ v_x \geq v_w \]  \hspace{1cm} (5.2)

Including the correct normalization of (4.7), this is equivalent to:

\[ p = \frac{k}{\omega} v_x \geq 1 \]  \hspace{1cm} (5.3)

and in terms of the action-angle coordinates, (4.16) gives:

\[ \sqrt{\frac{2I}{T}} \cos \psi \geq 1 \]  \hspace{1cm} (5.4)

A lower bound is obtained by taking the cosine to unity which gives:

\[ I \geq \frac{T}{2} \]  \hspace{1cm} (5.5)

This bound may be improved slightly using a trapping argument. If the particle moves at a velocity less than, but near the wave velocity, the two may still interact. This occurs because the particle may be trapped by the plasma wave. The width of this velocity window in the normalized units may be directly obtained from the Hamiltonian in (4.11) and (4.12).\textsuperscript{5,10} This width is \( \sqrt{\epsilon} \). Hence, a more accurate lower bound in velocity for interaction is:

\[ p \geq 1 - \sqrt{\epsilon} \]  \hspace{1cm} (5.6)
which is equivalent to:

$$\sqrt{\frac{2I}{T}} \cos \psi \geq 1 - \sqrt{\epsilon}$$  \hspace{1cm} (5.7)

using (4.16) and setting the cosine to unity and isolating the action gives a lower bound of:

$$I \geq \frac{T}{2} \left(1 - 2\sqrt{\epsilon} + \epsilon\right)$$ \hspace{1cm} (5.8)

### 5.2 A Condition for Primary Island Formation

We have already discussed the condition for island formation in Section 4.5. In this section, we will look more carefully at the condition for primary island formation. The condition for primary island formation is a special case of (4.42), the phase space resonance condition. For primary island formation, we take $n_0$, the number of cyclotron orbits of the particle before the same phase of the plasma wave is seen at the same position in space to one. In this case, the condition reads:

$$\psi = \frac{1}{n_0}$$ \hspace{1cm} (5.9)

This frequency has been calculated using Hamilton's equations in (4.19) and the result is:

$$\psi = \frac{1}{T} + \epsilon \frac{T}{\rho} \sum_m J'_m(\rho) \cos(m \psi - \tau)$$ \hspace{1cm} (5.10)

If we take $n$ as the nearest integer to $T$ so that $T = n + \delta$ with $|\delta| \leq \frac{1}{2}$, it is clear we should take $m_0 = n$ in (5.9). In this case, the perturbed normalized frequency is nearest to $\frac{1}{n}$. We are interested in the regime of island formation. This means that it is possible to isolate the resonance term from the summation in (5.10) using the techniques described in Section 3.3. In this regime:

$$\psi \approx \frac{1}{T} + \epsilon \frac{T}{\rho} J'_n(\rho) \cos(n \psi - \tau)$$ \hspace{1cm} (5.11)

It is clear how to proceed. We plug the value of the frequency from (5.11) into the condition for island formation in (5.9):

$$\epsilon \frac{T}{\rho} J'_n(\rho) \cos(n \psi - \tau) = \frac{1}{n} - \frac{1}{T} = \frac{\delta}{nT}$$ \hspace{1cm} (5.12)
This is then the condition for primary island formation. Notice that primary islands do not exist for all values of the non-linearity parameter if $\delta \neq 0$ and this is as we expect from the fixed point analyses in Section 4.2 and 4.4.

### 5.3 An Upper Bound for Stochasticity

The idea behind this chapter is to extend the condition for primary island formation to an upper bound for stochasticity. Coupled with the result of Section 5.1, this is enough to determine which regions of phase space are stochastic for a given non-linearity parameter. Notice that for the on-resonance case, the condition in (5.12) is trivial. In this case, $\delta = 0$ so that primary islands form at the zeros of the derivative of the Bessel function independent of the non-linearity parameter. For this reason, it is not possible to extend the condition in (5.12) to one for stochasticity in the on-resonance case. We focus on the off-resonance case. At this point, we notice that the Bessel functions are difficult to manipulate analytically. Hence, we perform an expansion of the Bessel function valid when the argument is much larger than the order:

$$J_n'(\rho) \approx -\frac{2}{\sqrt{\pi \rho}} \sin \left( \rho - \left( \frac{n}{2} + \frac{1}{4} \right) \pi \right), \quad \rho >> m \tag{5.13}$$

With this expansion, (5.12) becomes:

$$\epsilon \frac{T}{\rho} \sqrt{\frac{2}{\pi \rho}} \sin \left( \rho - \left( \frac{n}{2} + \frac{1}{4} \right) \pi \right) \cos(n\psi - \tau) = -\frac{\delta}{nT} \tag{5.14}$$

To convert this to an upper bound on stochasticity, we take the sine and cosine to unity and solve for the non-linearity parameter:

$$\epsilon = \delta \frac{\pi^{\frac{\lambda}{2}} \rho^{\frac{\lambda}{2}}}{nT^2 2^{\frac{\lambda}{2}}} \tag{5.15}$$

Substituting the value for the normalized Larmor radius gives:

$$\epsilon = \delta \frac{2^{\frac{\lambda}{2}} \pi^{\frac{\lambda}{2}} \rho^{\frac{\lambda}{2}}}{nT^\lambda} \tag{5.16}$$

At this point, the work becomes empirical. It is clear from the phase space plots and numerical simulations that island formation is related to stochasticity in a significant manner. As we described in Section 4.6, stochasticity begins in regions of phase space.
where neighboring islands overlap. For this reason, Karney extends the threshold for primary island formation in (5.16) to one for stochasticity by taking $\delta = \frac{1}{4}$. With this substitution, using results of simulations, Karney claims that (5.16) is a valid bound for stochasticity. Karney's extension is an upper bound for the stochastic phase space in action and it takes the form:

$$
\varepsilon = \frac{1}{4} \frac{2^{\frac{\gamma}{4}} \pi^{\frac{\gamma}{4}} I^{\frac{\gamma}{4}}}{nT^{\frac{\gamma}{4}}} \quad (5.17)
$$

In the original coordinates, this bound takes the form:

$$
e E_0 k \frac{2^{\frac{\gamma}{4}} \pi^{\frac{\gamma}{4}}}{m \omega^2} = 2^{\frac{\gamma}{4}} \pi^{\frac{\gamma}{4}} \left( \frac{\omega_0}{\omega} \right)^{\frac{\gamma}{4}} \left( \frac{\omega_0}{m \omega_0} \frac{k^2}{H_0} \right)^{\frac{\gamma}{4}} \quad (5.18)
$$

where $H_0$ is the energy of the unperturbed system (4.2). Solving for the amplitude of the electric field gives:

$$
E_0 = \frac{2^{\frac{\gamma}{4}} \pi^{\frac{\gamma}{4}}}{4en} m^{\frac{\gamma}{4}} \omega^{\frac{\gamma}{4}} \omega_0^{\frac{\gamma}{4}} k^{\frac{\gamma}{4}} H_0^{\frac{\gamma}{4}} \quad (5.19)
$$

We will derive a similar bound in a more satisfactory manner in the next chapter. Also, the comparison of this bound to numerical simulations may be found in Chapter 8 and Appendix C.
Chapter 6
The Chirikov Condition in Frequency

6.0 Introduction

In this section we develop an alternative upper bound for stochasticity. Although the analytical motivation for the lower bound of stochasticity in Chapter 5 is physically sound, the analytical motivation for the upper bound is suspect. Island formation is related to stochasticity, but it would be better if we could directly use a condition for stochasticity to derive an upper bound. Further, numerical investigations which will be discussed in Chapter 7 suggest that the upper bound of Chapter 5 can be improved. The goal of this section is to obtain another semi-empirical upper bound for stochasticity, but this upper bound will be derived using a Chirikov condition in frequency. We will analytically determine an upper bound valid for normalized actions much greater than the
frequency ratio for both the on and off-resonance case up to a numerical factor. Computer simulations will then be used to determine this numerical factor.

### 6.1 The Chirikov Condition in Frequency

The upper bound is determined using a Chirikov condition in frequency. The equivalence of the Chirikov condition in frequency to that in action for the standard map has been demonstrated. The approach used in this section has been motivated by the work of Sagdeev and Zaslavsky. In this section, we present an analytically sound technique for deriving the Chirikov condition in frequency which leads to an upper bound for stochasticity.

We recall the definition of the Hamiltonian from (4.17) and (4.18):

\[ H = \frac{I}{T} + \varepsilon \cos(p \sin \psi - \tau) = \frac{I}{T} + \varepsilon \sum_{n} J_{n}(p) \cos(m \psi - \tau) \]  \hspace{1cm} (6.1)

where the second form is derived using a Bessel function expansion. We are trying to calculate a Chirikov condition in frequency. The condition in frequency is similar to the condition in action. The motion near a phase space resonance is that of a non-linear oscillator. This motion has a distinct frequency. A neighboring resonance is associated with another distinct frequency. When the perturbation acts to alter the frequency of the particle enough so that it may jump from one resonance to the next, the Chirikov overlap condition in frequency is satisfied. The first step is to determine the separation in frequency of the phase space resonances of the system. This is easily calculated. Phase space resonances occur when (4.21):

\[ \frac{d(m \psi - \tau)}{d\tau} = m \psi - 1 = 0 \]  \hspace{1cm} (6.2)

or when a stationary phase condition is satisfied. This is equivalent to:

\[ \psi = \frac{1}{m} \Rightarrow \delta \psi = \left| \frac{1}{m} - \frac{1}{m \pm 1} \right| = \frac{1}{m(m \pm 1)} \]  \hspace{1cm} (6.3)

The second step is to determine the frequency width of the phase space resonances. After this has been determined, the Chirikov condition may then be applied. If the perturbation acts in such a manner as to allow the frequency of the particle to vary as much as \( \delta \psi \), the
resonance overlap condition will be satisfied. The frequency is calculated using Hamilton's equations and it is given by (4.19):

$$\frac{\partial H}{\partial t} = \frac{1}{T} + \epsilon \frac{T}{\rho} \sum_{m} J'_m(\rho) \cos(m\psi - \tau)$$

(6.4)

The important step for determining the frequency width is reducing the infinite summation in (6.4). We would like to define a finite bounded set of integers, $\Delta n$, such that:

$$\psi \approx \frac{1}{T} + \epsilon \frac{T}{\rho} \sum_{m \in \Delta n} J'_m(\rho) \cos(m\psi - \tau)$$

(6.5)

is a valid approximation to the frequency for large values of $\rho$. This has been done many times in this thesis by isolating a resonance term. This has been done in Section 3.3 using the methods of Lichtenberg and Lieberman for the case of oblique propagation, in Section 4.3 to determine the resonance Hamiltonian and in Section 5.2 to determine a condition for island formation. In this section, we will be more careful. A mathematical argument for reducing this summation is presented in Appendix B, but at this point, we will present a brief intuitive argument. The terms in this summation oscillate very quickly as we move away from $n = T$. The effect of these terms on the frequency becomes negligible as $m \to \infty$ because it averages out to zero. In the regime of large $\rho$, it is possible to perform an expansion of the Bessel function using (5.13) so that:

$$\psi = \frac{1}{T} + \epsilon \frac{T}{\rho} \sqrt{\frac{2}{\pi \rho}} \sum_{m \in \Delta n} \sin \left( \rho - \frac{m\pi}{2} - \frac{\pi}{4} \right) \cos(m\psi - \tau)$$

(6.6)

We define $S = 2 \sum_{m \in \Delta n} \sin \left( \rho - \frac{m\pi}{2} - \frac{\pi}{4} \right) \cos(m\psi - \tau)$. It is clear that $S$ need not be negligible and it is never infinite. Consider the case when $\rho = \frac{\pi}{4}, \tau = \frac{\pi}{2}, \psi = \frac{\pi}{2}$:

$$S = 2 \sum_{m \in \Delta n} \sin \left( \frac{m\pi}{2} \right) \cos \left( \frac{m\pi}{2} - \frac{\pi}{2} \right) = |\Delta n|$$

(6.7)

We precede without evaluating $S$. We are forced to do this because we cannot rigorously determine the set $\Delta n$. With this notation:

$$\psi = \frac{1}{T} + \frac{1}{2} \epsilon T \frac{2}{\pi \rho^3} S$$

(6.8)
It is clear from this expression that:

\[ \Delta \psi = \varepsilon T \frac{2}{\pi \rho^3} S \]  \hspace{1cm} (6.9)

It is now possible to apply the Chirikov overlap condition:

\[ \delta \psi = \Delta \psi \]  \hspace{1cm} (6.10)

Recall that:

\[ \delta \psi = \frac{1}{m(m \pm 1)} \]  \hspace{1cm} (6.11)

but for an upper bound on the stochastic phase space, the resonances of interest are for \( m \approx n \) and this gives:

\[ \delta \psi = \frac{1}{n(n \pm 1)} \]  \hspace{1cm} (6.12)

Hence, the overlap condition gives:

\[ \frac{1}{n(n \pm 1)} = \varepsilon T \frac{2}{\pi \rho^3} S \]  \hspace{1cm} (6.13)

and after some algebra:

\[ \varepsilon = \frac{\pi \gamma 2 \gamma I \gamma}{T \gamma n(n \pm 1) S} \]  \hspace{1cm} (6.14)

where again, \( T = n \) and this is valid for large normalized action. This is close to the expression of Karney (5.17) which is found by replacing \( n(n \pm 1) \) by \( T^2 \) and taking \( S = 4 \).

This form has been used to numerically estimate \( S \) from simulations. Doing so, a crude method for determining \( S \) as a function of the nearness to resonance and the value of \( n \) is described by:

\[ S = \delta_i -.05(n - 5) \]  \hspace{1cm} (6.15)

where \( \delta_i \) depends on the nearness to resonance and takes the value 3.25 when \( T - n \) is 0, it takes the value 4 when \( T - n \) is .1, it is 3.75 when \( T - n \) is .2, 4 when \( T - n \) is .3 and 5 when \( T - n \) is .4. It is clear that this method for estimating \( S \) is not valid for arbitrary frequency ratios. As the frequency gets very large, it is clear that the value of \( S \) predicted by (6.15) will become negative which is not a physical result. The limits of this estimation...
have not been determined. In the original coordinates of the system, the condition in (6.14) takes the form:

\[ \frac{eE_0 k}{m \omega^2} = \frac{\pi \sqrt{2 \pi}}{n(n \pm 1) S} \left( \frac{\omega}{\omega_o} \right)^{\frac{X}{2}} \left( \frac{\omega - k}{\omega_o m \omega \omega_o} H_0 \right)^{\frac{X}{2}} \]  

(6.16)

where $H_0$ is the energy of the unperturbed system (4.2). Solving for the electric field gives:

\[ E_0 = \frac{\pi \sqrt{2 \pi}}{n(n \pm 1) S} \left( \frac{\omega}{\omega_o} \right)^{\frac{X}{2}} m^{\frac{X}{2}} \omega \sqrt{k} X H_0 \]  

(6.17)

The comparison of this bound to both that suggested by Karney (5.17) and numerical simulations may be found in Chapter 8 and Appendix C.
Chapter 7
Numerical Results

7.0 Introduction
Two different methods for determining which region of phase space has been presented in this thesis. The first was presented in Chapter 5, and it is based on the work performed by Karney. The second is presented in Chapter 6 and it is based on the work performed during my graduate studies at MIT. As has already been mentioned, the tool we use to generate data for this system is numerical simulations. In this chapter, we will discuss the techniques used to gather this data.

7.1 The Data
The data from the numerical simulations was calculated using the codes listed in Appendix D. The code numerically integrates the equations of motion given a set of initial
conditions and the values of the frequency ratio and the non-linearity parameter. The output is a phase space plot. The strategy for collecting data was to choose an appropriate initial condition, one located in the largest stochastic region and then to examine which part of phase space this condition would fill. An upper and lower bound for the stochastic region was then determined by qualitatively examining the output plot and estimating the upper and lower bounds of the filled phase space.

In some cases, it was not exactly clear which part of phase space was stochastic. In particular, in many cases, islands and stochasticity inhabit the same regions of phase space. In these cases, a region was judged stochastic as long as the width of the island structures was smaller than the width of the stochastic region separating the islands. Examples of the phase space plots generated and the regions of stochasticity defined are illustrated in Figures 7.1 and 7.2.

The data was collected for several different values of the non-linearity parameter and the frequency ratio. The non-linearity parameter was varied among four to six values for a given frequency ratio. Also, the frequency ratios were varied between 5 and 35 in steps of 5 where the ratios were taken to be integer as well as .1, .2, .3, and .4 above the integer value.
Figure 7.1: An example of a phase space plot for a frequency ratio of 25.0 and non-linearity parameter of 0.0075. The x axis is the angle while the y axis is the action. The points are intersections of one trajectory on the torus with a surface of constant time. The estimated region of stochasticity contains actions between 18 and 31.
Figure 7.2: An example of a phase space plot for a frequency ratio of 25.3 and non-linearity parameter of .01. The x axis is the angle while the y axis is the action. The points are intersections of one trajectory on the torus with a surface of constant time. The estimated region of stochasticity contains for actions between 15 and 54.
Chapter 8

Results and Conclusion

8.1 Results

For each group of simulations run for a given frequency ratio, the bound predicted by Karney (5.8) and (5.17) was plotted against the bound predicted here (6.14) and (6.15). On the same plot, the data computed numerically was also included. Karney's bound is represented by the dashed curve, the bound from Chapter 6 is represented by the solid curve and the numerical data is represented by points. Figures 8.1-8.4 are a sample of these plots and a complete collection of all the plots can be found in Appendix C. In Figure 8.1, both bounds are similar for a frequency ratio of 25.4. In Figure 8.2, we see that the bound derived in Chapter 6 is much better than that of Karney. Similarly in Figures 8.3 and 8.4 the bound in Chapter 6 shows superior performance, the first in the on-resonance case and the second for a low frequency ratio.
Figure 8.1: The predicted stochastic regions for a frequency ratio of 25.4. Karney's bounds are represented by the dashed curve and that of Chapter 6 is represented by the solid curve. The numerical data are represented by the points. In this case, both bounds are reasonable and they are roughly the same.
Figure 8.2: The predicted stochastic regions for a frequency ratio of 35.3. Karney's bounds are represented by the dashed curve and that of Chapter 6 is represented by the solid curve. The numerical data are represented by the points. In this case, the bound of Chapter 6 is reasonable while that of Karney predicts stochasticity in too large a region of phase space.
Figure 8.3: The predicted stochastic regions for a frequency ratio of 5.1. Karney's bounds are represented by the dashed curve and that of Chapter 6 is represented by the solid curve. The numerical data are represented by the points. In this case, the bound of Chapter 6 is reasonable, while that of Karney predicts stochasticity in too small a region of phase space.
Figure 8.4: The predicted stochastic regions for a frequency ratio of 20. This is the on-resonance case. Karney's bounds are represented by the dashed curve and that of Chapter 6 is represented by the solid curve. The numerical data are represented by the points. In this case, the bound in Chapter 6 is reasonable, while that of Karney predicts stochasticity in too large a region of phase space.
It is clear from these plots that Karney’s bound is not adequate. It does not show any sensitivity to the nearness to resonance and not enough sensitivity to the magnitude of the frequency ratio. Given the simplicity of the rule for determining the numerical factor $S$ in (6.15), the improvement of the bound presented here (6.14) is remarkable. In almost every case, it is better than Karney’s bound. Further, the bound was valid analytically for values of action much larger than the frequency ratio. Numerical simulations support that it is valid for lower actions as well.

The bound presented here is numerically superior to Karney’s bound, and it should also be clear that it is analytically superior. Karney’s bound is an extension from a threshold for island formation. The bound determined here comes from a Chirikov condition for stochasticity. Although a parameter was determined empirically, the bound was not realized through an ad-hoc extension of island formation coupled with an empirically determined parameter.

8.2 Conclusions

- We have found a semi-empirical practical solution for determining the upper bound in action of the stochastic region for the case of perpendicular propagation. It is valid both on and off-resonance for reasonable values of the frequency ratio.
- Numerically, this formulation is superior to the bound determined by Karney.
- The analytical motivation for this formulation is better than that of the work of Karney.

8.3 Future Work

- Although the bounds for stochasticity in Chapter 6 work quite well for the range of frequency ratios explored here, it is clear that there are limitations. By increasing the frequency ratio, the simple formula for determining the numerical parameter, $S$, given by (6.15) eventually breaks down. A better numerical formulation of $S$ and the limits of the current formulation would be desirable.
• This work is semi-empirical. A full analytical solution to the problem would be desirable.

• This work is valid analytically only for normalized actions much greater than the frequency ratio. A threshold valid analytically for a larger range of action would be desirable.
Appendix A

Including Harmonics in the Perturbation

A.0 Introduction

In this section we will use standard methods for dealing with resonances as presented by Lichtenberg and Lieberman\textsuperscript{5} to analyze a more complicated system. The system consists of a constant magnetic field which is perturbed by an infinite collection of plasma waves with varying amplitudes that propagate across the magnetic field. This problem is similar to that we are studying in this thesis except that all harmonics of the perturbation are included. By including the harmonics there will exist terms in the perturbation that are in resonance with some harmonic of the cyclotron frequency for any frequency ratio. To analyze this problem, the Chirikov condition will be used to determine a threshold for stochasticity. The problem will be solved without specifying the amplitudes of the harmonics. This was done with the intention of determining a threshold for the original problem by setting the amplitudes of all non-fundamental harmonics to zero. Unfortunately, this is not possible. However, the result is itself of interest.
A.1 The Chirikov Condition for First Order Islands

We work with the normalizations in Chapter 4. Including all harmonics of the perturbing wave is a simple extension of the problem treated there. The new perturbation looks like (4.12) and takes the form:

$$H_1 = \varepsilon \sum_{m \neq 0} \alpha_m \cos(q - m\tau)$$  \hspace{1cm} (A.1)

where the \( \{\alpha_m\} \) are weights which determine the strength of the \( m^{th} \) harmonic. The Hamiltonian for such a system takes the form:

$$H = \frac{I}{T} + \varepsilon \sum_{l,m} \alpha_m J_l(\sqrt{2IT}) \cos(l\psi - m\tau)$$  \hspace{1cm} (A.2)

where we have used equations (4.17) and (4.18). Hamilton's equations give:

$$\dot{\psi} = \frac{\partial H}{\partial I} = \frac{1}{T} + O(\varepsilon) = \frac{1}{T}$$  \hspace{1cm} (A.3)

and of course, \( \dot{\tau} = 1 \). To switch to the notation of Lichtenberg and Lieberman,\(^4\) simply define:

$$J_1 = I, \quad J_2 = H, \quad \theta_1 = \psi, \quad \theta_2 = \tau$$  \hspace{1cm} (A.4)

Further, if \( H_{l,m} = \alpha_m J_l(\sqrt{2IT}) \) where \( J_1 \) and \( J_2 \) represent the action not to be confused with the function \( J_l \) which is a Bessel function of the first kind, the Hamiltonian looks like:

$$H = \frac{J}{T} + \varepsilon \sum_{l,m} H_{l,m}(J_1) \cos(l\theta_1 - m\theta_2)$$  \hspace{1cm} (A.5)

and

$$\omega_1 = \dot{\theta}_1 = \frac{1}{T}, \quad \omega_2 = \dot{\theta}_2 = 1$$  \hspace{1cm} (A.6)

so that:

$$\frac{\omega_2}{\omega_1} = T = \frac{r}{s} \text{ for } r, s \in \mathbb{Z}$$  \hspace{1cm} (A.7)

In equation (A.7) we use the mathematical result that any real number is arbitrarily close to a rational number. Consider the generating function:

$$F_2 = (r\theta_1 - s\theta_2)\tilde{J}_1$$  \hspace{1cm} (A.8)

which corresponds to the canonical transformation:
\[
\frac{\partial F_2}{\partial J_1} = \Theta_1 - s\Theta_2 = \bar{\Theta}_1, \quad \frac{\partial F_2}{\partial \Theta_1} = r\bar{J}_1 = J_1, \quad \frac{\partial F_2}{\partial \tau} = \frac{\partial F_2}{\partial \Theta_2} = -s\bar{J}_1 \quad (A.9)
\]

so that:

\[
\bar{H}_0 = \frac{r\bar{J}_1}{T} + \frac{\partial F_2}{\partial \tau} = \left( \frac{r}{T} - s \right)\bar{J}_1 \quad (A.10)
\]

But, by definition, \( T = \frac{r}{s} \), so that \( \bar{H}_0 = 0 \). This is the case referred to as intrinsic degeneracy and the corresponding methods will be used to analyze this system. With the canonical transformation, the perturbation takes the form:

\[
\bar{H}_1 = \varepsilon \sum_{l,m} H_{l,m} \left( \bar{J}_1 \right) \cos \left[ \left( \frac{\bar{\Theta}_1}{r} + \frac{s}{r} \bar{\Theta}_2 \right) - m\bar{\Theta}_2 \right] \quad (A.11)
\]

As in Section 3.4, we time average the system to isolate the resonance term. Time averaging or averaging over \( \bar{\Theta}_2 \) gives:

\[
\bar{H}_1 = \varepsilon \sum_{l,m} H_{l,m} \left( \cos \left( \frac{1}{r} \left[ l\bar{\Theta}_1 + (l s - m r)\bar{\Theta}_2 \right] \right) \right) \quad (A.12)
\]

The time average leaves only the cosine terms independent of \( \bar{\Theta}_2 \) or those with \( ls - mr = 0 \). This implies that \( l = pr, \ m = ps \) with \( p \in \mathbb{Z} \). Hence,

\[
\bar{H}_1 = \varepsilon \sum_p H_{pr,ps} \cos \left( p\bar{\Theta}_1 \right) = \varepsilon \sum_p \alpha_{ps} J_{ps} \left( \sqrt{2rT}\bar{J}_1 \right) \cos \left( p\bar{\Theta}_1 \right) \quad (A.13)
\]

The \( \alpha_{ps} \) represent the strength of the harmonics so that it is reasonable to assume that the \( \alpha_{ps} \) fall off rapidly with increasing \( p \). This means that to close approximation:

\[
\bar{H}_1 = \varepsilon \alpha s J_1 \left( \sqrt{2rT}\bar{J}_1 \right) \cos \bar{\Theta} = \bar{H} \quad (A.14)
\]
Hamilton’s equations give:

\[ \dot{J}_1 = -\frac{\partial H}{\partial \theta_1} = \epsilon \alpha J_r(\sqrt{2rTJ_1}) \sin \bar{\theta}_1 = O(\epsilon) \quad (A.15) \]

\[ \dot{\bar{\theta}}_1 = \frac{\partial H}{\partial J_1} = \epsilon \alpha \sqrt{\frac{rT}{2J_1}} J'_r(\sqrt{2rTJ_1}) \cos \bar{\theta}_1 = O(\epsilon) \quad (A.16) \]

The elliptic fixed points take the form:

\[ \{ (\theta_1, J_1) \bigg| \dot{\theta}_1 = 0, \ J'_r(\sqrt{2rTJ_1}) = 0 \} \quad (A.17) \]

and they will be denoted as \( \{ (\bar{\theta}_{10}, \bar{J}_{10}) \} \). If \( \bar{J}_1 = J_{10} + \Delta J_1 \) and \( \bar{\theta}_1 = \theta_{10} + \Delta \bar{\theta}_1 \), then expanding about the elliptic fixed points gives:

\[ J'_r(\sqrt{2rT\bar{J}_1}) = J'_r(\sqrt{2rT\bar{J}_{10}}) + \frac{rT}{2J_{10}} J'_r(\sqrt{2rT\bar{J}_{10}}) \Delta \bar{J}_1 \]

\[ + \frac{1}{2} \left[ \frac{rT}{2J_{10}} J'_r(\sqrt{2rT\bar{J}_{10}}) - \frac{rT}{8J_{10}^3} J'_r(\sqrt{2rT\bar{J}_{10}}) \right] (\Delta \bar{J}_1)^2 \quad (A.18) \]

But, by assumption:

\[ J'_r(\sqrt{2rT\bar{J}_{10}}) = 0 \quad (A.19) \]

so that:

\[ J'_r(\sqrt{2rT\bar{J}_{10}}) = J'_r(\sqrt{2rT\bar{J}_{10}}) + \frac{rT}{4J_{10}} J'_r(\sqrt{2rT\bar{J}_{10}}) (\Delta \bar{J}_1)^2 \quad (A.20) \]

Expanding the cosine:

\[ \cos \bar{\theta}_1 = 1 - \frac{1}{2} (\Delta \bar{\theta}_1)^2 + \frac{1}{4!} (\Delta \bar{\theta}_1)^4 \quad (A.21) \]

means that the Hamiltonian takes the form:

\[ \tilde{H}_1 = \epsilon \alpha J_r(\sqrt{2rT\bar{J}_{10}}) + \epsilon \alpha J'_r(\sqrt{2rT\bar{J}_{10}}) (\Delta \bar{J}_1)^2 \]

\[ - \epsilon \alpha \left[ J'_r(\sqrt{2rT\bar{J}_{10}}) + \frac{rT}{4J_{10}} J'_r(\sqrt{2rT\bar{J}_{10}}) (\Delta \bar{J}_1)^2 \right] \left[ \frac{1}{2} (\Delta \bar{\theta}_1)^2 - \frac{1}{4!} (\Delta \bar{\theta}_1)^4 \right] \quad (A.22) \]
Keeping the lowest order in $\Delta \tilde{J}$ and $\Delta \tilde{\theta}$ and also ignoring the constant terms gives:

$$\Delta \tilde{H}_0 = \varepsilon \alpha \frac{rT}{4 J_{10}} J_1' \left( \sqrt{2 r T \tilde{J}_{10}} \right) (\Delta \tilde{J})^2 - \frac{1}{2} \varepsilon \alpha \cdot J_1' \left( \sqrt{2 r T \tilde{J}_{10}} \right) (\Delta \tilde{\theta})^2 \quad (A.23)$$

Defining $G = \varepsilon \alpha \frac{rT}{2 J_{10}} J_1' \left( \sqrt{2 r T \tilde{J}_{10}} \right)$ and $F = -\varepsilon \alpha \cdot J_1' \left( \sqrt{2 r T \tilde{J}_{10}} \right)$ leaves the Hamiltonian which describes the motion of the pendulum:

$$\Delta \tilde{H}_0 = \frac{1}{2} G (\Delta \tilde{J})^2 + \frac{1}{2} F (\Delta \tilde{\theta})^2 \quad (A.24)$$

The island widths for the pendulum are well known:

$$\Delta \tilde{J}_i = 2 \sqrt{\frac{F}{G}} = \sqrt{\frac{8 J_{10} J_1' \left( \sqrt{2 r T \tilde{J}_{10}} \right)}{r T J_1' \left( \sqrt{2 r T \tilde{J}_{10}} \right)}} \quad (A.25)$$

Further, the separation between islands is simply the separation of the fixed points which are solutions to:

$$J_1' \left( \sqrt{2 r T \tilde{J}_{10}} \right) = 0 \quad (A.26)$$

A Chirikov condition for first order island separation may now be calculated. However, there is a problem. It is known that stochasticity for the on-resonance case is not due to island overlapping of first order islands. In fact, the Chirikov condition here derived is independent of the non-linearity parameter. Higher order island calculations must be computed. This calculation follows.

**A.2 The Chirikov Condition for Higher Order Islands**

The Hamiltonian left from the first order island manipulations is:

$$\Delta \tilde{H} = \Delta H_0 + \Delta H_1 \quad (A.27)$$

where:

$$\Delta H_0 = \frac{1}{2} G (\Delta \tilde{J})^2 + \frac{1}{2} F (\Delta \tilde{\theta})^2 \quad (A.28)$$

and

$$\Delta H_1 = F \left[ -\frac{1}{4!} (\Delta \tilde{\theta})^4 + \frac{1}{6!} (\Delta \tilde{\theta})^6 - \cdots \right] \quad (A.29)$$
The strategy for working with higher order resonances or islands is to find a set of action-angle coordinates for the unperturbed system, $\Delta H_0$, so that the methods of Lichtenberg and Lieberman may again be applied. The first step is to find a set of action-angle coordinates. Consider the generating function:

$$F_i = \frac{1}{2} R (\Delta \tilde{\theta}_1)^2 \cot \theta$$  \hspace{1cm} (A.30)

where $R = \sqrt{\frac{F}{G}}$ which gives the canonical transformation:

$$\theta = \sin^{-1} \left( \sqrt{\frac{R}{2J}} \Delta \tilde{\theta}_1 \right), \quad \Delta \tilde{\theta}_1 = \sqrt{\frac{2J}{R}} \sin \theta$$  \hspace{1cm} (A.31)

$$J = \left[ \frac{1}{2} G (\Delta \tilde{J}_1)^2 + \frac{1}{2} F (\Delta \tilde{\theta}_1)^2 \right] (FG)^{-1}, \quad \Delta \tilde{J}_1 = \sqrt{2RJ} \cos \theta$$  \hspace{1cm} (A.32)

then clearly $\Delta H_0 = \sqrt{FG}J$. Here is the new set of action-angle coordinates. Directly substituting for $\Delta \tilde{\theta}_1$ gives for the perturbation:

$$\Delta H_1 = -F \frac{1}{4!} \left( \frac{2J}{R} \right)^2 \sin^4 \theta + F \frac{1}{6!} \left( \frac{2J}{R} \right)^3 \sin^6 \theta - \cdots$$  \hspace{1cm} (A.33)

Averaging over $\theta$ leaves for the perturbation:

$$\Delta \tilde{H}_1 = \langle \Delta H_1 \rangle_\theta = \sqrt{FG}J - F \left( \frac{2J}{R} \right)^2 \langle \sin^4 \theta \rangle$$  \hspace{1cm} (A.34)

and because $\langle \sin^4 \theta \rangle = \frac{3}{8}$:

$$\Delta \tilde{H}_1 = \sqrt{FG}J - \frac{GJ^2}{16}$$  \hspace{1cm} (A.35)

This system is independent of the angle variable. It is now possible to analyze the remaining terms of the perturbation, $H'_i = \tilde{H}_i - \tilde{H}_1$. Recall that:

$$\tilde{H}_i = \varepsilon \sum_{i,m} H_{i,m} \cos \left[ \frac{1}{r} \left( \Delta \tilde{\theta}_1 + (ls - mr) \tilde{\theta}_2 \right) \right]$$  \hspace{1cm} (A.36)

and,

$$\tilde{H}_i = \varepsilon \sum_{p} H_{pr,pt} \cos \left( p \tilde{\theta}_1 \right)$$  \hspace{1cm} (A.37)
so that:

$$H'_1 = \varepsilon \sum_{l,m} H_{l,m} \cos\left[\frac{1}{r} \left( l \tilde{\theta}_1 + (l s - m r) \tilde{\theta}_2 \right) \right]$$ \hspace{1cm} (A.38)

where the prime on the summation means that it is to be performed over the set of integer pairs \(\{(l, m) | l s + m r \neq 0\}\). Expanding about the elliptic fixed points gives:

$$H'_1 = \varepsilon \sum_{l,m} H_{l,m} \left( J_{10} + \Delta J_1 \right) \cos\left[\frac{l}{r} \Delta \tilde{\theta}_1 + \left( \frac{l}{r} s - m \right) \tilde{\theta}_2 \right]$$ \hspace{1cm} (A.39)

Taking \(\Delta J_1\) small, it is possible to make the approximation:

$$H_{l,m} \left( J_{10} + \Delta J_1 \right) = H_{l,m} \left( J_{10} \right)$$ \hspace{1cm} (A.40)

The system then takes the form:

$$K_0 = \sqrt{FG} \frac{J - GJ^2}{16}, \hspace{0.5cm} K_1 = H'_1 = \varepsilon \sum_{l,m} H_{l,m} \left( J_{10} \right) \cos\left[\frac{l}{r} \Delta \tilde{\theta}_1 + \left( \frac{l}{r} s - m \right) \tilde{\theta}_2 \right]$$ \hspace{1cm} (A.41)

Replacing \(\Delta \tilde{\theta}_1\) with the new action-angle variables gives:

$$K_1 = \varepsilon \sum_{l,m} H_{l,m} \left( J_{10} \right) \cos\left[\frac{l}{r} \sqrt{\frac{2J}{R}} \sin \theta + \left( \frac{l}{r} s - m \right) \tilde{\theta}_2 \right]$$ \hspace{1cm} (A.42)

and using the Bessel function expansion for the cosine (3.21) gives:

$$K_1 = \varepsilon \sum_{l,m,n} \alpha_{m} \frac{J_{l}}{J_{l}} \left( \sqrt{2J} \overline{J_{10}} \right) J_{n} \left( \frac{l}{r} \sqrt{\frac{2J}{R}} \right) \cos\left[ n \theta + \left( \frac{s}{r} - m \right) \tilde{\theta}_2 \right]$$ \hspace{1cm} (A.43)

The frequencies of the system are:

$$\tilde{\omega}_2 = \tilde{\theta}_2 = \dot{\tilde{\omega}}_2 = 1, \hspace{0.5cm} \tilde{\omega}_1 = \dot{\tilde{\omega}}_1 = \frac{\partial K_0}{\partial J} = \frac{\sqrt{FG} - GJ}{8}$$ \hspace{1cm} (A.44)

Take a resonance of the form:

$$\tilde{\omega}_2 = p \tilde{\omega}_1 \hspace{1cm} (A.45)$$

then consider the generating function:

$$F_2 = (p \theta - q \tilde{\theta}_2) J_1$$ \hspace{1cm} (A.46)
This gives the canonical transformation:

\[ \phi_1 = p\theta - q\tilde{\theta}_2, \quad pI_1 = J, \quad K' = K - qI_1 \quad (A.47) \]

The unperturbed system takes the form:

\[ K_0 = p\sqrt{FG}I_1 - \frac{Gp^2I_1^2}{16} - qI_1 \quad (A.48) \]

But the resonance condition implies that:

\[ q = \tilde{\omega}_1, \quad p = \sqrt{FG}p - p^2 \frac{GJ}{8} \quad (A.49) \]

Notice this condition depends on the non-linearity parameter since both \( F \) and \( G \) depend linearly on \( \varepsilon \). Plugging equation (A.49) into (A.48) gives:

\[ K_0 = \frac{p^2GI_1^2}{16} \quad (A.50) \]

The perturbed system takes the form:

\[ K_1 = \varepsilon \sum_{j,m,n} \alpha_m J_j \left( \sqrt{2rTJ_{10}} \right) J_n \left( \frac{l}{r} \sqrt{\frac{2pl}{J}} \right) \cos \left( \frac{nq}{p} + \left( \frac{nq}{p} + \frac{r}{m} - m \right) \phi_2 \right) \quad (A.51) \]

Again taking a time average leaves only the cosine terms independent of \( \tau = \tilde{\theta}_2 = \phi_2 \).

This requires that:

\[ nq = p \left( m - \frac{r}{s} \right) \quad (A.52) \]

which is satisfied when \( n = jp, \quad l = kr, \quad m = jq + ks \) for \( j, k \in \mathbb{Z} \). The averaged perturbation takes the form:

\[ \bar{K}_1 = \langle K_1 \rangle_{q_2} = \varepsilon \sum_{j,k} \alpha_{jk} J_{jk} \left( \sqrt{2rTJ_{10}} \right) J_{jk} \left( k \sqrt{\frac{2pl}{J}} \right) \cos \left( j\phi_1 \right) \quad (A.53) \]

Keeping the slowly oscillating terms gives:

\[ \bar{K}_1 = \varepsilon \sum_{k>0} \alpha_{k} J_{k} \left( \sqrt{2rTJ_{10}} \right) + \varepsilon \sum_{k>0} \alpha_{q+k} J_{k} \left( \sqrt{2rTJ_{10}} \right) J_{p} \left( k \sqrt{\frac{2pl}{J}} \right) \cos \phi_1 \quad (A.54) \]
To simplify notation, define:

\[ K_{0,0} = \sum_{k \neq 0} \alpha_{k} J_{kr} \left( \sqrt{2rT_{10}} \right), \quad K_{1,1} = \sum_{k} \alpha_{k+\theta} J_{kr} \left( \sqrt{2rT_{10}} \right) J_{r} \left( k \sqrt{\frac{2p_{10}}{R}} \right) \]  

(A.55)

Then, the system takes the form:

\[ \overline{K} = \frac{p^{2}G l^{2}}{16} + \varepsilon K_{0,0} + \varepsilon K_{1,1} \cos \phi_{1} \]  

(A.56)

The fixed points are found using Hamilton's equations:

\[ \frac{\partial K}{\partial I_{10}} = 0 = \frac{pG l_{10}}{8} + \varepsilon \frac{\partial K_{1,1}}{\partial I_{10}} \cos \phi_{10} \]  

(A.57)

\[ \frac{\partial K}{\partial \phi_{10}} = 0 = \varepsilon K_{1,1} \sin \phi_{10} \]  

(A.58)

Notice that \( \dot{\phi}_{1} = O(1) \), this is the case of accidental degeneracy. As before, the elliptic points occur when \( \phi_{10} = 0, \pi \). Hence, the action at an elliptic fixed point satisfies the equation:

\[ \frac{pG l_{10}}{8} + \varepsilon \sum_{k} k \sqrt{\frac{p}{2l_{10}R}} \alpha_{k+\theta} J_{k} \left( \sqrt{2rT_{10}} \right) J'_{r} \left( k \sqrt{\frac{2p_{10}}{R}} \right) = 0 \]  

(A.59)

Expanding about the elliptic fixed points with \( I_{1} = I_{10} + \Delta I_{1}, \phi_{1} = \phi_{10} + \Delta \phi_{1} \) gives:

\[ \overline{K}_{0} = \frac{p^{2}G l_{10}^{2}}{16} + \frac{p^{2}G l_{10}^{2}}{8} \Delta I_{1} + \frac{p^{2}G}{8} \left( \Delta I_{10} \right)^{2} \]  

(A.60)

\[ \overline{K}_{1} = \varepsilon K_{0,0} + \left[ \varepsilon K_{1,1} \left( I_{10} \right) + \varepsilon \frac{\partial K_{1,1}}{\partial I_{10}} \right] \Delta I_{1} \cos \phi_{1} \]  

(A.61)

But, from the fixed point condition, \( \frac{p^{2}G l_{10}^{2}}{8} \) is on the order of second order island widths so that it may be ignored, and if the constant terms are also ignored, to lowest order the system takes the form:

\[ \overline{K} = \frac{p^{2}G}{8} \left( \Delta I_{10} \right)^{2} + \varepsilon K_{1,1} \left( I_{10} \right) \cos \phi_{1} \]  

(A.62)
Defining:

\[ A = \frac{p^2 G}{4} = \varepsilon \alpha_s \frac{p^2 r T}{8 J_{10}} J_r' \left( \sqrt{2 r T J_{10}} \right) \]  

(A.63)

\[ B = -\varepsilon \sum_k \alpha_{q+kr} J_{kr} \left( \sqrt{2 r T J_{10}} \right) J_r \left( k \sqrt{\frac{2 p I_{10}}{R}} \right) \]  

(A.64)

where \( R \) has been defined as \( R = \frac{2 J_{10} j_r \left( 2 r T J_{10} \right)}{r T J_r' \left( \sqrt{2 r T J_{10}} \right)} \), the system has the simple form:

\[ \Delta K = \frac{1}{2} A (\Delta I_1)^2 - B \cos \theta_1 \]  

(A.65)

again that of a pendulum. The island widths are now described as:

\[ \Delta I = 2 \sqrt{\frac{B}{A}} = \sqrt{\frac{32 J_{10} \sum \alpha_{q+kr} J_{kr} \left( \sqrt{2 r T J_{10}} \right) J_r \left( k \sqrt{\frac{2 p I_{10}}{R}} \right)}{\alpha_s r^2 r T J_r' \left( \sqrt{2 r T J_{10}} \right)}} \]  

(A.66)

and the island separations are the separation of the fixed points, solutions to:

\[ \varepsilon \alpha_s \frac{r T p I_{10}}{16 J_{10}} J_r' \left( \sqrt{2 r T J_{10}} \right) + \varepsilon \sum_k \alpha_{q-kr} J_{kr} \left( \sqrt{2 r T J_{10}} \right) J_r' \left( k \sqrt{\frac{2 p I_{10}}{R}} \right) = 0 \]  

(A.67)

In these equations, \( r \) and \( s \) are determined by the resonance condition for first order islands, (A.7), \( p \) and \( q \) are determined for the resonance condition for second order islands, (A.49), \( I_{10} \) is determined by (A.59) and \( J_{10} \) is determined by (A.17). Notice that (A.67) is a very difficult equation to solve. This equation can be simplified depending on the choice of harmonic amplitudes. In particular, choosing the amplitudes to be non-zero for only a finite number of harmonics will terminate the infinite summation in this equation.

There are still two problems with the higher order separations. The first problem is the existence of \( \alpha \) terms in the expressions. This is the reason that this approach could not be used to determine a threshold for the system with only one perturbing wave. The second problem is that these terms are again independent of \( \varepsilon \). Although this is true
superficially, closer inspection reveals that there is an $\varepsilon$ dependence for $p$ and $q$ in the resonance condition (A.49). Expanding this condition using the values of $F$ and $G$ give:

$$q = \varepsilon\alpha \frac{rT}{2J_0} J \left( \sqrt{2rTJ_0} \right) J' \left( \sqrt{2rTJ_0} \right) p - p^2 \varepsilon\alpha, \quad \frac{rT}{2J_0} \frac{J'' \left( \sqrt{2rTJ_0} \right)}{8} J$$

(A.50)

This solves the independence of the Chirikov condition on $\varepsilon$. 
Appendix B

Reducing the Summation in Chapter 6

B.1 Reducing the Summation

The point in this section is to argue the validity of equation (6.5). To do this, we need to prove that there exists a finite bounded set, $\Delta n$, such that:

$$
\frac{T}{\rho} \sum_{m} J'_m(\rho) \cos(m\psi - \tau) = \frac{T}{\rho} \sum_{m \in \Delta n} J'_m(\rho) \cos(m\psi - \tau) \tag{B.1}
$$

We recall the Bessel function identity of (3.25):

$$
\exp i(\rho \sin \psi - \tau) = \sum_{m} J_m(\rho) \exp i(m\psi - \tau) \tag{B.2}
$$
and taking the derivatives of both sides with respect to action as well as taking the real parts gives:

\[
\frac{T}{\rho} \sin \psi \cos(\rho \sin \psi - \tau) = \frac{T}{\rho} \sum_m J'_m(\rho) \cos(m\psi - \tau) \tag{B.3}
\]

Notice that for \( \rho \neq 0 \), it is clear that \( \frac{T}{\rho} \sin \psi \cos(\rho \sin \psi - \tau) \) is well defined for all values of \((I, \psi, \tau)\) so it must be the case that \( \frac{T}{\rho} \sum_m J'_m(\rho) \cos(m\psi - \tau) \) converges for all \((I, \psi, \tau)\). This means that for any \( \varepsilon > 0 \) there exists \( N \) such that:

\[
\sum_{|m| > N} J'_m(\rho) \cos(m\psi - \tau) < \varepsilon \tag{B.4}
\]

This is a mathematical result. Unfortunately, we do not have uniform convergence. This means that we can find such an \( N \) for any \( \rho \), but we cannot find one such that it is true for all \( \rho \). We fix \( \rho \) and take \( \varepsilon = \varepsilon^2 \) and find the corresponding value of \( N \). This is enough to terminate the summation. Now, what happens as \( \rho \) increases? There are three cases to consider:

i) for \( m \ll \rho \), \( J'_m(\rho) = \sqrt{\frac{2}{\pi \rho}} \sin \left( \rho - \frac{m\pi}{2} - \frac{\pi}{4} \right) \) which decreases for increasing \( \rho \).

ii) for \( m \approx \rho \), \( J_m(\rho) = \frac{1}{\Gamma \left( \frac{2}{3} \right)} \left( \frac{2}{9\rho} \right)^{1/3} \), where \( \Gamma(x) \) is the gamma function, and the contribution of these terms decreases for increasing \( \rho \). Hence, the contribution due to \( J'_m(\rho) = -J_{m+1}(\rho) + \frac{m}{\rho} J_m(\rho) \) decreases for increasing \( \rho \).

iii) for \( m \gg \rho \), \( J_m(\rho) \approx \left( \frac{1}{2\rho} \right)^m \frac{1}{m!} \approx \left( \frac{\rho}{m} \right)^m \) which is negligible always and again so too is \( J'_m(\rho) \).

These expansions may be found in Abramowitz and Stegun,\(^9\) and they show that if \( \rho \) is increased enough, the original value of \( N \) is still valid to bound our summation. This was
our goal. So, it is valid to approximate the frequency for large \( \rho \) by:

\[
\dot{\psi} = \frac{1}{T} + \varepsilon \frac{T}{\rho} \sum_{m \in \Delta n} J'_m(\rho) \cos(m\psi - \tau)
\]  \hspace{1cm} (B.5)

where \( \Delta n = \{m, |m| < N\} \), a finite bounded set.
Appendix C

A Complete Collection of Data

Comparing Karney's Bound to that of Chapter 6

C.0 Introduction

In this appendix, we present all of the data we collected when comparing Karney's bound, (5.8) and (5.17), to that of Chapter 6, (6.14). Each figure compares the bounds in action as a function of the non-linearity parameter derived by Karney which are represented by the dashed curves, to the bound derived in Chapter 6, to the numerical data collected using the techniques described in Chapter 7. The figures are ordered by frequency ratio starting with a ratio of 5.0 and ending with a ratio of 35.4.
Figure C.1: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.2: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.3: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.4: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.5: A comparison of Karney’s bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney’s bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.6: A comparison of Karney’s bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney’s bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.7: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.8: A comparison of Karney’s bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney’s bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.9: A comparison of Karney’s bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney’s bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.10: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.11: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.12: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.13: A comparison of Karney’s bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney’s bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.14: A comparison of Karney’s bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney’s bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.15: A comparison of Karney’s bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney’s bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.16: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.17: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.19: A comparison of Karney’s bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney’s bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.20: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.21: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.22: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.23: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.24: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.25: A comparison of Karney’s bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney’s bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.26: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.27: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.28: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.29: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.30: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.31: A comparison of Karney’s bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney’s bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.32: A comparison of Karney’s bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney’s bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.33: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.34: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.35: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Figure C.36: A comparison of Karney's bounds in action on the region of stochasticity, (5.8) and (5.17), to that derived in Chapter 6, (6.14). The plots compare the predicted region of stochasticity in action as a function of the non-linearity parameter against the numerical data. Karney's bounds are represented by the dashed curves, while that in Chapter 6 is represented by the solid curve. The numerical data is represented by the points.
Appendix D
Numerical Code

D.0 Introduction

In this appendix, we present the code used to collect data for the system. The code was written by Dr. Ram and there are two main programs. The first is used to numerically integrate the equations of motion for the full system and the second is used to make phase space plots. The programs are written in FORTRAN and are compiled with cft77 available on the cray machines. Together, the programs take as input a set of initial conditions in action-angle phase space, the frequency of the plasma wave, the cyclotron frequency and the non-linearity parameter, and the output is an action-angle phase space plot.
C# LINEAR OSCILLATOR PERTURBED BY ONE WAVE

C# (ALL THIS IS IN THE NORMALIZED SET OF COORDINATES)

C#

OUTPUTS Q, P WHICH ARE TRANSLATED TO

C#

I, PSI IN THE PLOTTING ROUTINE

C#

C#

real q(101),p(101),act(51),ang(51)
real arg(101),c(101),s(101),xi(101)
real q1(101),q2(101),q3(101),q4(101)
real q5(101),q6(101),q7(101),q8(101)
real p1(101),p2(101),p3(101),p4(101)
real p5(101),p6(101),p7(101),p8(101)

C#

C............. normalized amplitude of the perturbing wave ..........
C............ oscillator frequency and the wave frequency ............
C...... (in the program the ratio of the two is taken since the ......
C..... oscillator frequency is normalized to the wave frequency) ..... namelist /input/ eps,wosc,wwave
C
C... initial array of actions; number of actions to be considered .... namelist /acti/ act,nact
C
C.... initial array of angles; number of angles to be considered ..... namelist /phas/ ang,nang
C
C................ number of steps in one two-pi period ..............
C... number of time steps later when orbit values should be stored ...
C....... for surface of sections ntime should equal ndtpi2 .......... namelist /info/ ndtpi2,ntime,norb
C
C............. kpr = 1 ==> print information (otherwise store) .......... namelist /outd/ kpr
C
C open(3,file='wavi')
C open(7,file='wavo')
C open(4,file='wavdat',form='unformatted')
C
C read (3,input)
C read (3,acti)
C read (3,phas)
C read (3,info)
C read (3,outd)
C
C write (7,input)
C write (7,acti)
C write (7,phas)
C write (7,info)
C
pi=3.141592653589793238462643
twopi=6.283185307179586476925287
normalized oscillator frequency

\[ w_0 = \frac{\omega_{osc}}{\omega_{wave}} \]

total number of particles (or initial conditions)

\[ npart = nact \times nang \]

write (4) \( \epsilon, \omega_{osc}, \omega_{wave} \)
write (4) \( npart, \text{norb}, w_0 \)

write (7,101) \( w_0 \)

converting the input action and angle data into \((q,p)\)

\[ \text{ipart} = 0 \]
\[ \text{do } 1 \text{ nacton}=1, nact \]
\[ \text{action} = \text{act}(\text{nacton}) \]
\[ \text{do } 2 \text{ nangle}=1, nang \]
\[ \text{angle} = \text{ang}(\text{nangle}) \times 2\pi \]
\[ \text{ipart} = \text{ipart} + 1 \]
\[ q(\text{ipart}) = \sqrt{2 \times \text{action} / w_0} \times \sin(\text{angle}) \]
\[ p(\text{ipart}) = \sqrt{2 \times \text{action} \times w_0} \times \cos(\text{angle}) \]

\[ \text{2 continue} \]
\[ \text{1 continue} \]

time step for each integration

\[ dt = \frac{2\pi}{\text{float}(n\text{dpi2})} \]

parameters for the Taylor's series expansion

\[ c_2 = 0.5 \]
\[ c_3 = 1/6. \]
\[ c_4 = 1/24. \]
\[ c_5 = 1/120. \]
\[ c_6 = 1/720. \]
\[ c_7 = 1/5040. \]
\[ c_8 = 1/40320. \]

if \((kpr.eq.1)\) then
\[ \text{do } 3 \text{ i}=1, npart \]
\[ \text{write (7,102) i,q(i),p(i)} \]
\[ 3 \text{ continue} \]
else
\[ \text{do } 4 \text{ i}=1, npart \]
\[ \text{write (4) q(i),p(i)} \]
\[ 4 \text{ continue} \]
end if

all orbits have been set to start at time zero

t = -dt

\[ \text{do } 5 \text{ iorb}=1, \text{norb} \]
\[ \text{do } 6 \text{ itime}=1, \text{ntime} \]
t = t + dt

C
do 7 np = 1, npart

C
q1(np) = p(np)

C
arg(np) = amod(q(np) - t, twopi)
c(np) = eps * cos(arg(np))
s(np) = eps * sin(arg(np))
xi(np) = q1(np) - 1.

C
p1(np) = q(np) * w0**2 + s(np)

C
q2(np) = p1(np)
p2(np) = q1(np) * w0**2 + c(np) * xi(np)

C
q3(np) = p2(np)
p3(np) = q2(np) * w0**2 - s(np) * xi(np) ** 2 +
       c(np) * q2(np)

C
q4(np) = p3(np)
p4(np) = q3(np) * w0**2 - c(np) * xi(np) ** 3 -
       3. * q2(np) * s(np) * xi(np) +
       c(np) * q3(np)

C
q5(np) = p4(np)
p5(np) = q4(np) * w0**2 + s(np) * xi(np) ** 4 -
       6. * c(np) * q2(np) * xi(np) ** 2 -
       4. * q3(np) * s(np) * xi(np) -
       3. * q2(np) ** 2 * s(np) +
       c(np) * q4(np)

C
q6(np) = p5(np)
p6(np) = q5(np) * w0**2 + c(np) * xi(np) ** 5 +
       10. * q2(np) * s(np) * xi(np) ** 3 -
       10. * c(np) * q3(np) * xi(np) ** 2 +
       (-15. * c(np) * q2(np) ** 2 -
       5. * q4(np) * s(np)) * xi(np) -
       10. * q2(np) * q3(np) * s(np) +
       c(np) * q5(np)

C
q7(np) = p6(np)
p7(np) = q6(np) * w0**2 - s(np) * xi(np) ** 6 +
       15. * c(np) * q2(np) * xi(np) ** 4 +
       20. * q3(np) * s(np) * xi(np) ** 3 +
       (45. * q2(np) ** 2 * s(np) -
       15. * c(np) * q4(np)) * xi(np) ** 2 +
       (-60. * c(np) * q2(np) * q3(np) -
       6. * q5(np) * s(np)) * xi(np) -
       15. * c(np) * q2(np) ** 3 +
       (-15. * q2(np) * q4(np) - 10. * q3(np) ** 2) *
       s(np) + c(np) * q6(np)
\begin{verbatim}
q8(np)=p7(np)
p8(np)=-q7(np)*w0**2-c(np)*xi(np)**7+
  21.*q2(np)*s(np)*xi(np)**5+
  35.*c(np)*q3(np)*xi(np)**4+
  (105.*c(np)*q2(np)**2+21.*q2(np)*s(np)*xi(np)**5+
  35.*c(np)*q3(np)*xi(np)**4+
  (105.*c(np)*q2(np)**2*s(np)+
  (-105.*c(np)*2q2(np)*q4(np)-
  70.*c(np)*q2(np)*q4(np)-
  7.*q6(np)*s(np)*xi(np)-
  105.*c(np)*q2(np)**2*q3(n)+
  (-21.*q2(np)*q5(n)-35.*q3(np)*q4(np)))*
  s(np)+c(np)*q7(np)

C
q(np)=q(np)+q 1l(np)*dt+
  co2*q2(np)*dt**2+co3*q3(np)*dt**3+
  co4*q4(np)*dt**4+co5*q5(np)*dt**5+
  co6*q6(np)*dt**6+co7*q7(np)*dt**7+
  co8*q8(np)*dt**8
p(np)=p(np)+p 1(np)*dt+
  co2*p2(np)*dt**2+co3*p3(np)*dt**3+
  co4*p4(np)*dt**4+co5*p5(np)*dt**5+
  co6*p6(np)*dt**6+co7*p7(np)*dt**7+
  co8*p8(np)*dt**8

7 continue
C
6 continue
C
if (kpr.eq.1) then
  do 8 i=l,npart
    write (7,102) i,q(i),p(i)
  8 continue
else
  do 9 i=1,npart
    write (4) q(i),p(i)
  9 continue
end if
C
5 continue
C
101 format(1x,"normalized wave frequency = ",g18.10)
102 format(1x,"particle = ",i3," (x, v) = (",g18.10,"",g18.10,"))")
C
C
stop
end
C
C
\end{verbatim}
THIS PROGRAM IS FOR PLOTTING THE STUFF CREATED BY THE FILE WEBSTOC.F FOR THE LINEAR OSCILLATOR INTERACTING WITH SINGLE PLANE WAVE

PLOTS ACTION VERSUS ANGLE (NORMALIZED TO 2*PI)

character*60 string(1)

real q(201),p(201)
real act(201),ang(201)

kax = 1 ==> xmin, xmax in the graphs are given by xmin0, xmax0 ...
otherwise default values in the program are used ...
kay = 1 ==> ymin, ymax in the graphs are given by ymin0, ymax0 ...
= 2 ==> set ymin=0. and use default value for ymax ...
otherwise default values in the program are used ...
namelist /xax/ kax,xmin0,xmax0
namelist /yax/ kay,ymin0,ymax0

open(4,file='wavdat',form='unformatted')
open(5,file='pltdat')
open(7,file='pltin')

read (7,xax)
read (7,yax)

pi=3.141592653589793238462643
twopi=6.283185307179586476925287

if (kay.eq.1) then
  ymin=ymin0
  ymax=ymax0
if (kax.eq.1) then
  xmin=xmin0
  xmax=xmax0
else
  xmin=0.
  xmax=1.
end if

read (4) eps,wosc,wave
read (4) npart,norb,w0

do 1 i=1,npart
  read (4) q(i),p(i)
  act(i)=0.5*w0*(q(i)**2+(p(i)/w0)**2)
1 continue
if (kax.eq.0) then
  xmin=0.
  xmax=1.
else
  xmin=xmin0
  xmax=xmax0
end if
ymin=act(1)
ymax=act(1)
C
do 2 i=2,npart
  ymin=amin1(ymin,act(i))
  ymax=amax1(ymax,act(i))
2 continue
C
do 3 i=1,norb
  do 4 j=1,npart
    read (4) q(j),p(j)
    act(j)=0.5*wO*(q(j)**2+(p(j)/wO)**2)
    ymin=amin1(ymin,act(j))
    ymax=amax1(ymax,act(j))
  4 continue
3 continue
C
if (kay.eq.2) ymin=0.
C
rewind (4)
C
8 call ncarcgm(1,'web.plot')
C
write (5,101) xmin,xmax
write (5,102) ymin,ymax
101 format("xmin, xmax = ",2g18.10)
102 format("ymin, ymax = ",2g18.10)
C
call maps(xmin,xmax,ymin-0.005*(ymax-ymin),ymax,  
% -1.0,-1.0,-1.0,-1.0)
call dders(-1)
C
read (4) eps,wosc,wave
read (4) npart,norb,w0
C
narr=npart
do 5 i=0,norb
  do 6 j=1,npart
    read (4) q(j),p(j)
    act(j)=0.5*wO*(q(j)**2+(p(j)/wO)**2)
    if (p(j).eq.0.) then
      if (q(j).ge.0) then
        ang(j)=(0.5*pi)/twopi
      else
        ang(j)=(1.5*pi)/twopi
      end if
    go to 7
  6 continue
5 continue
end if

psi = atan(abs(wO*q(j)/(p(j))))
if (p(j).gt.0.) then
  if (q(j).ge.0.) then
    ang(j) = psi/twopi
  else
    ang(j) = (twopi-psi)/twopi
  end if
else
  if (q(j).ge.0.) then
    ang(j) = (pi-psi)/twopi
  else
    ang(j) = (pi+psi)/twopi
  end if
end if

continue

if ((ang(j).lt.xmax).and.(ang(j).gt.xmin)) then
  if ((act(j).lt.ymax).and.(act(j).gt.ymin)) then
    call points(ang(j), act(j), 1, -1, -1, 0., 0.)
  end if
end if

continue

continue

call setlch(xmin,ymin-0.2*(ymax-ymin+1),0,2,0,-1)
write (string, 103) eps, wosc, wave
103 format("eps, wO, w = ",g12.5,1x,g12.5,1x,g12.5)
call wrtstr(string,1)

C............... for terminating the plots .................
C
call plote()
C
C
stop
end

C

C*******************************************************************************
C
C*******************************************************************************
C
C*******************************************************************************
C
C*******************************************************************************
Bibliography


