A THEORY OF NONLINEAR SYSTEMS

by

Amar Gopal Bose

S. B., Massachusetts Institute of Technology
(1952)

S. M., Massachusetts Institute of Technology
(1952)

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF SCIENCE
at the
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
June, 1956

Signature of Author_____
Department of Electrical Engineering, May 14, 1956

Certified by_____
Thesis Supervisor

Accepted by_____
Chairman, Departmental Committee on Graduate Students

A THEORY OF NONLINEAR SYSTEMS

by

Amar Gopal Bose

Submitted to the Department of Electrical Engineering on May 14, 1956, in partial fulfillment of the requirements for the degree of Doctor of Science.

ABSTRACT

The first section of this thesis describes the Wiener theory of nonlinear system characterization and discusses some of its important concepts. Following the lines of this theory a theory is developed for the experimental determination of optimum time-invariant nonlinear systems. The systems are optimum in a weighted mean square sense in which the weighting function is at our disposal.

The design of nonlinear systems is regarded as the problem of mapping the function space of the past of the input onto a line that corresponds to the amplitude of the filter output. By choosing a series expansion for this mapping operation that partitions the function space into non-overlapping cells, an orthogonal representation for nonlinear systems is obtained that leads to convenient apparatus for the determination of optimum systems. General methods are described for applying this theory to determine systems having a performance that is superior to that of given linear or nonlinear systems. A criterion is established relative to which two systems are defined as "nearly equivalent" and the approximation of nonlinear systems by linear and simple nonlinear ones is discussed. The theory is extended to include the problem of multiple nonlinear prediction and apparatus for the determination of optimum predictors is indicated.

Thesis Supervisor: Y. W. Lee
Title: Associate Professor of Electrical Engineering

Acknowledgment

The author wishes to express his deep appreciation to Professor Y. W Lee, who first suggested a study of the Wiener theory of nonlinear system characterization and supervised the research that resulted from this study. The constructive criticism of Professor Lee in the research and in the writing of this report has led the work to its present form. The author is also indebted to Professor Norbert Wiener, who has been a source of inspiration through his explanations and discussions of his theory.

Contents

Introduction

A physically realizable nonlinear system, like a linear one, is a system whose present output is a function of the past of its input. We may regard the system as a computer that operates on the past of one time function to yield the present value of another time function. Mathematically we say that the system performs a transformation on the past of its input to yield its present output. When this transformation is linear (the case of linear systems) we can take advantage of the familiar convolution integral to obtain the present output from the past of the input and the system is said to be characterized by its response to an impulse. That is, the response of a linear system to an impulse is sufficient to determine its response to any input. When the transformation is nonlinear we no longer have a simple relation like the convolution integral relating the output to the past of the input and the system can no longer be characterized by its response to an impulse since superposition does not apply. Wiener has shown, however, that we can characterize a nonlinear system by a set of coefficients and that these coefficients can be determined from a knowledge of the response of the system to shot noise excitation. Thus, shot noise occupies the same position as a probe for investigating nonlinear systems that the impulse occupies as a probe for investigating linear systems. The first section of this thesis is devoted to the Wiener theory of nonlinear system characterization. Emphasis is placed on important concepts of this theory that are used in succeeding chapters to develop a theory for determining optimum nonlinear systems.

I. The Wiener Theory of Nonlinear System Characterization and Synthesis

1.1 General Remarks

The objectives of Wiener's method are: to obtain a set of coefficients which characterize a time-invariant nonlinear system, and to present a procedure for synthesizing the system from a knowledge of its characterizing coefficients. An operator relating the output to the past of the input of a nonlinear system is defined in such a way that the characterizing coefficients can be evaluated experimentally.

The method is confined to those nonlinear systems whose present behavior depends less and less upon the remote past of the input as we push this past back in time. More precisely, attention is restricted to those systems whose present output is influenced to an arbitrarily small extent by that portion of the past of the input beyond some arbitrarily large but finite time. Further, we shall restrict our attention to those nonlinear systems that operate on continuous time functions to yield continuous time functions. This is clearly no physical restriction since physical time functions, though they may change very rapidly, are continuous. The reasons for these restrictions will become apparent in the development of the theory that follows.

According to Wiener the most general probe for the investigation of nonlinear systems is gaussian noise with a flat power density spectrum because there is a finite probability that this noise will, at some time, approximate any given time function arbitrarily closely over any finite time interval. Gaussian noise with a flat power density spectrum can be approximated by the output of a shot noise generator. Hence, if two

systems have the same response to shot noise they will have the same response for any input and we say that the systems are equivalent. The Wiener theory of nonlinear system classification is based on this property of the shot noise probe. A given system is characterized by exciting it with shot noise and measuring certain averages of products of its output with functions of the shot noise input which can be generated in the laboratory. The measured quantities are numerically equal to the coefficients in the Wiener nonlinear operator. Once these coeffieients are determined a system can be synthesized that yields the same response to shot noise as does the given system. Hence the two systems are equivalent.

Recognizing that the present output of a nonlinear system is a function of the past of its input, Wiener formulated his nonlinear operator by first characterizing the past of the time function on which it operates by a set of coefficients and then expressing the result of the operation (the system output) as an expansion of these coefficients.[1] In the development which follows we shall treat these problems separately; first the problem of characterizing the past of a time function by a set of coefficients and then the problem of expressing a nonlinear function of these coefficients.

1.2 Definitions

To simplify the description of the method, it is convenient at this point to define certain quantities and relations.

A. The nth Laguerre polynomial is defined as[2]

$$L_n(x) = \frac{1}{(n-1)!}\, e^x \frac{d^{(n-1)}}{dx^{(n-1)}} \left(x^{(n-1)} e^{-x}\right) \qquad n = 1, 2, \ldots$$

B. The normalized Laguerre functions $h_n(x)$ are defined as

$$h_n(x) = \begin{cases} e^{-x/2} L_n(x) & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{1}$$

The following orthogonality relation exists for these functions:

$$\int_0^\infty h_m(x)\, h_n(x)\, dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{2}$$

C. The nth Hermite polynomial is defined as[3]

$$F_n(x) = (-1)^{(n-1)} e^{x^2} \left(\frac{d}{dx}\right)^{(n-1)} e^{-x^2} \qquad n = 1, 2, 3, \ldots$$

D. The normalized Hermite polynomials $\eta_n(x)$ are defined as

$$\eta_n(x) = \frac{F_n(x)}{[2^{(n-1)}(n-1)!\ (\pi)^{1/2}]^{1/2}} \tag{3}$$

E. The normalized Hermite functions are defined

$$\psi_n(x) = e^{-x^2/2}\, \eta_n(x) \tag{4}$$

These functions form a normal orthogonal set over the interval $-\infty$ to ∞. Consequently we have the relation

$$\int_{-\infty}^\infty \eta_m(x)\, \eta_n(x)\, e^{-x^2}\, dx = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \tag{5}$$

1.3 Characterizing the Past of a Time Function

Given a time function x(t) our object is to determine a set of coefficients which characterize its past. The coefficients are said to characterize the

past of x(t) if we can construct this past from a knowledge of them. Let us here confine our attention to real time functions x(t) having the property

$$\int_{-\infty}^{\infty} x^2(t)\, dt < \infty$$

The past of such time functions can be expanded in a complete set of orthogonal functions. Further, from a knowledge of the coefficients of this expansion we can construct the time function almost everywhere.[4] Because of their realization as the impulse response of rather simple networks Wiener chose to expand the past of x(t) in terms of Laguerre functions. These functions form a complete set over the interval 0 to ∞ and have the orthogonality property indicated in Eq. (2). The expansion of the past of x(t) in terms of the Laguerre functions is

$$x(-t) = \sum_{n=1}^{\infty} u_n\, h_n(t) \qquad t \geq 0 \qquad (6)$$

where the present time is t = 0 and the u_n are the Laguerre coefficients of the past of x(t). Taking advantage of the orthogonality property of Eq. (2) we obtain the following expression for the u_n.

$$u_n = \int_0^{\infty} x(-t)\, h_n(t)\, dt \qquad (7)$$

These Laguerre coefficients are readily generated in practice as the outputs of a rather simple network whose input is x(t). This network, shown in Fig. 1, is called a Laguerre network. It is a constant impedance lossless ladder structure terminated in its characteristic impedance and

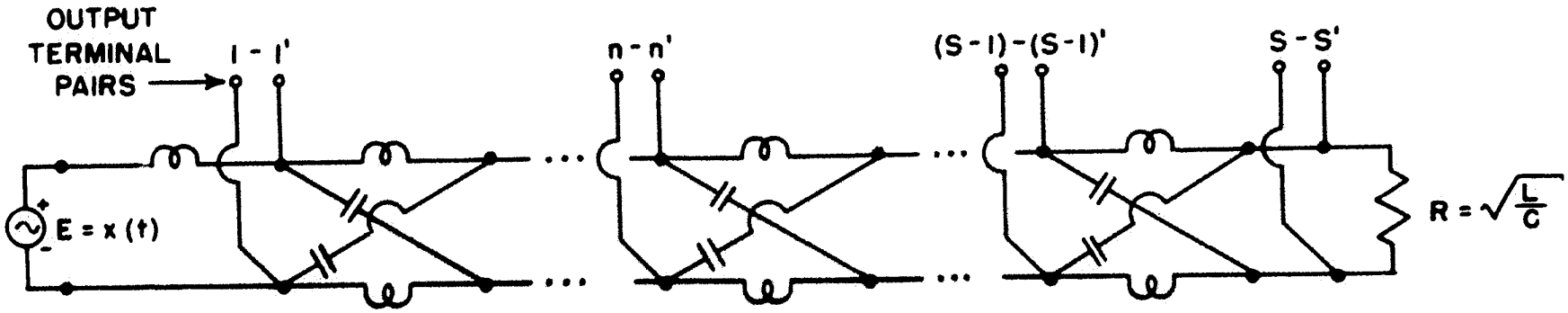

Fig. 1. LAGUERRE NETWORK
For R = L = C = 1 the impulse response at the nth output terminal pair is given by Eq. (1)

preceded by a series inductance. For a detailed description of Laguerre networks, their analysis and synthesis, see reference 2. For our purposes it is sufficient to know that the impulse response of the Laguerre network at the nth output terminal pair on open circuit is $h_n(t)$ for $n = 1, 2, 3, \ldots$. We must now show that if $x(t)$ is applied to the input of this network the output at the nth terminal pair at time $t = 0$ is the nth Laguerre coefficient u_n of the past of $x(t)$ up to the time $t = 0$. To this end we consider the block diagram of the Laguerre network shown in Fig. 2. For simplicity only the nth output terminal is shown. The network input is $x(t)$. Its output $r_n(t)$ is given by the convolution of $x(t)$ with $h_n(t)$. That is,

$$r_n(t) = \int_0^\infty x(t-\tau)\, h_n(\tau)\, d\tau$$

At time $t = 0$ the output is

$$r_n(0) = \int_0^\infty x(-\tau)\, h_n(\tau)\, d\tau \qquad (8)$$

But the right side of this equation is seen to be equivalent to the expression for u_n given in Eq. (7). Hence we see that if $x(t)$ is applied to the input of a Laguerre network the output of the nth terminal pair at time $t = 0$ is equal to the nth Laguerre coefficient of the past of $x(t)$ up to the time $t = 0$. In general, the output of the nth terminal pair of the Laguerre network at any time t is equal to the nth Laguerre coefficient of the past of the input up to the time t.

1.4 Properties of the Laguerre Coefficients of a Shot Noise Process

Since the probe for the investigation of nonlinear systems in the Wiener theory is shot noise it will be necessary in our development of this theory

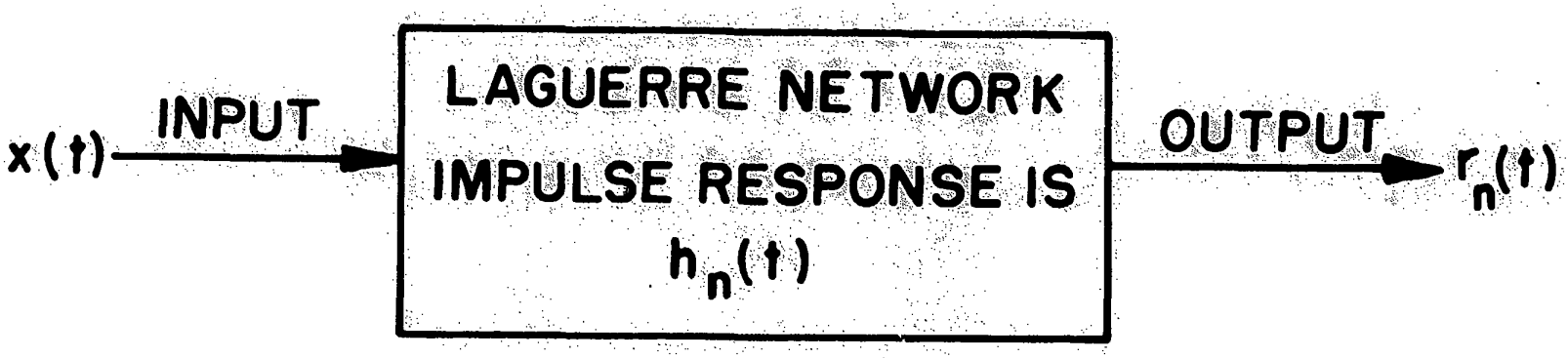

Fig. 2. Block diagram of Laguerre network showing nth output.

to make use of several properties of the Laguerre coefficients of a shot noise process.

When the input to a Laguerre network is shot noise the outputs (the Laguerre coefficients of the past of the shot noise input) have the following three properties of interest:

1. They are gaussianly distributed.
2. They are statistically independent.
3. They all have the same variance.

The first property follows from the well-known result that the response of a linear system to a gaussian input is gaussian[5] (recall that shot noise is a gaussian time function with a flat power density spectrum).

The second property is proved as follows: Consider the Laguerre functions $h_m(t)$ and $h_n(t)$. Let $H_m(\omega)$ and $H_n(\omega)$ be the Fourier transforms of $h_m(t)$ and $h_n(t)$ respectively. $H_m(\omega)$ and $H_n(\omega)$ are then the transfer functions from the input of the Laguerre network to the mth and nth output terminal pairs. The cross power density spectrum $\Phi_{nm}(\omega)$ of the mth and nth output time functions can be expressed as

$$\Phi_{nm}(\omega) = H_m(\omega)\, H_n^*(\omega)\, \Phi_{ii}(\omega) \tag{9}$$

where $\Phi_{ii}(\omega)$ is the input power density spectrum and the asterisk denotes the complex conjugate of $H_n(\omega)$.[6] The cross correlation function $\phi_{nm}(\tau)$ of these output time functions is given by the Fourier transform of $\Phi_{nm}(\omega)$ as follows:

$$\phi_{nm}(\tau) = \overline{f_n(t)\, f_m(t+\tau)} = \int_{-\infty}^{\infty} \Phi_{nm}(\omega)\, e^{j\omega\tau}\, d\omega \tag{10}$$

in which the bar indicates averaging with respect to time. Using Eq. (9),

Eq. (10) becomes

$$\phi_{nm}(0) = \overline{f_n(t)\, f_m(t)} = \int_{-\infty}^{\infty} H_m(\omega)\, H_n^*(\omega)\, \Phi_{ii}(\omega)\, d\omega \tag{11}$$

for $\tau = 0$.

If $\Phi_{ii}(\omega)$ is a constant, then Eq. (11) can be written

$$\phi_{nm}(0) = \overline{f_n(t)\, f_m(t)} = \Phi_{ii}(\omega) \int_{-\infty}^{\infty} H_m(\omega)\, H_n^*(\omega)\, d\omega \tag{12}$$

We now make use of the Parseval theorem to express the integral in Eq. (12) as follows:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} H_m(\omega)\, H_n^*(\omega)\, d\omega = \int_0^{\infty} h_n(t)\, h_m(t)\, dt \tag{13}$$

Using the orthogonality property of the Laguerre functions (Eq. 2) in (13) and (12) we have the result

$$\overline{f_n(t)\, f_m(t)} = \begin{cases} 2\pi\Phi_{ii}(\omega) & m = n \\ 0 & m \neq n \end{cases} \tag{14}$$

when $\Phi_{ii}(\omega)$ is a constant. Note that if $\Phi_{ii}(\omega)$ is a constant it can have no impulse at the origin and thus the input and output time functions of the Laguerre network must have zero means. Hence we have shown that the outputs of the Laguerre network are linearly independent when the power density spectrum of the input is flat. (Note that this is true whether or not the input time function is gaussian and that it also holds for any orthogonal set of networks, not only the Laguerre network.)

In the case of shot noise input the Laguerre coefficients are gaussian time functions (property 1 above) and linear independence implies statistical independence, proving property 2.

Property 3 can be proved by solving for the variance of the nth Laguerre coefficient in terms of the power density spectrum of the nth output of the network. However it can be seen very simply by recalling that the Laguerre network, except for its first series inductance, is a constant resistance lossless structure terminated in its characteristic resistance. If in Fig. 1 we look to the right at any of the output terminal pairs n-n we see the characteristic resistance of the network. Since the structure is lossless the same power flows through each section and since the impedance at each section is resistive and the same for each section the mean square value of every Laguerre coefficient is the same. For shot noise input the mean value of each coefficient is zero. Hence the variance $\sigma_n^2 = \overline{u_n^2(t)} - \overline{u_n(t)}^2$ is the same for all Laguerre coefficients. In particular if the level of the shot noise input to the network of Fig. 1 is properly adjusted all the Laguerre coefficients will have $\sigma^2 = 1$. In the development of the Wiener theory which follows we shall assume this to be the case.

In section 1.3 we restricted our attention to time functions that are squared integrable over the interval $-\infty$ to ∞. Then in the present section we speak of applying shot noise to the input of the Laguerre network. This is justified by the fact that the past of any physical time function that we can generate as an input to our Laguerre network is squared integrable since it starts at some time in the finite past.

Any practical application of the Wiener theory must of course use only a finite number of Laguerre coefficients to characterize the past of the system input. Since all the Laguerre functions decay exponentially (Eq. 1),

for any finite number of these functions there exists some time in the finite past such that the present outputs of the Laguerre network are influenced to an arbitrarily small extent by the behavior of the input prior to this time. That is, for all practical purposes the outputs of the Laguerre network are not cognizant of the past of the input beyond some finite time. Hence, as mentioned in section 1.1 the application of the Wiener theory is restricted to systems whose present output is influenced to an arbitrarily small extent by that portion of the past of the input beyond some arbitrarily large but finite time.

1.5 The Wiener Nonlinear Operator

Since the Laguerre coefficients characterize the past of a time function, any quantity dependent only on the past of this time function can be expressed as a function of these coefficients. Thus for the nonlinear system with input x(t) and output y(t) we can write

$$y(t) = F[u_1, u_2, \ldots, u_s, \ldots] \tag{15}$$

in which the u's are the Laguerre coefficients of x(t) at time t.

To put Eq. (15) in a more useful form we must choose an expansion for the function F of the Laguerre coefficients. These coefficients can take on any real value from $-\infty$ to ∞. The Hermite functions are chosen for the expansion because they form a complete orthonormal set over the interval $-\infty$ to ∞ and, as we shall see, are particularly adapted to a gaussian distribution. The expansion of Eq. (15) in terms of normalized Hermite functions which are defined in Eq. (4) reads

$$y(t) = \lim_{s\to\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \cdots \sum_{h=1}^{\infty} a_{i,j,\ldots,h}\, \eta_i(u_1)\, \eta_j(u_2) \cdots \eta_h(u_s)\, e^{-\frac{(u_1^2+u_2^2+\ldots+u_s^2)}{2}} \tag{16}$$

This equation expresses the amplitude of the time function y(t) as a function of the Laguerre coefficients of the past of the time function x(t). It can be simplified by letting $V(\alpha)$ represent the product of polynomials $\eta_i(u_1)\ \eta_j(u_2)\ldots \eta_h(u_s)$ and A_α represent the corresponding coefficient $a_{i,j,\ldots,h}$. Then Eq. (16) becomes

$$y(t) = \lim_{s\to\infty} \sum_{\alpha} A_\alpha\, V(\alpha)\, e^{-\frac{u_1^2+u_2^2+\ldots+u_s^2}{2}} \tag{17}$$

The behavior of any system of the class of systems considered in the Wiener theory can be expressed in the form of Eq. 17. The coefficients A_α are said to characterize the system because the complete expression relating the output of the system y(t) to the past of its input x(t), for any input time function, is known when the A_α's are known.

We now come to the problem of characterizing a given nonlinear system, that is, the problem of evaluating the A_α's corresponding to a given nonlinear system. The object is to obtain an expression for the A_α's suitable for experimental evaluation. To obtain such an expression Wiener multiplies both sides of (17) by $V(\beta)$ and then makes use of the gaussian distribution of the Laguerre coefficients of a shot noise process to obtain equation (26) for the A_α's. However, we shall take a different approach to arrive at Eq. (26) that will give us a better physical understanding of the Wiener method.

In the practical case we will always use a finite number of Laguerre coefficients and Hermite functions. Then the sum on the right side of (17) does not yield y(t) exactly but only approximates it. We can regard the finite sum

$$\sum_{a} A_a V(a)\, e^{-\frac{u_1^2+\ldots+u_s^2}{2}} \tag{18}$$

as representing the output of a nonlinear system in terms of s Laguerre coefficients of its input and a finite number of Hermite functions. We want to choose the A_a's so that this sum best approximates the output y(t) of the given system with respect to some error criterion when both systems have the same input. Since, according to section 1.1 the most general time function is shot noise we shall let it be the common input. We define

$$\mathcal{E} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{\frac{u_1^2+\ldots+u_s^2}{2}} \left[y(t) - \sum_{a} A_a V(a)\, e^{-\frac{u_1^2+\ldots+u_s^2}{2}} \right]^2 dt \tag{19}$$

as the error between the outputs of the two systems. The justification of the choice of this weighted mean square error is that it leads to convenient independent expressions for the A_a's as we shall see. We now minimize $\mathcal{E}$ with respect to the A_a's. In particular, for the coefficient A_β we have

$$\frac{\partial \mathcal{E}}{\partial A_\beta} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} -2\, V(\beta) \left[y(t) - \sum_{a} A_a V(a)\, e^{-\frac{u_1^2+\ldots+u_2^2}{2}} \right] dt \tag{20}$$

For the error to be a minimum with respect to A_β we must set Eq. (20) to zero. This gives Eq. (21).

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y(t)\, V(\beta)\, dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} V(\beta) \sum_{a} A_a V(a)\, e^{-\frac{u_1^2+\ldots+u_s^2}{2}} dt \tag{21}$$

We have seen (section 1.4) that the Laguerre coefficients of the past of a shot noise process are statistically independent, normalized, gaussian variables. Thus the joint distribution of the Laguerre coefficients is

$$P(u_1, \ldots, u_s) = (2\pi)^{-s/2} \, e^{-\frac{u_1^2 + \ldots + u_s^2}{2}} \tag{22}$$

Our knowledge of this distribution is helpful in evaluating the integral on the right side of Eq. (21). Taking advantage of the ergodic theorem we can replace the time average of the right side of Eq. (21) by the corresponding ensemble average with the result

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} y(t) \, V(\beta) \, dt = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} V(\beta) \sum_{\alpha} A_\alpha \, V(\alpha) \, e^{-\frac{u_1^2 + \ldots + u_s^2}{2}} \times P(u_1, \ldots, u_s) \, du_1 \ldots du_s \tag{23}$$

After using (22) in (23) and interchanging the order of integration and summation we obtain

$$(2\pi)^{s/2} \, \overline{y(t)V(\beta)} = \sum_{\alpha} A_\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} V(\alpha) \, V(\beta) \, e^{-(u_1^2 + \ldots + u_s^2)} \, du_1 \ldots du_s \tag{24}$$

in which the bar above $y(t)V(\beta)$ indicates the time average of this quantity. Since $V(\alpha)$ and $V(\beta)$ are products of Hermite polynomials of the Laguerre coefficients we can separate Eq. (24) into a product of integrals each of which involves only one Laguerre coefficient as in Eq. (25)

$$(2\pi)^{s/2}\,\overline{y(t)V(\beta)}$$

$$= \sum_{a} A_a \int_{-\infty}^{\infty} \eta_i(u_1)\,\eta_{i'}(u_1)\,e^{-u_1^2}\,du_1 \cdots \int_{-\infty}^{\infty} \eta_h(u_s)\,\eta_{h'}(u_s)\,e^{-u_s^2}\,du_s \qquad (25)$$

In this equation the unprimed subscript indicies belong to those Hermite polynomials that make up $V(\alpha)$ while the primed indicies belong to those Hermite polynomials that make up $V(\beta)$. Recalling the orthogonality property of the Hermite functions (Eq. 5) it is clear that unless all the primed indicies i', j', ..., h' in Eq. (25) are equal to the corresponding unprimed indicies i, j, ..., h, in other words unless β equals α, at least one of the integrals will be zero. By the same token, if $\beta = \alpha$ then all the integrals have the value unity. Hence Eq. (25) reduces to

$$(2\pi)^{s/2}\,\overline{y(t)V(\beta)} = A_\beta \qquad (26)$$

which provides the basis for the experimental determination of the characterizing coefficients A_α. The reason for the choice of Hermite functions to expand the right side of Eq. (15) now becomes apparent. The joint gaussian probability density of the Laguerre coefficients of the shot noise input (Eq. 22) supplies the necessary exponential weighting factor in Eq. (23) to enable us to take advantage of the orthogonality of the Hermite functions in evaluating the coefficients A_α.

This approach to the Wiener theory clearly points out that, for any given number of Laguerre coefficients and Hermite functions, this theory determines that system whose output best approximates (in the weighted mean square sense of Eq. 19) the output of the given system for shot noise input to both systems. As the number of Laguerre coefficients and Hermite functions is increased, the output (for shot noise input) of any system

of the Wiener class can be approximated with vanishing error. And, from the discussion of section 1.1 if two systems have the same response to shot noise then they have the same response to any common input and can be considered to be equivalent.

1.6 The Experimental Apparatus for Characterizing and Synthesizing Nonlinear Systems

Equation (26) provides the basis for the experimental determination of the characterizing coefficients A_α. The apparatus for the determination of the coefficients A_α is shown in Fig. 3. The output of a shot noise generator is fed simultaneously into the given nonlinear system and into the Laguerre network. The output of the given nonlinear system is y(t). The outputs of the Laguerre network are fed into a device involving multipliers and adders. This device generates products of Hermite polynomials (the V's) whose arguments are the Laguerre coefficients. Each output of this Hermite polynomial generator, when multiplied by y(t) and averaged, yields, by Eq. (26), one of the characterizing coefficients of the given nonlinear system.

Having described the method for determining the characterizing coefficients of a nonlinear system we now turn our attention to the Wiener method of synthesis of nonlinear systems from their characterizing coefficients. The general representation of a nonlinear system is given by Eq. (17) which is the guide for the synthesis problem. This equation tells us that, for each α, we must generate $V(\alpha)$ and multiply it by A_α and the exponential $\exp -(u_1^2 + \ldots + u_s^2)/2$. Then each product must be added to give the system output y(t). In practice, the number of multipliers is reduced if we first form the sum of the

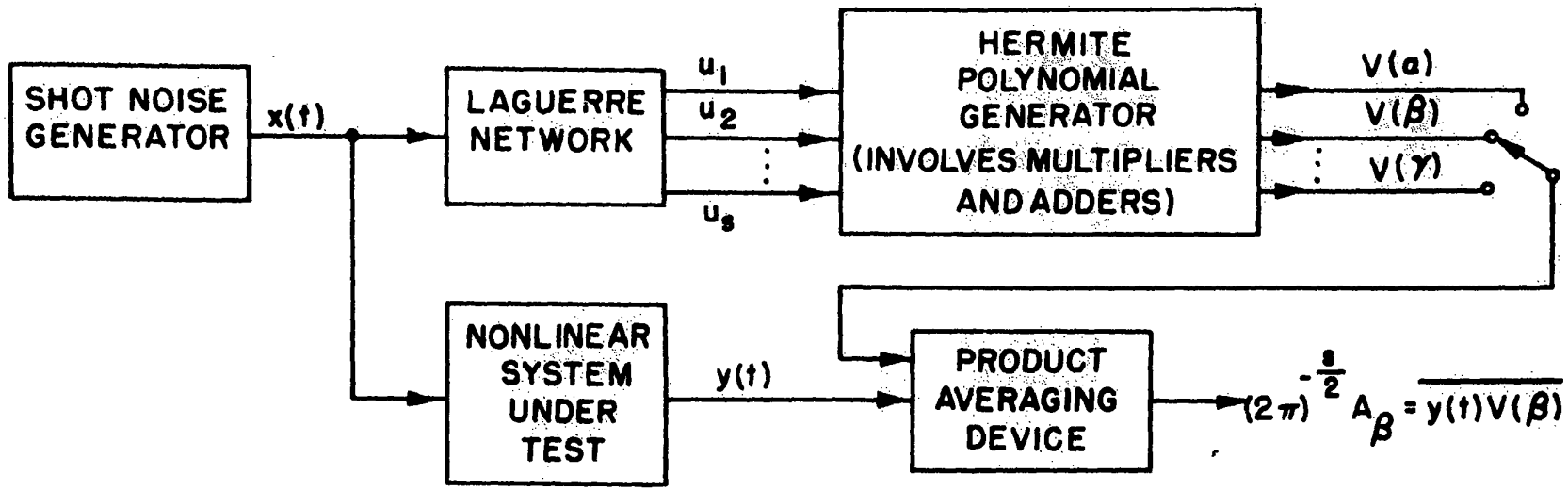

Fig. 3. Block diagram of the circuitry for the characterization of nonlinear systems.

products $A_\alpha V(\alpha)$ and then multiply by the exponential function.

The exponential function, $\exp -(u_1^2 + \ldots + u_s^2)/2$, can be obtained as the product of s exponential function generators whose inputs are respectively u_1 through u_s. Such generators give an output of $\exp(-u^2/2)$ when the input is u. They are realizable, among other ways, in the form of a small cathode-ray tube with a special target to generate the $\exp(-u^2/2)$ function.

The block diagram of the apparatus for the synthesis procedure is shown in Fig. 4.

1.7 Example

In order to fix ideas, let us consider a simple example which is particularly suited to characterization and synthesis by the Wiener method. It should be emphasized that the Wiener method is an experimental method and that, for the purpose of illustrating mathematically how the method works, only the simplest of examples can be handled analytically. Let the example be a nonlinear system that contains no storage elements. Further let its output-input characteristic (transfer characteristic) be given by the equation

$$y(t) = e^{-x^2(t)/2} \tag{27}$$

In both the characterization and synthesis procedures described, the function of the Laguerre network is to introduce dependence of the system output on the past of its input. The nonlinearity is brought about by the Hermite polynomial generator. For the simple example under consideration there is no dependence upon the past and thus we can bypass the Laguerre network. In the experimental procedure, the fact that this

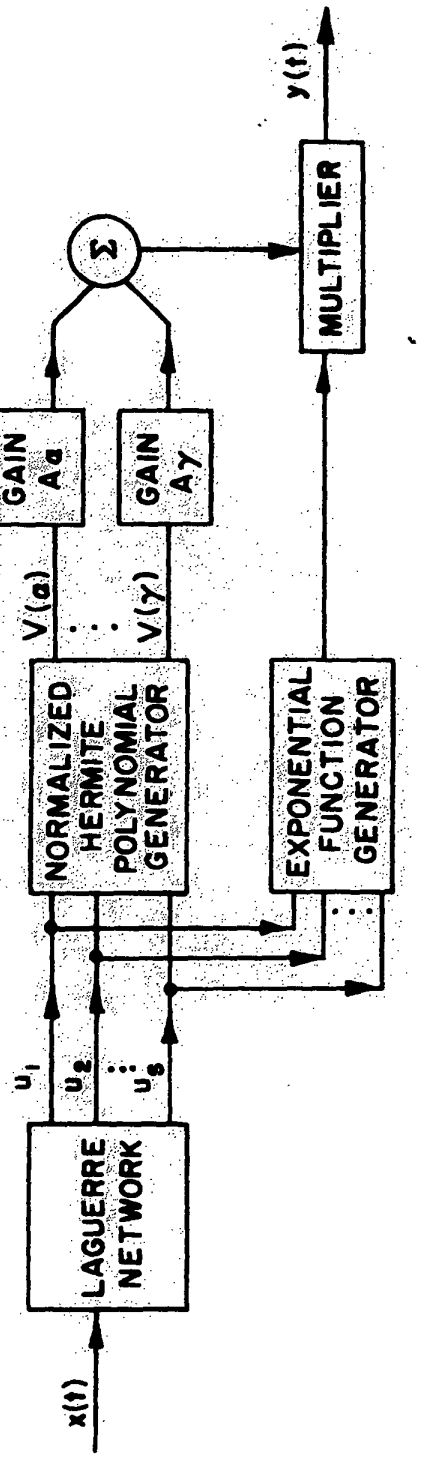

Fig. 4. Block diagram of the circuitry for the synthesis of nonlinear systems.

given nonlinear system has no storage could be determined from the results of a priori tests made on the system.

We notice that as a result of bypassing the Laguerre network the variables u_1 through u_s (Fig. 3) are replaced by the single variable x(t) as shown in Fig. 5. Equation (16) then becomes:

$$y(t) = \sum_i a_i \eta_i(x) e^{-x^2/2} \tag{28}$$

and Eq. (26) becomes

$$a_i = (2\pi)^{1/2} \overline{y(t) \eta_i(x)} \tag{29}$$

Let us make use of the ergodic theorem to evaluate this time average as an ensemble average. Using Eq. (27) we can write

$$a_i = (2\pi)^{1/2} \int_{-\infty}^{\infty} \eta_i(x) e^{-x^2/2} P(x) dx \tag{30}$$

But since, in the test setup (Fig. 5), x(t) is the output of an ideal shot-effect generator, the probability density of x is

$$P(x) = (2\pi)^{-1/2} e^{-x^2/2} \tag{31}$$

Thus

$$a_i = \int_{-\infty}^{\infty} \eta_i(x) e^{-x^2} dx \tag{32}$$

Referring to Eq. (3) and the definition of the Hermite polynomial, it is seen that

$$\eta_1(x) = \pi^{-1/4} \tag{33}$$

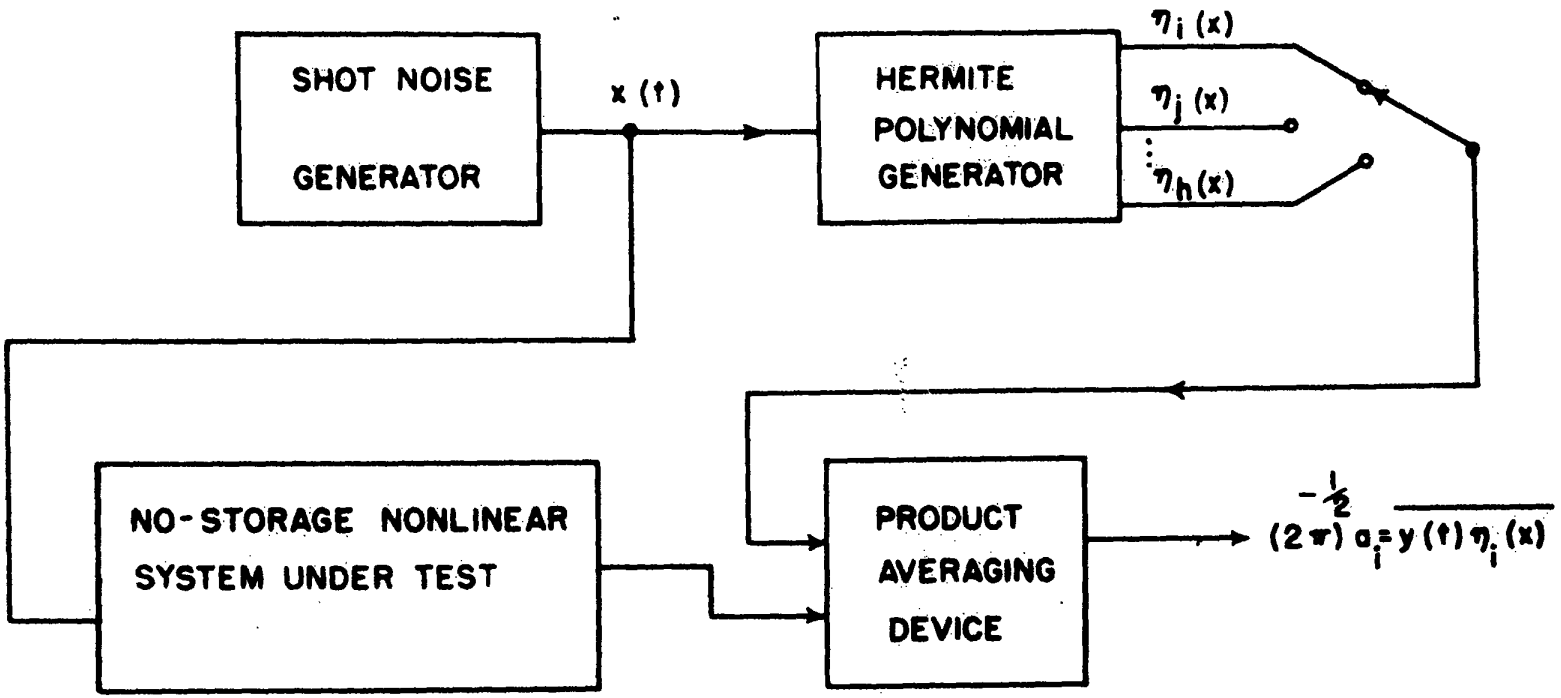

Fig. 5. Block diagram of the circuitry for the characterization of no-storage nonlinear systems.

With this result Eq. (32) can be written

$$a_i = \pi^{1/4} \int_{-\infty}^{\infty} \eta_i(x)\, \eta_1(x)\, e^{-x^2}\, dx \tag{34}$$

As a consequence of the orthogonality of the Hermite functions (Eq. 5) we have the result

$$a_1 = \pi^{1/4} \tag{35}$$

$$a_i = 0 \qquad i \neq 1$$

These coefficients serve to characterize the simple nonlinear system of this example.

Now let us synthesize the system from these coefficients. The guide for the synthesis is Eq. (28) which corresponds to Eq. (17) for the more complicated case involving storage. Since, from Eq. (35) only one coefficient is different from zero, the sum in Eq. (28) has only one term and can be written

$$y(t) = a_1\, \eta_1(x)\, e^{-x^2/2} \tag{36}$$

The synthesis of the system amounts to generating $\eta_1(x)$ and $e^{-x^2/2}$ and forming the product indicated in Eq. (36). The formal synthesis of the system according to the block diagram of Fig. 4 is shown in Fig. 6(a). Since $\eta_1(x)$ is just a constant, independent of x, the system is seen to be equivalent to that of Fig. 6(b). We see that for the simple example considered the synthesized system consists solely of the "function generator," a component which in the more complicated case will form only a part of the synthesized system.

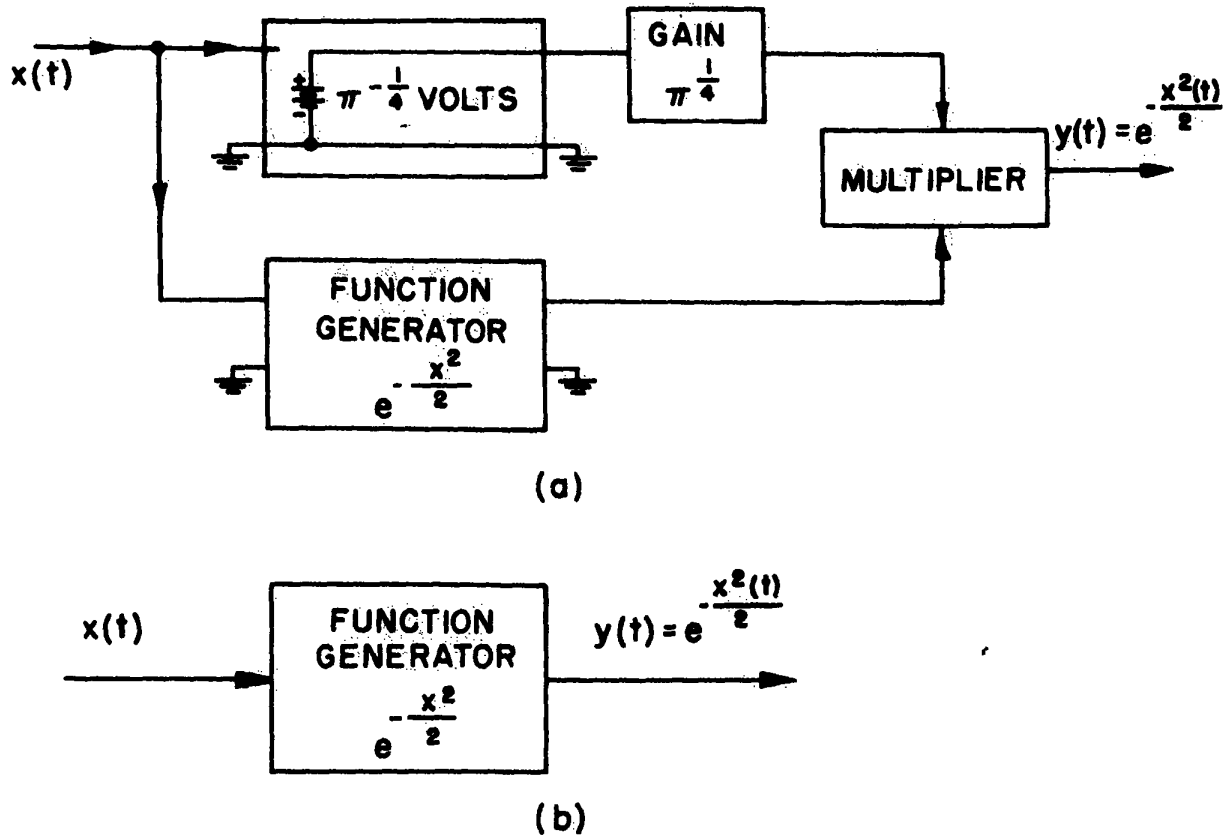

Fig. 6. (a) Synthesis of the nonlinear characteristic $y = \exp -(x^2/2)$ according to the general procedure of Fig. 4.

(b) Equivalent reduced form of part (a).

1.8 Observations and Comments

It can be seen from Eq. (16) that if we choose to represent the past of the system input by s Laguerre coefficients and if, furthermore, we decide to let the Hermite polynomial indicies, i, j, ..., h (Eq. 16), range from 1 to n we have n^s coefficients A_α to evaluate. Without a doubt this number can become quite large in many cases of practical interest. However, with the freedom that exists in nonlinear systems we can hardly expect to apply such a general approach without a great deal of effort. At present, the large number of multipliers that are required for the generation of the Hermite polynomials and their products is the principal deterrent to the practical application of the Wiener method of characterization and synthesis. It is safe to say that, at present, the Wiener theory is of greater theoretical than practical interest.

One of the most significant contributions of the Wiener theory is that it shows us that any nonlinear system, of the broad class of systems considered by this theory, can be synthesized as a linear network with multiple outputs cascaded with a nonlinear circuit that has no memory of the past (Fig. 4). The linear network (the Laguerre network) serves to characterize the past of the input and the nonlinear no-storage circuit performs a nonlinear operation on the present outputs of the linear network to yield the system output. Thus, regardless of how the linear and nonlinear operations occur in any given circuit the same over-all operation can be achieved by a linear operation followed by a nonlinear one as shown in Fig. 4.

Another important contribution of the Wiener theory is the concept of the shot noise probe for a nonlinear system. Just as the response of a linear system to an impulse is sufficient to characterize the system so Wiener has

shown that the response of a nonlinear system to shot noise is sufficient to characterize it.

In the Wiener theory two parameters remain free; the time scale factor of the Laguerre functions and the scale factor in the argument of the Hermite functions. For convenience both have been taken as unity in the preceding development. We may choose these as we desire in order to reduce the apparatus necessary to perform a given operation. Unfortunately we have no simple procedure for determining the optimum values of these scale factors to enable us to do the best job with a given number of Laguerre coefficients and Hermite functions. We shall see a possible approach to this problem when we discuss a similar but somewhat simpler problem that arises in connection with the determination of optimum filters by the theory developed in the following sections.

Since linear systems form such an important class of systems in engineering it is proper that we ask of any nonlinear theory, "How conveniently does this theory handle linear and nearly linear systems?" Although the Wiener theory includes within its scope linear as well as nonlinear systems it is not particularly suited for application to the former. The reason for this can be seen by observing the form of the general Wiener system (Fig. 4). We note that the exponential function generator bypasses the Hermite polynomial generator. In order for the system of Fig. 4 to represent a linear system the operation from the output of the Laguerre network to the output of the system must be linear. This means that the gain coefficients A_a must have values which cause cancellation of the output of the exponential function generator and give the desired linear operation on the Laguerre coefficients. To achieve this cancellation effect will in general

require a very large number of Hermite functions and even then we have the unfavorable situation of obtaining a desired output that may be the small difference of two large quantities. The nonlinear theory that is developed in the following sections does not suffer from this difficulty and, as we shall see, can be readily applied to linear and nearly linear systems as well as to general nonlinear systems.

II. The Filter Problem

2.1 Objectives and Assumptions

In part I we have seen how we can synthesize general nonlinear systems from a knowledge of their characterizing coefficients. We now turn our attention to the problem of determining optimum nonlinear systems or filters.

We shall deal with time-invariant nonlinear systems that operate on statistically stationary time functions. The filter problem as considered here is one of determining that system, of a class of systems, that operates on the past of a given input time function x(t) to yield an output y(t) that best approximates a given desired output z(t) with respect to some error criterion. When the optimum filter is chosen from the class of linear systems and when the mean square error criterion is adopted Wiener has shown that this optimum filter is determined by the autocorrelation function of the input time function and the cross-correlation function of the input with the desired output.[7] Since these correlation functions determine the optimum mean-square linear filter, the same linear filter is optimum for all time functions having these same correlation functions in spite of the fact that other statistical parameters of these time functions may be very different. It is in the search for better filters that we turn to nonlinear filters which make use of more statistical data than just first order correlation functions.

As pointed out by Zadeh[8] there have been two distinct modes of approach to the optimum nonlinear filter problem. One approach parallels the approach of Wiener to linear systems by choosing the

form or class of filters and then finding the optimum member of this class by minimizing the mean square error between the desired output and the actual system output. The other approach formulates an appropriate statistical criterion and then determines the optimum filter for this criterion with little or no restrictions placed upon the form of the filter. Both these approaches yield equations for optimum filters in terms of higher order statistics (higher order distribution functions or correlation functions) of the input and desired output. In applying these approaches we are in general faced with two problems. First we must obtain the necessary statistical data about the input and desired output and then we must solve the design equations, which usually are quite complex, for the optimum filter in terms of this data. In nonlinear filter problems we find that the amount of statistical data we require in the design of the filter usually far exceeds that which is available to us and we find it necessary to make certain simplifying assumptions or models of the signal and noise processes in order to calculate the required distributions.

Instead of assuming a statistical knowledge of the filter input and desired output the approach to the nonlinear filter problem developed in this work assumes that we have at our disposal an ensemble member of the filter input time function x(t) and the corresponding ensemble member of the desired filter output z(t). By recording or making direct use of a portion of the given filter input time function, we obtain the ensemble member of x(t). The ensemble member of z(t) can be determined in different ways depending upon the problem. For pure prediction problems z(t) is obtained directly from x(t) by a time shift. For filter problems involving the separation of signal from noise at the reciever in a communication link we can, in the program for the design of the

filter, record a portion of the desired signal z(t) at the transmitter and the corresponding portion of x(t) at the receiver. For radar type problems, in the program for the design of the filter z(t) can be generated corresponding to signals x(t) received from known typical targets.

Since the ensembles of x(t) and z(t) contain all the statistical information concerning the filter input and desired output and since we shall make direct use of these time functions in our filter determination it is not necessary to make any assumptions about the distributions of x(t) and z(t). Thus, for example, in the problem of designing a filter to separate signal from noise we need make no assumptions about the statistics of the signal or noise or about how the two are mixed.

We note that in most practical cases our assumption of having a portion of x(t) and z(t) does not restrict us any more than the usual assumptions of knowing the higher order probability densities of the input and desired output do; for at present, except in very simple cases, the only practical way of obtaining these statistics is to measure them from ensembles of x(t) and z(t) when these ensembles are available. When they are available, the present approach makes measurements on them that directly yield optimum filters instead of first measuring the distributions and then solving design equations in terms of these measured values.

2.2 Relation to the Characterization Problem

The Wiener theory of nonlinear system characterization and synthesis provides us with a physically realizable operator on the past of a time function that includes within its scope a very large class of nonlinear systems. Hence it is of interest to investigate the possibility of determining the optimum nonlinear filter (for a given task and a particular

error criterion) from the class of systems of the Wiener theory.

Figure 3 shows the experimental procedure for obtaining the characterizing coefficients for a given nonlinear system (the system labeled "Nonlinear System Under Test"). Notice that the A_α's are completely determined from a knowledge of the response of the given system to a shot noise input. In fact, the presence of this system is not necessary in the experimental procedure of Fig. 3 if we have recordings of an ensemble member of the shot noise input x(t) and the corresponding output y(t). By feeding the recording of x(t) into the Laguerre network and the recording of y(t) into the product averaging device in place of the output of the given system we obtain the A_α's that correspond to the given system; that is, we obtain the A_α's that correspond to the system which operates on the shot noise x(t) to yield y(t). This arouses our curiosity concerning the possibility of determining the A_α's for the optimum filter problem directly from a knowledge of an ensemble member of its input and its desired output time functions without actually having the filter at our disposal. To this end let us consider the optimum filter problem and see how it differs from the characterization problem discussed above.

Unlike the characterization problem, in the determination of an optimum filter we do not have at our disposal the system labeled "Nonlinear System Under Test" in Fig. 3. In the filter problem this system would be the optimum filter; exactly what we are searching for. Consider the following problem. Suppose that we want to find a nonlinear filter whose input is a white gaussian time function x(t) and whose desired output is the stationary random time function z(t). Suppose also that we have at our disposal an ensemble member of

x(t) and the corresponding ensemble member of z(t). We excite the Laguerre network of Fig. 3 with x(t) and feed z(t) into the product averaging device in place of y(t) as shown in Fig. 7. From the discussion above it is clear that if the desired filter, which operates on x(t) to yield z(t), is a member of the class of systems considered in the Wiener theory, the test procedure of Fig. 7 will yield the A_α's corresponding to this system. We can then synthesize it in the general form of Fig. 4. However it will usually happen that the desired system is not even physically realizable, let alone a member of the Wiener class of nonlinear systems. In this case the derivation of section 1.5 shows that the procedure of Fig. 7 will yield that system of the Wiener class (having as many Laguerre coefficients and Hermite functions as used in the test apparatus) whose output best approximates z(t) in a weighted mean square sense. Thus for the special case of white gaussian filter input we can adapt the Wiener method of characterization to the experimental determination of optimum nonlinear filters.

2.3 The Need for a General Orthogonal Representation

When the given filter input is not shot noise we can no longer apply the method described above to determine the optimum filter. Recall that the orthogonality relations which led to Eq. (26) for the A_α's depended upon the fact that the Laguerre coefficients were gaussianly distributed and statistically independent, and this fact, in turn, depended on the fact that the input to the Laguerre network was shot noise. When x(t) is not shot noise we no longer obtain independent relations (Eq. 26) for the A_α's and the procedure for determining them shown in Fig. 7 is

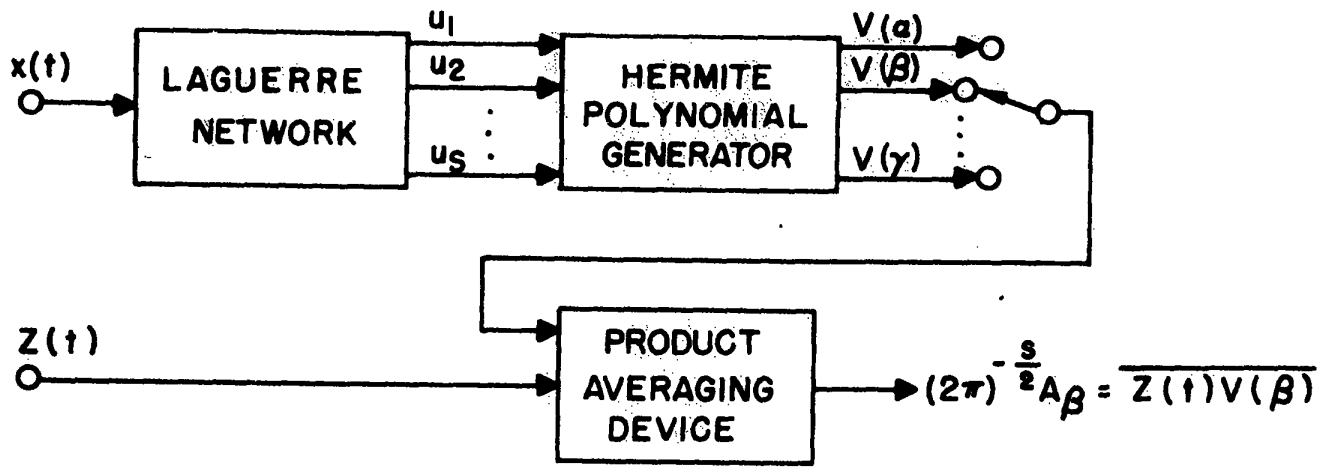

Fig. 7. Block diagram of experimental apparatus to determine an optimum filter when the given filter input x(t) is shot noise. z(t) is the desired filter output.

no longer valid. Thus we appreciate the need of an expression for a nonlinear operator in which the terms in its series representation are orthogonal in time, irrespective of the nature of the input time function. The development and application of such an operator is the subject of the following sections.

III. Optimum Nonlinear Filters

3.1 Object

The object of the work in this section is to develop an orthogonal representation for nonlinear systems that enables the convenient determination of optimum nonlinear filters. The development is best described if, before proceeding to the general filter, we first examine the class of no-storage nonlinear filters.

3.2 The No-Storage Nonlinear Filter

By a no-storage system we mean one whose output, at any instant, is a unique function of the value of its input at the same instant. We call the input-output characteristic of this system the transfer characteristic.

Let $x(t)$ and $z(t)$ be the given filter input and desired filter output time functions, respectively. We assume that $x(t)$ and $z(t)$ are bounded, continuous time functions. This is clearly no restriction in the practical case and it enables us to confine our attention to approximating desired filter transfer characteristics that are bounded and continuous. Since $x(t)$ is bounded, there exists an a and b such that $a \leq x(t) \leq b$ for all t. Now consider a set of n functions $\phi_j(x)$ $(j = 1, \ldots, n)$ over the interval (a, b). These functions are defined as follows

$$\phi_j(x) = \begin{cases} 1 \text{ for } x_j - \frac{w}{2} \leq x < x_j + \frac{w}{2}, & j = (1, \ldots, n-1) \\ \quad \text{and } x_j - \frac{w}{2} \leq x \leq b, & j = n \\ 0 \text{ for all other } x \end{cases} \qquad \begin{aligned} w &= \frac{b-a}{n} \\ x_j &= a + w\left(j - \tfrac{1}{2}\right) \end{aligned} \tag{37}$$

A plot of the jth function of this set of functions is shown

in Fig. 8. (A separate definition is given for $\phi_n(x)$ in order to include the point b. In practical application of these functions we simply generate n gate functions of equal width that cover the interval (a, b).) Clearly this set of functions is normal and orthogonal over the interval (a, b). We shall refer to these functions as "gate functions." Let us define y as a gate function expansion of x as follows

$$y = \sum_{j=1}^{n} a_j \, \phi_j(x) \tag{38}$$

It is clear that by taking n sufficiently large y can be made to approximate any single-valued continuous function of x arbitrarily closely everywhere on the interval (a, b).

When x is a function of time it is convenient to write Eq. (38) as

$$y(t) = \sum_{j=1}^{n} a_j \, \phi_j[x(t)] \tag{39}$$

As a consequence of the non-overlapping property of the gate functions along the x axis the $\phi_j[x(t)]$ will, for any single valued time function x(t), form an orthogonal set in time as well as an orthonormal set in x. Further this time domain orthogonality holds for any bounded weighting function G(t). That is

$$G(t) \, \phi_j[x(t)] \, \phi_k[x(t)] = \begin{cases} 0 & j \neq k \\ G(t) \, \phi_j^2[x(t)] & j = k \end{cases} \tag{40}$$

Relation (39) specifies the form of an equation that defines a no-storage

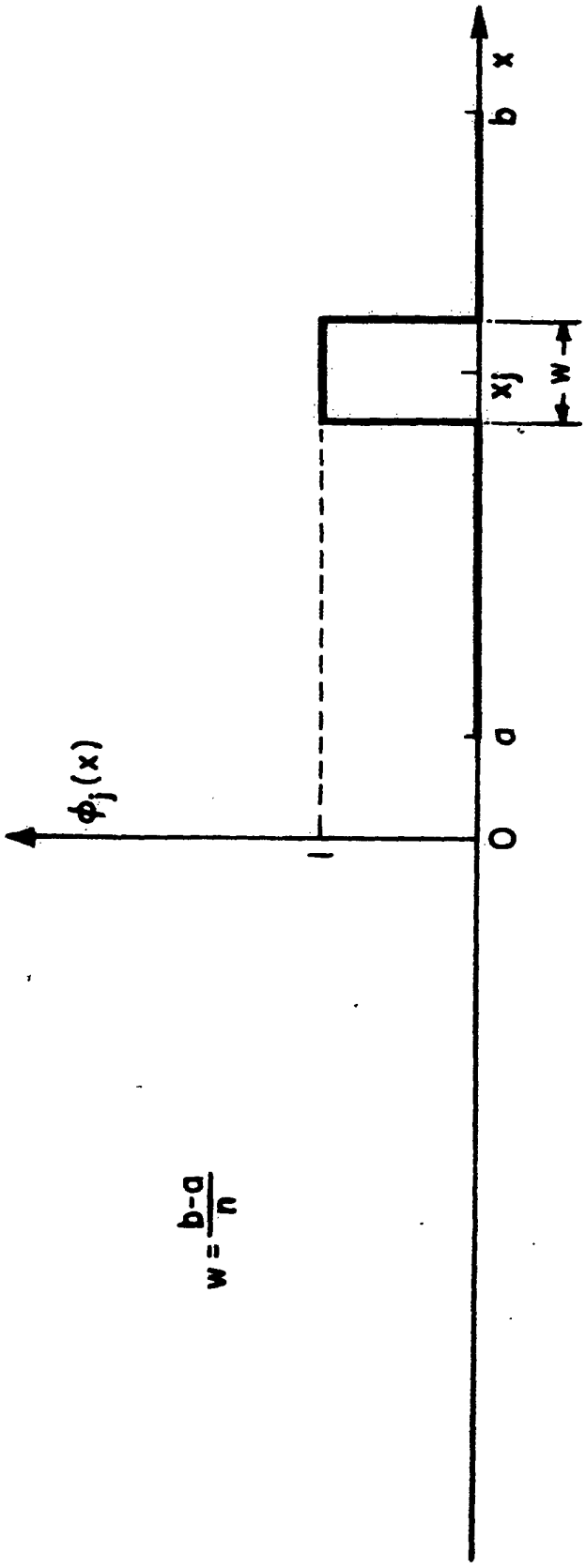

Fig. 8. Gate function $\phi_j(x)$.

nonlinear system. The determination of an optimum no-storage filter for a given error criterion consists of choosing the a_j's in such a manner that, for a given x(t), the error between y(t) and the desired output z(t) is a minimum. We adopt a weighted mean square error criterion in which G(t) is, as we shall discuss later, a non-negative weighting function at our disposal. More specifically we minimize the error

$$\mathcal{E} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} G(t) \left\{ Z(t) - \sum_{j=1}^{n} a_j \, \phi_j[x(t)] \right\}^2 dt \tag{41}$$

with respect to the n coefficients a_j. Differentiating with respect to a_k and setting the result to zero we obtain

$$\frac{\partial \mathcal{E}}{\partial a_k} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} -2G(t) \, \phi_k[x(t)] \left\{ z(t) - \sum_{j=1}^{n} a_j \, \phi_j[x(t)] \right\} dt = 0 \qquad k = (1, \ldots, n) \tag{42}$$

Denoting the operation of time averaging by a bar above the averaged variable Eq. (42) can be written

$$\overline{G(t) \, \phi_k[x(t)] \sum_{j=1}^{n} a_j \, \phi_j[x(t)]} = \overline{z(t) \, G(t) \, \phi_k[x(t)]} \tag{43}$$

Making use of the time domain orthogonality of the gate functions (Eq. 40), Eq. (43) reduces to

$$a_k \, \overline{G(t) \, \phi_k^2[x(t)]} = \overline{z(t) \, G(t) \, \phi_k[x(t)]} \tag{44}$$

It follows from the definition of the $\phi_j(x)$ given in Eq. (37) that $\phi_j^2[x(t)] = \phi_j[x(t)]$ so that Eq. (44) is equivalent to the equation

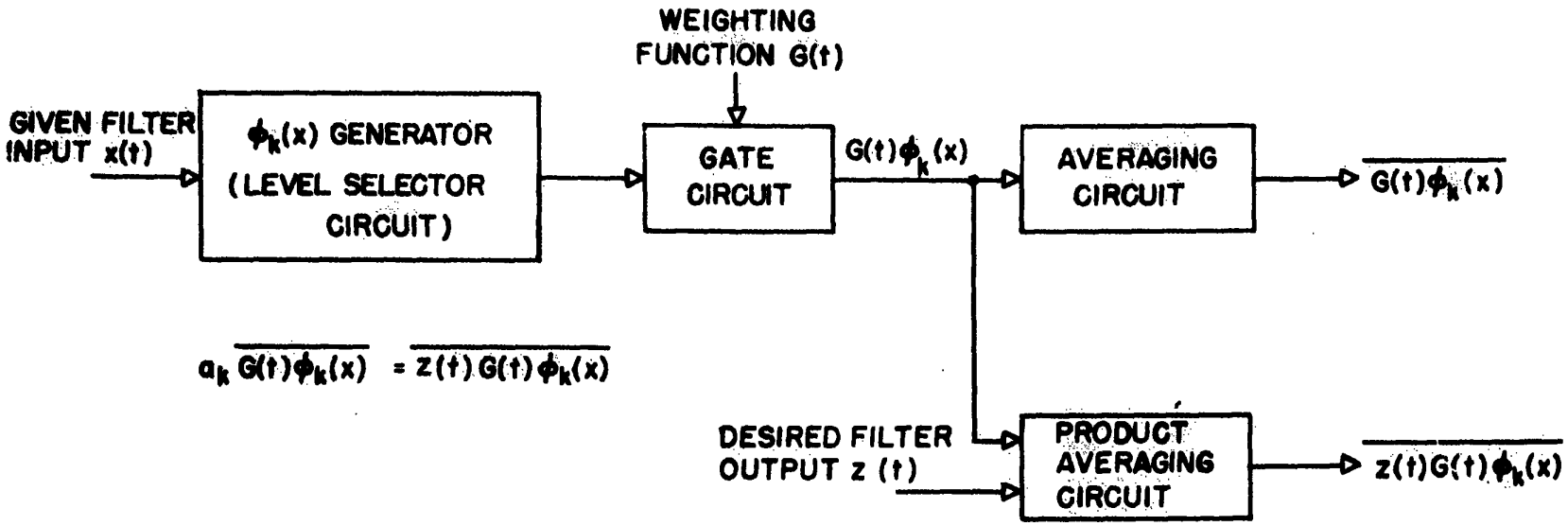

Fig. 9. Experimental procedure for determination of the optimum filter coefficients for no-storage filters. x(t) and z(t) are corresponding ensemble members of the stationary input and desired output time functions respectively.

$$a_k \overline{G(t)\,\phi_k[x(t)]} = \overline{z(t)\,G(t)\,\phi_k[x(t)]} \tag{45}$$

This equation provides a convenient experimental means of determining the desired coefficients a_k. The experimental procedure for the evaluation of these coefficients is shown in Fig. 9. An ensemble member of x(t) is fed into a level selector circuit and the corresponding ensemble member of z(t) is fed into the product averaging device. The output of the level selector circuit is unity whenever the amplitude of x(t) falls within the interval of the kth gate function and zero at all other times. This output is used to gate the weighting function G(t). The output of the gate circuit is then averaged and also multiplied by z(t) and averaged to yield the two quantities necessary to determine a_k in Eq. (45).

From a knowledge of the a_k we can directly construct a stepwise approximation, like that of Fig. 10, to the desired optimum transfer characteristic (see Eq. 38). The synthesis of the filter can be carried out formally according to Eq. (38) by using level selector circuits and an adder as shown in Fig. 11, or we can synthesize the optimum characteristic by any of the other available techniques such as piecewise linear approximations or function generators.

In order to become more familiar with the operation and terminology of this method let us consider a very simple example. In this example we shall do analytically what, in practice, the experimental procedure of Fig. 9 does for us. Suppose we are given an ensemble member of x(t) and the corresponding ensemble member of z(t). Further suppose that the desired filter output z(t) is equal to f[x(t)] where f is a continuous real function of x. We desire to verify that the filter determined by the procedure of Fig. 9 is actually a stepwise approximation to the transfer

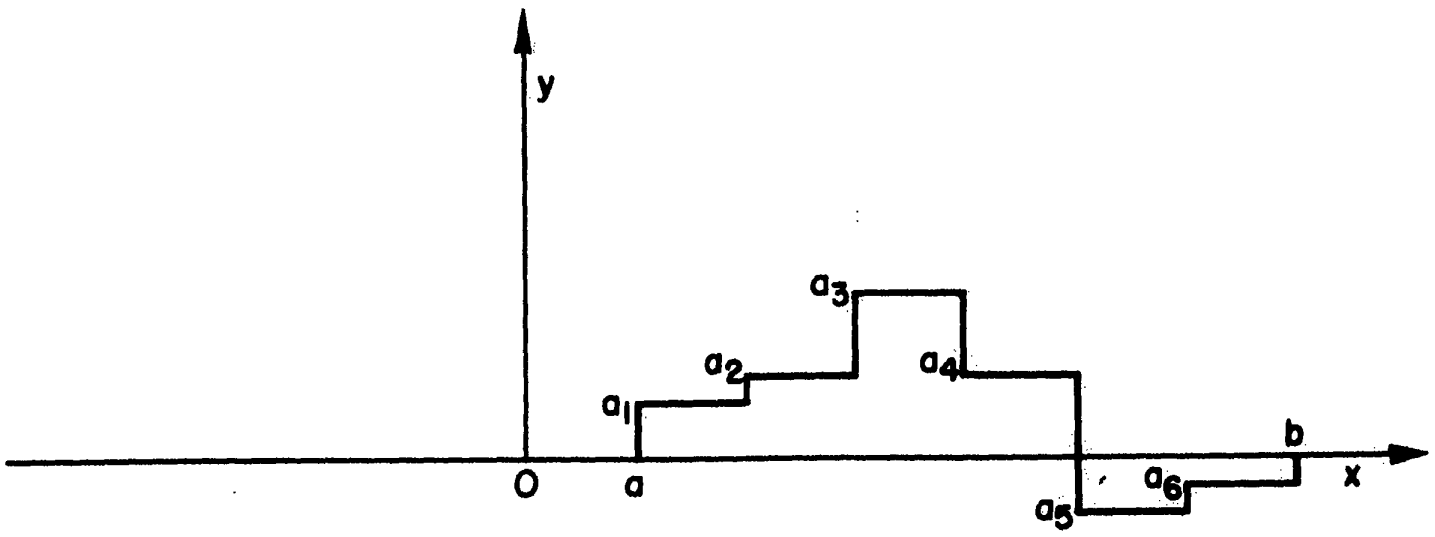

Fig. 10. Example of a stepwise representation of a transfer characteristic over the interval (a, b). The a_k's are the optimum filter coefficients evaluated by the procedure indicated in Fig. 9.

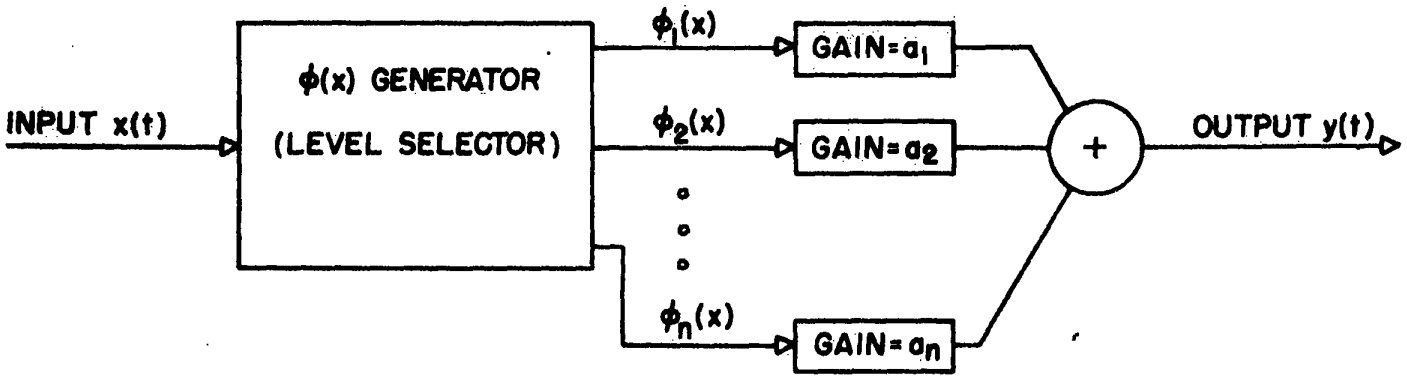

Fig. 11. Example of the formal synthesis of no-storage filters according to Eq. (38).

characteristic f(x). For simplicity let us assume that n has been chosen sufficiently large so that the function f(x) is approximately constant over the width of the gate functions and let us choose G(t) equal to a constant so that the conventional mean square error criterion results. For these conditions whenever $\phi_k[x(t)]$ has a non-zero value x must lie in the interval of width w about x_k and z(t) is approximately equal to $f(x_k)$. Equation (45) becomes

$$a_k \overline{\phi_k[x(t)]} \approx f(x_k) \overline{\phi_k[x(t)]} \tag{46}$$

from which we obtain the relation

$$a_k \approx f(x_k) \tag{47}$$

for the a_k which shows (see Eq. 38) that they determine a filter that is a stepwise approximation to the desired transfer characteristic f(x). (A closer examination of this example shows that the same results are obtained for any weighting function G(t). This is because for this example the desired filter is a member of the class of no-storage filters and hence as $n \to \infty$ the error $\mathcal{E}$ in Eq. (41) can be made zero for any G(t).)

In addition to knowing that as $n \to \infty$ the gate function expansion (Eq. 38) can approximate any continuous transfer function arbitrarily closely, it is of practical interest to investigate how the expansion converges for small n as n is increased when the coefficients are chosen to minimize the mean square error. This is most easily done with the aid of an example. Let the transfer characteristic of Fig. 12 be the one that we desire to approximate. The simplest gate function expansion is that for which n = 1. The best mean square approximation clearly occurs for $a_1 = (y_1 + y_2)/2$. For n = 2 the best

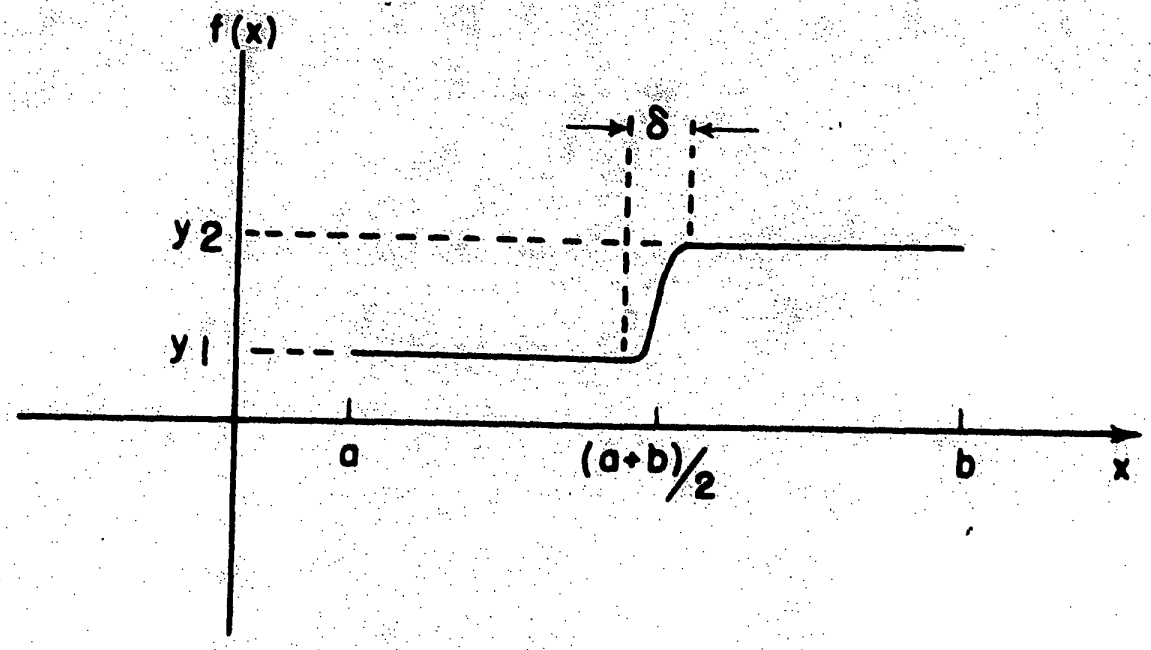

Fig. 12. A transfer characteristic of a no-storage nonlinear system.

approximation is seen to occur for $a_1 = y_1$ and $a_2 = y_2$. This approximation is considerably better than that for n = 1. Now consider n = 3. The best mean square approximation is, by inspection $a_1 = y_1$, $a_2 = (y_1 + y_2)/2$ and $a_3 = y_2$. But this is seen to be a worse approximation than that for n = 2 ! For n = 4 we clearly must do at least as well as for n = 2 since $a_1 = a_2 = y_1$ and $a_3 = a_4 = y_2$ constitute a possible solution. Again, for this example, the approximation for n = 5 is inferior to that for n = 2 or 4 but better than the n = 3 approximation. A moment's reflection reveals that the reason for this peculiar convergence is that the function f(x) changes appreciably in an interval that is small compared to the width of the gate functions and hence the position of the gate functions along the x axis is critical. For this example when n is even one gate function ends at x = (a+b)/2 and another begins, thus providing a nice fit to f(x). For n odd one gate function straddles the point x = (a+b)/2 and because of symmetry it will have a coefficient equal to $(y_1 + y_2)/2$. As we increase n beyond the point where the width (w = (b-a)/n) of the gate functions becomes less than δ, the position of the gate functions becomes less and less critical, the oscillatory behavior disappears and the expansion converges to f(x) everywhere.

From this simple example we can draw some general conclusions regarding the convergence of the gate function expansion to continuous functions. When the desired function changes appreciably in an interval of x comparable to or smaller than w it may happen that an increase in n will result in a poorer approximation. However, if n is increased by an integral factor the approximation will always be at least as good as that before the increase. Further, if n is taken large enough so that the function is essentially constant over any interval of width w then

any increase in n will yield at least as good an approximation as before the increase. Thus in the practical application of this theory if we increase n and get an inferior filter we should not be alarmed. It is merely an indication that the desired filter characteristic has a large slope over some interval. By further increasing n the desired characteristic will be obtained.

In the discussion above it was assumed for convenience that each gate function had the same width w. This is not a necessary restriction however. It is sufficient to choose them so that they cover the interval (a, b) and do not overlap. Thus if we have some a priori knowledge about the optimum transfer characteristic we may be able to save time and work in determining it by judiciously choosing the widths of the $\phi_j(x)$'s. In fact, after evaluating any number m of the a_k's we are free to alter the widths of the remaining functions $\phi_j(x)$ $(j > m)$ as we proceed. This flexibility is permissible because in taking advantage of it we do not disturb the time domain orthogonality of the gate functions.

3.3 Linear and Nonlinear Systems from the Function Space Point of View

In section 1.3 we saw how we can characterize the past of a time function by the coefficients of a complete set of orthogonal functions such as the Laguerre functions. Let us now think of a function space which has as a basis the Laguerre functions. Just as in a vector space a given vector can be represented as a linear combination of the basis vectors so in function space a given function (satisfying appropriate regularity conditions) can be represented as a linear combination of the functions that form the basis of the space. We can think of the Laguerre coefficients of a function x(t) as being the scalar components of x(t) along the respective basis

vectors. At any instant, the past of x(t) is represented by the point in function space corresponding to the tip of the vector whose scalar components are the Laguerre coefficients of the past of x(t).

We have also discussed that any function of the past of x(t) can be expressed as a function of the Laguerre coefficients of this past. In terms of the function space then, a function of the past of x(t) can be expressed as a function of position in this space. We say that we generate the desired function of the past of x(t) by a transformation that maps the function space onto a line — the line corresponds to the amplitude of the desired function. This concept provides a powerful tool in the study of linear and nonlinear systems. To better understand it let us consider the Wiener theory in this light. The output of the general Wiener nonlinear system is expressed (Eq. 17) as a Hermite function expansion of the Laguerre coefficients of the past of the input time function. The Laguerre functions form the basis of the function space of the past of the input and the Hermite function expansion maps this space onto a line — the amplitude of the system output.

Several important concepts follow immediately from this viewpoint. The first, as was made evident by the Wiener theory, is that any system (of the broad class considered in the Wiener theory) can be represented by the cascade of a linear system followed by a no-memory nonlinear system. The outputs of the linear system characterize the past of the input as a point in function space and the no-memory nonlinear system maps this space onto a line. Secondly we see that in principle (we assume that the complete set of Laguerre functions is used) the difference between any two systems is accounted for by a difference in the no-memory part that performs the mapping. For example, if the mapping is linear

(we shall discuss this case in a later section) then a linear system is represented, if it is not then a nonlinear system is represented. Since the difference between two systems is just in this mapping, the problem of determining an optimum system for a desired performance and given error criterion becomes that of determining the optimum no-memory system which maps the function space onto the output.

Finally we see that this function space point of view provides the key for finding a general orthogonal expansion for the output of a nonlinear system. For reasons that will become evident in the next section, we desire to obtain a series expansion for the output of a nonlinear system in which the terms are mutually orthogonal in time. Furthermore, we require that this orthogonality be independent of the input time function. Clearly this is achieved by choosing a mapping that partitions the function space into non-overlapping cells and by letting each term in the series expansion represent the system output for a particular cell in the function space. Since at any instant the past of the input is represented by only one point in the function space, only one term in the series expansion will be non-zero at any instant: thus all the terms are mutually orthogonal in time. The gate function expansion for the no-storage filter (Eq. 39) is recognized to be an application of this approach in the simple case for which the input space is just a line. The next section applies this approach to the more general case of a finite dimensional space. (Note: Although the function space of which we have spoken is infinite dimensional we shall continue to use the term even when we speak of a finite number of Laguerre functions.)

3.4 The General Nonlinear Filter Involving Storage

The class of nonlinear systems considered in this section is the same as that of the Wiener theory. Without introducing any physical restriction we shall, for convenience, assume that the given filter input x(t) is bounded. As in the Wiener theory we characterize the past of x(t) by its Laguerre coefficients. It is easily shown, as follows, that these coefficients are bounded if x(t) is bounded. Recall that the Laguerre coefficients of x(t) are given by the convolution of x(t) with the Laguerre functions. That is

$$u_n(t) = \int_0^\infty x(t-\tau)\, h_n(\tau)\, d\tau \tag{48}$$

where h_n is defined in Eq. (1).

It follows that

$$\left|u_n(t)\right| \leq \int_0^\infty \left|x(t-\tau)\right| \left|h_n(\tau)\right| \, d\tau \tag{49}$$

Assume $\left|x(t)\right| \leq M$ for all t. Then

$$\left|u_n(t)\right| \leq M \int_0^\infty \left|h_n(\tau)\right| \, d\tau \tag{50}$$

But from Eq. (1) we see that the Laguerre functions are polynomials multiplied by decaying exponentials and hence they are absolutely integrable. Thus the Laguerre coefficients are bounded if x(t) is bounded.

Now consider the function space formed by s Laguerre coefficients

We divide the bounded region over which each Laguerre coefficient ranges into n intervals and define a set of gate functions, as in Eq. (37), for each coefficient. (It is only for convenience in notation that we choose the same number of gate functions for each Laguerre coefficient.) In the preceding section we saw that if we choose an expansion of these coefficients that partitions this function space into non-overlapping cells and is such that each term in the expansion represents the system output for one cell in the function space, an orthogonal expansion is obtained. To this end consider the expansion

$$y(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \cdots \sum_{h=1}^{n} a_{i,j,\ldots,h}\, \phi_i(u_1)\, \phi_j(u_2) \cdots \phi_h(u_s) \qquad (51)$$

in which the ϕ's are the gate functions defined in Eq. (37). Let us examine a typical term in this expansion. The term

$$a_{i,j,\ldots,h}\, \phi_i(u_1)\, \phi_j(u_2) \cdots \phi_h(u_s) \qquad (52)$$

is non-zero only when the amplitude of u_1 is in the interval for which $\phi_i(u_1)$ is unity and the amplitude of u_2 is in the interval for which $\phi_j(u_2)$ is unity, and so on for each Laguerre coefficient. The collection of these intervals defines a cell in the function space and thus the term in Eq. (52) is non-zero only when this cell is occupied. Hence the expansion (Eq. 51) divides the function space into non-overlapping cells and each term represents y when the corresponding cell in the function space of the input is occupied. Thus the terms are mutually orthogonal in time for any x(t). It is clear that as the width of the gate functions is decreased, by increasing n, the cells become smaller and y can be made to approximate any continuous function of the u's everywhere with vanishing error.

If $\Phi(\alpha)$ represents the function $\phi_i(u_1)\,\phi_j(u_2)\ldots\phi_h(u_s)$ and A_α represents the corresponding coefficient $a_{i,j,\ldots,h}$, Eq. (51) takes the simplified form

$$y(t) = \sum_{\alpha} A_\alpha\,\Phi(\alpha) \tag{53}$$

This equation is the desired orthogonal representation for nonlinear systems involving storage. We now proceed to determine the A_α's for the optimum filter problem. As in the case of the no-storage filter (Eq. 41) we adopt a weighted mean square error criterion and minimize the error

$$\mathcal{E} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} G(t) \left\{ z(t) - \sum_{\alpha} A_\alpha\,\Phi(\alpha) \right\}^2 dt \tag{54}$$

with respect to the coefficients A_α. For the coefficient A_β we have

$$\frac{\partial \mathcal{E}}{\partial A_\beta} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} -2G(t)\,\Phi(\beta) \left\{ z(t) - \sum_{\alpha} A_\alpha\,\Phi(\alpha) \right\} dt \tag{55}$$

For the error to be a minimum with respect to A_β we set this equation to zero. The result is

$$\overline{G(t)\,\Phi(\beta) \sum_{\alpha} A_\alpha\,\Phi(\alpha)} = \overline{z(t)\,G(t)\,\Phi(\beta)} \tag{56}$$

Taking advantage of the time domain orthogonality of the Φ's this equation reduces to

$$A_\beta\,\overline{G(t)\,\Phi^2(\beta)} = \overline{z(t)\,G(t)\,\Phi(\beta)} \tag{57}$$

Since the Φ's are products of gate functions they can only take on the values zero or unity, hence Eq. (57) is equivalent to

$$A_\beta \overline{G(t)\,\Phi(\beta)} = \overline{z(t)\,G(t)\,\Phi(\beta)} \qquad (58)$$

which forms the basis for the experimental procedure for determining the optimum filter coefficients.

The apparatus for the determination of the optimum filter coefficients is shown in Fig. 13. An ensemble member of x(t) is fed into the Laguerre network and the corresponding ensemble member of z(t) is fed into the product averaging device. The outputs of the Laguerre network are fed into a no-memory nonlinear circuit consisting of level selectors and gate or coincidence circuits. This circuit generates the Φ's. Since the Φ's are either zero or unity they can be multiplied by G(t) in a simple gate circuit. The output of this gate circuit is averaged and also multiplied by z(t) and averaged to yield the two quantities necessary to find the optimum coefficients according to Eq. (58).

Having determined the optimum coefficients, the nonlinear system can be synthesized formally according to Eq. (53) as indicated in Fig. 14. In Fig. 14 we note that the operation from the outputs of the Laguerre network to the system output y(t) is a no-memory operation. That is y is an instantaneous function of the Laguerre coefficients. Once the A's are known this function is directly specified and any other method of synthesizing no-storage systems for a prescribed operation can be used. One such method is described in reference 9.

In the procedure described above for determining and synthesizing optimum nonlinear filters the use of gate functions in the expansion of Eq. (51) is of central importance. Let us examine some of the consequences of this:

1. The use of gate functions provides us with a series representation

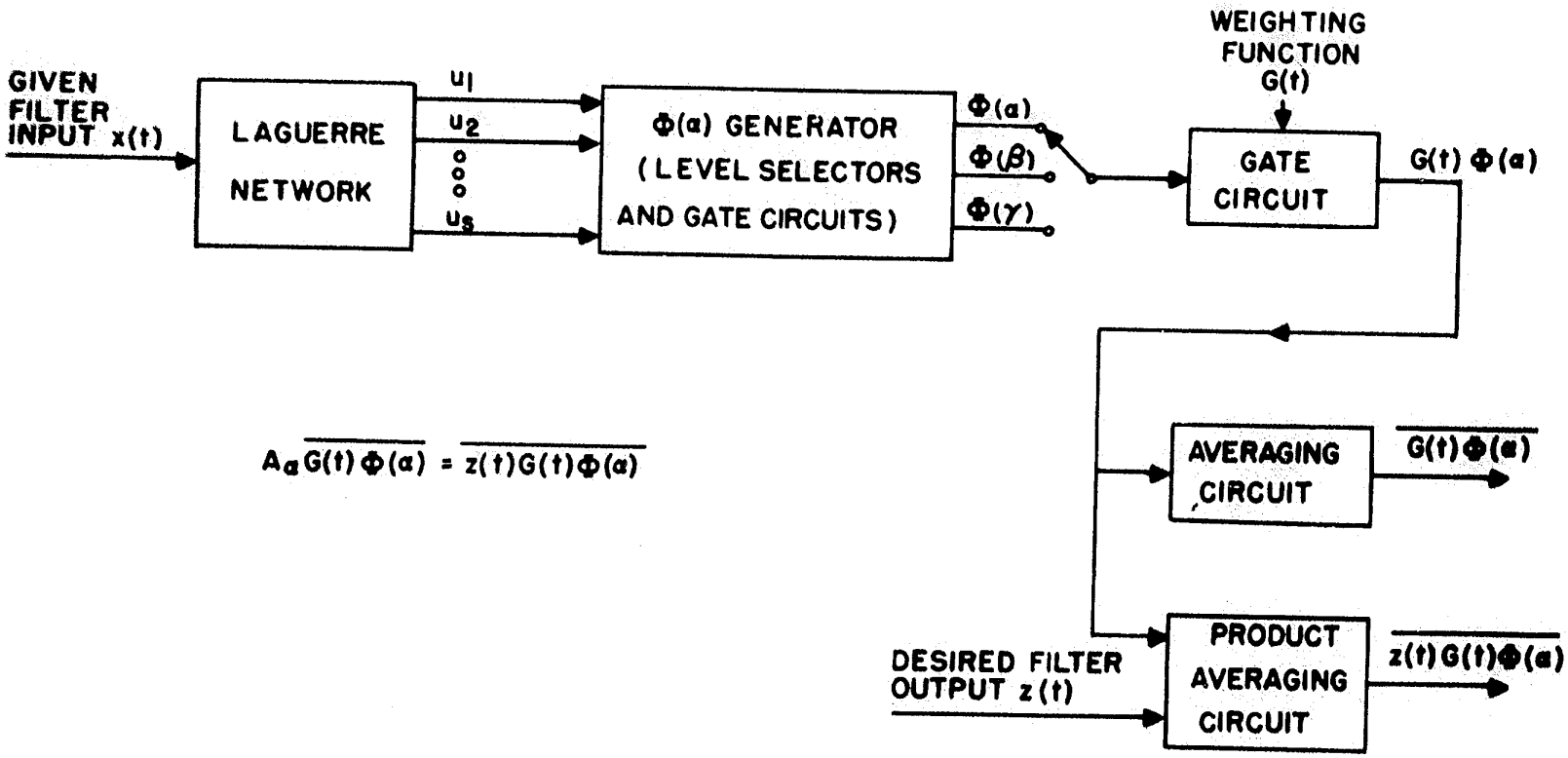

Fig. 13. Experimental procedure for the determination of optimum nonlinear filters involving storage.

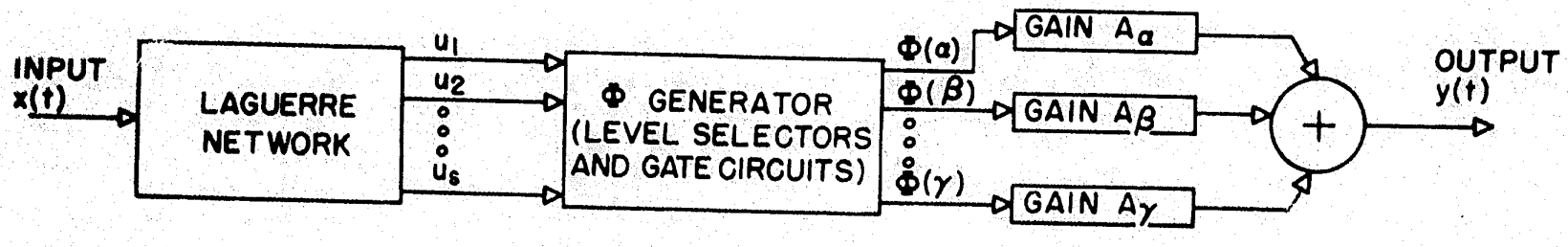

Fig. 14. Synthesis of general optimum nonlinear filters according to Eq. (53).

for the output of the filter in which the time domain orthogonality of the terms of the series is independent of the filter input. This enables us to obtain the optimum filter coefficients for arbitrary filter inputs without solving simultaneous equations.

2. Since the gate functions are orthogonal with respect to any weighting factor, we can determine optimum filters for weighted mean square error criteria.

3. In most series representations of a function we encounter the difficulty that over some region of the independent variable small differences of two or more large terms are necessary to represent the desired function. In the gate function expansion (Eq. 53) only one term has a non-zero value at any one instant of time; so this difficulty does not arise.

4. In general, expansions that represent nonlinear functions involve the use of multipliers in the experimental setup. (For example, if a Taylor series or Hermite function expansion is used.) The use of gate functions replaces the multipliers by simpler level selectors and coincidence circuits.

3.5 The Error Criterion

An error weighting function G(t) appears in the error expressions Eq. (41) and Eq. (54) for the no-storage and the general filter respectively. The choice of this weighting function will, of course, depend upon the particular problem. It may be chosen as a function of the past, present, and/or future of x(t) and z(t) and can be generated in the laboratory from the recorded ensemble members of x(t) and z(t). If G(t) is a constant

then the mean-square error criterion results. Other choices for G(t) enable us to design filters for different error criteria and to introduce a priori information into the filter design. In this section a few choices of G(t) are discussed. We restrict G(t) to be non-negative since the concept of negative error is not meaningful.

One choice of G(t) is illustrated by the following example. Let the signal component z(t) of the filter input x(t) consist of amplitude modulated pulses occurring periodically. x(t) is z(t) corrupted in some way by noise as shown in Fig. 15. We assume that we know when the signal pulses occur. Our object is to determine their amplitude. The optimum mean-square filter, of a given class of filters, for recovering z(t) from x(t) is the one for which the time average of $[z(t) - y(t)]^2$ is a minimum for all filters of the class. (In this expression y(t) is the filter output.) However, we are actually interested in minimizing the error between z(t) and y(t) only during the time when signal pulses are present. By choosing a G(t) that is a constant during the time intervals when signal pulses occur and zero at all other times (Fig. 15) we can design just such a filter. In general, if both these filters have the same degree of freedom (i.e., the same number of Laguerre coefficients and gate functions) the performance of the one designed with the weighting function mentioned above will be superior to that of the mean-square filter since all the freedom of the former is used to minimize the error over the time intervals of interest. Thus through G(t) we have introduced a priori information (about the periodic occurrence of the signal pulses) into the filter design to obtain a better filter.

In other problems it may be desirable to choose G(t) to be a function of the magnitude of the difference between the present values of x(t) and

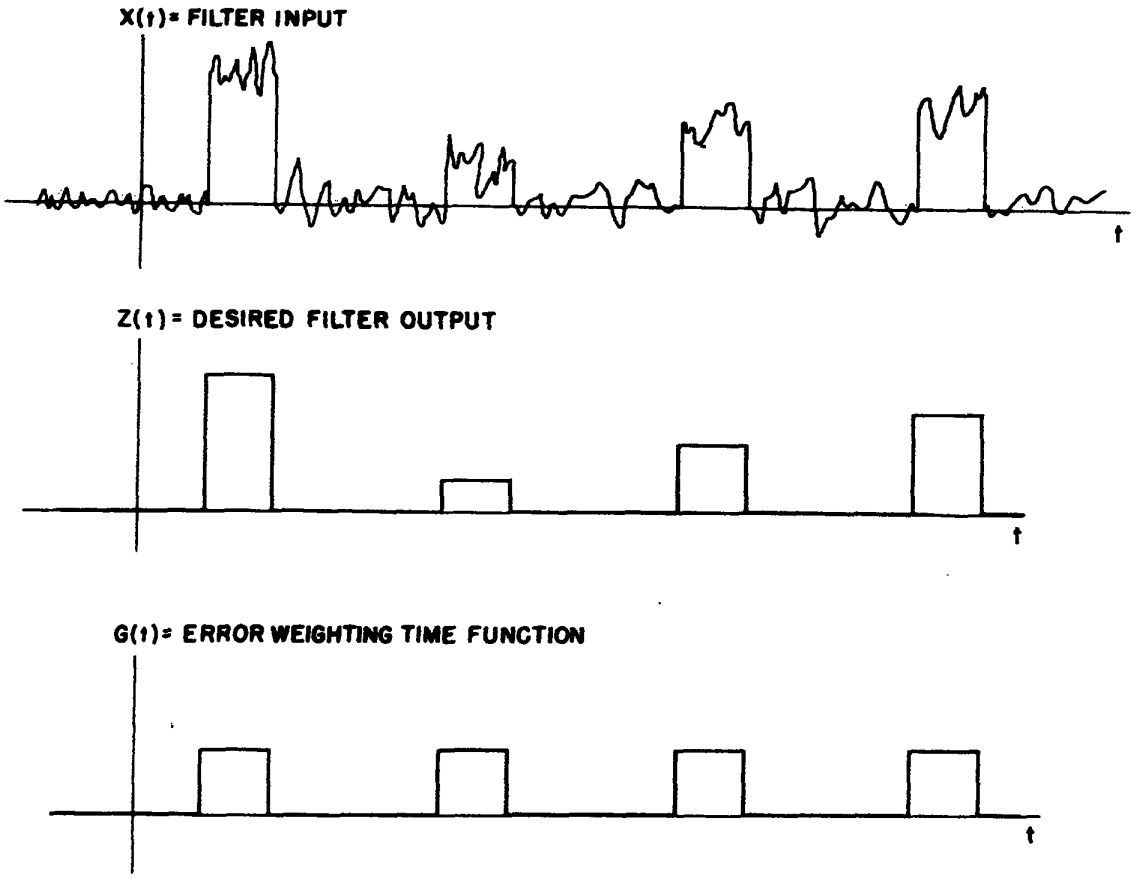

Fig. 15. An example of the use of the error weighting time function.

z(t) so that the freedom of the filter is used to do a better job, on the average, when $|x(t) - z(t)|$ lies in certain ranges, at the expense of its performance when this difference lies in other ranges. For example, we might desire that the filter output be as close as possible to the desired output, on the average, when the difference between the input and desired output is small and, when this difference is large, we might choose to attach less significance to the filter output. In such a case we could let G(t) be $|x(t) - z(t)|^{-n}$. For large n this G(t) weights small errors between x(t) and z(t) much more heavily than large errors.

The choices of G(t) are limited only by the ingenuity of the designer to best make use of the data at his disposal. By precisely defining the object of the particular filter problem and carefully studying the nature of the problem he may often be able to choose a G(t) that yields a far better filter than the mean-square filter.

3.6 Minimum Error Determination

Paralleling the Wiener approach to linear filters we shall find an expression for the minimum error of nonlinear filters that can be evaluated from a knowledge of the input and desired output time functions. The general expression for the error between the desired output and the actual nonlinear system output is given by Eq. (54) which is repeated below for convenience.

$$\mathcal{E} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} G(t) \left\{ z(t) - \sum_{a} A_a \Phi(a) \right\}^2 dt \qquad (54)$$

We have seen (Eq. 55) that for this error to be a minimum with respect to the A_a's we must have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} G(t)\, \Phi(\beta) \left[z(t) - \sum_{a} A_a\, \Phi(a) \right] dt = 0 \qquad \text{for all } \beta \tag{59}$$

and hence

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_{\beta} A_\beta\, \Phi(\beta)\, G(t) \left[z(t) - \sum_{a} A_a\, \Phi(a) \right] dt = 0 \tag{60}$$

Now Eq. (54) can be written as follows:

$$\mathcal{E} = \overline{G(t)\, z(t) \left[z(t) - \sum_{a} A_a\, \Phi(a) \right]} - \overline{\sum_{\beta} A_\beta\, \Phi(\beta)\, G(t) \left[z(t) - \sum_{a} A_a\, \Phi(a) \right]} \tag{61}$$

But from Eq. (60) we see that the term on the right side of Eq. (61) is zero for the optimum filter. Using this fact and inserting the expression for the optimum filter coefficients (Eq. 58) into Eq. (61) we obtain the desired expression for the minimum error.

$$\mathcal{E}_{min} = \overline{z^2(t)\, G(t)} - \sum_{a} \frac{\overline{z(t)\, G(t)\, \Phi(a)}^2}{\overline{G(t)\, \Phi(a)}} \tag{62}$$

This equation expresses the error of the optimum system, having a given number of Laguerre coefficients and gate functions, in terms of the filter input and desired output time functions. If, in Eq. (62), $\Phi(a)$ is changed to $\phi_j(x)$ and the summation is taken over j then we have the minimum error expression for no-storage filters.

With the addition of a squaring device at the output of the product averaging circuit in Fig. 13 the quantities necessary to determine $\mathcal{E}_{min}$ can be evaluated and $\mathcal{E}_{min}$ can thus be found without first constructing

the optimum filter. Similar apparatus could be built to automatically evaluate $\mathcal{E}_{min}$ upon application of x(t) and z(t). For those filters having a sufficiently small number of A_a's (for example, no-storage filters and simple filters involving one or two Laguerre coefficients) all the terms in the sum (Eq. 62) could be evaluated simultaneously and added. This would give a rapid way of finding $\mathcal{E}_{min}$. When the number of coefficients becomes very large then, to save equipment at the expense of time, the terms in the sum could be evaluated sequentially. This apparatus would be useful in deciding a priori the complexity of the nonlinear filter to use for a particular problem. It would also enable us to decide whether or not it is worthwhile to construct a complicated nonlinear filter to replace a simple linear or nonlinear one. Since such apparatus would make use of the same measurements that determine the A_a's, if after measuring its error we decided to build the filter we could construct it without further measurements.

3.7 The Statistical Approach

We can shed additional light on the filter theory that was developed in the previous sections by formulating the same problem on a statistical basis. As before, we shall characterize the past of the filter input by s Laguerre coefficients u_1 through u_s and determine the optimum nonlinear operator that relates these coefficients to the system output for a weighted mean square error criterion.

Consider an ensemble of the Laguerre coefficients $u_1(t), \ldots, u_s(t)$ and corresponding ensembles of the system output y(t), the desired output z(t), and the weighting function G(t). We shall regard $u_1, \ldots, u_s$, y, z, and G as random variables. We want to find the y, as a function of the u's, that

minimizes the error

$$\mathcal{E} = \int_G \int_z \int_{u_1} \cdots \int_{u_s} G(z-y)^2 \, P(z, G, u_1, \ldots, u_s) \, du_1 \ldots du_s \, dz \, dG \tag{63}$$

This expression is seen to correspond to the weighted mean square criterion of Eq. (54). The y that minimizes this expression is found by direct application of the calculus of variations. Setting the variation of $\mathcal{E}$ to zero we obtain

$$\delta(\mathcal{E}) = \int_G \int_z \int_{u_1} \cdots \int_{u_s} -2G(z-y) \, P(z, G, u_1, \ldots, u_s) \, \delta(y) \, du_1 \ldots du_s \, dz \, dG = 0 \tag{64}$$

or the equivalent expression

$$\delta(\mathcal{E}) = \int_G \int_z \int_{u_1} \cdots \int_{u_s} -2G(z-y) \, P(z, G|u_1, \ldots, u_s) \, dz \, dG \, \delta(y) \, P(u_1, \ldots, u_s) \times du_1 \ldots du_s = 0 \tag{65}$$

which must be true for all $\delta(y)$ where $\delta(y)$ is the variation in y. Equation (65) will hold for all $\delta(y)$ if we set

$$\int_G \int_z G(z-y) \, P(z, G|u_1, \ldots, u_s) \, dz \, dG = 0 \tag{66}$$

From this equation we obtain the equation

$$y \int_G \int_z G \, P(z, G|u_1, \ldots, u_s) \, dz \, dG = \int_G \int_z G \, z \, P(z, G|u_1, \ldots, u_s) \, dz \, dG \tag{67}$$

Performing the integration with respect to z in the left side of this equation we obtain the result

$$y = \frac{\int_G \int_z G\, z\, P(z, G|u_1, \ldots, u_s)\, dz\, dG}{\int_G G\, P(G|u_1, \ldots, u_s)\, dG} \tag{68}$$

This equation, though it is certainly not very suggestive of a filter design, is the desired relation between the optimum filter output and the s Laguerre coefficients of the past of the filter input. It should be noted that in deriving Eq. (68) no restrictions have been made on the relationship between y and the s Laguerre coefficients and hence this equation yields the optimum y in terms of these coefficients. Equation (68) takes on the more familiar form

$$y = \int_z z\, P(z|u_1, \ldots, u_s)\, dz \tag{69}$$

when G is a constant, corresponding to the mean square error criterion. In this case we have the result that the optimum output for a given past of the input is just the conditional mean of the desired output given this past of the input.

Let us now investigate the relation between the result of the statistical approach (Eq. 68) and that of the so-called time domain approach (Eq. 58). For convenience this latter equation is repeated below

$$A_a\, \overline{G(t)\, \Phi(a)} = \overline{G(t)\, z(t)\, \Phi(a)} \tag{58}$$

We shall express both these time averages as ensemble averages and then compare the result to Eq. (68). The average on the left side of Eq. (58)

is equal to

$$\overline{G(t)\,\Phi(a)} = \int_G \int_{u_1} \cdots \int_{u_2} G\,\Phi(a)\,P(G|u_1,\ldots,u_s)\,P(u_1,\ldots,u_s)\,dG\,du_1\ldots du_s \tag{70}$$

For the present let us assume that the width of the gate functions is so small that the u's are essentially constant over each cell in the function space. Then Eq. (70) can be written

$$\overline{G(t)\,\Phi(a)} \approx P(u_{1_j},\ldots,u_{s_h})\,(\Delta u_1,\ldots,\Delta u_s)\int_G G\,P(G|u_{1_j},\ldots,u_{s_h})\,dG \tag{71}$$

in which the factor multiplying the integral is just the probability that the ath cell in the function space is occupied.

By a similar procedure we have for the average on the right side of Eq. (58)

$$\overline{G(t)\,z(t)\,\Phi(a)} \approx P(u_{1_j},\ldots,u_{s_h})(\Delta u_1,\ldots,\Delta u_s)\int_G\int_z G\,z\,P(zG|u_{1_j},\ldots u_{s_h})\,dG\,dz \tag{72}$$

Using Eqs. (71) and (72) in Eq. (58) we obtain the result

$$A_a \approx \frac{\int_G\int_z G\;z\;P(zG|u_{1_j},\ldots,u_{s_h})\;dG\;dz}{\int_G G\;P(G|u_{1_j},\ldots,u_{s_h})\;dG} \tag{73}$$

Recall that A_a is just the system output when the ath cell in the function space is occupied. Thus as the cells become smaller the system determined by Eq. (58) approaches the optimum system of Eq. (68).

Let us now remove the restriction on the size of the cells. For simplicity in interpreting the results we shall let G = 1, corresponding to the mean square error criterion. Equation (58) becomes

$$A_a \overline{\Phi(a)} = \overline{z(t)\,\Phi(a)} \tag{74}$$

Expressing the time averages as ensemble averages we have for $\overline{\Phi(a)}$

$$\overline{\Phi(a)} = \int_{u_1} \cdots \int_{u_s} \Phi(a)\, P(u_1, \ldots, u_s)\, du_1 \ldots du_s \tag{75}$$

But this is just the probability that the ath cell is occupied. That is

$$\overline{\Phi(a)} = P(a\text{th cell}) \tag{76}$$

For the time average on the right side of Eq. (74) we have

$$\overline{z(t)\,\Phi(a)} = \int_z \int_{u_1} \cdots \int_{u_s} z\,\Phi(a)\, P(u_1, \ldots, u_s|z)\, P(z)\, dz\, du_1 \ldots du_s \tag{77}$$

Integrating over the u's we obtain

$$\overline{z(t)\,\Phi(a)} = \int_z z\, P(a\text{th cell}|z)\, P(z)\, dz \tag{78}$$

in which $P(a\text{th cell}|z)\,dz$ is the probability that the ath cell is occupied given that z is in the interval dz about z. It is convenient to rewrite Eq. (78) in the form

$$\overline{z(t)\,\Phi(a)} = P(a\text{th cell}) \int_z z\, P(z|a\text{th cell})\, dz \tag{79}$$

Using Eqs. (79) and (76) in Eq. (74) we obtain the result

$$A_a = \int_z z\, P(z \,|\, a\text{th cell})\, dz \tag{80}$$

In words, A_a is equal to the conditional mean of the desired output given that the ath cell in the input function space is occupied. Hence the result of the filter theory developed in the previous sections can be interpreted (for G(t) = 1) as a procedure that quantizes the function space of the input and assigns an output to each cell equal to the conditional mean of the desired output given that this cell is occupied.

It is interesting to note that although we can interpret this filter theory either in the time domain or on a statistical basis, the former leads directly to associated equipment for the filter determination and synthesis while the latter just expresses a mathematical relationship between the quantities involved in the problem. For example, in the simple case in which G(t) = 1 Eq. (80) might suggest that we evaluate $P(z \,|\, a\text{th cell})$ for all cells and for all z and then perform the indicated integration. However, the time domain approach directly shows us that the convenient quantities to measure are those of Eqs. (75) and (79) which look quite formidable from the statistical point of view. Further when we introduce G(t) the problem appears considerably more complicated from the statistical point of view but as we have seen from the time domain approach it only involves the addition of a single gate circuit in the experimental apparatus (Fig. 13).

3.8 Optimum Nonlinear Filters for a Maximum Probability Criterion

In this section we discuss a method for determining a nonlinear filter

whose output is the most probable value of the desired output given the past of the input. It is shown that such a filter can be determined with a simple modification of the apparatus (Fig. 13) used to determine optimum filters for a weighted mean square criterion.

As in the case of the weighted mean square criterion we let the form of the nonlinear operator be that of Eq. (53). At any instant the system output is equal to the coefficient of the term that corresponds to the occupied cell in the function space of the past of the input. Hence for the maximum probability criterion we must choose each coefficient A_α to be equal to the most probable value of the desired output given that the αth cell is occupied.

An experimental procedure for determining maximum probability filters is shown in Fig. 16. The desired output z(t) is fed into a level selector circuit. If the amplitude of z(t) is in the amplitude interval corresponding to the level z_j then the output at the jth terminal of the level selector is unity. Otherwise this output is zero. The outputs of the level selector circuit, along with the output of the Φ generator, are fed into gate circuits. The output of the jth gate circuit is unity when z(t) is in the amplitude interval about z_j and, simultaneously, the αth cell in the function space is occupied. It is zero at all other times. The time average of this output is the probability of the simultaneous occurrence of these two events. We shall write this probability as $P(z_j, \alpha\text{th cell})$. But

$$P(z_j, \alpha\text{th cell}) = P(z_j | \alpha\text{th cell})\, P(\alpha\text{th cell}) \tag{81}$$

so that the output of the jth averaging circuit is proportional to $P(z_j|\alpha\text{th cell})$. If terminal k has the largest output of all the m terminals then z_k is the most probable value of z(t) given that the αth cell is occupied. (This of

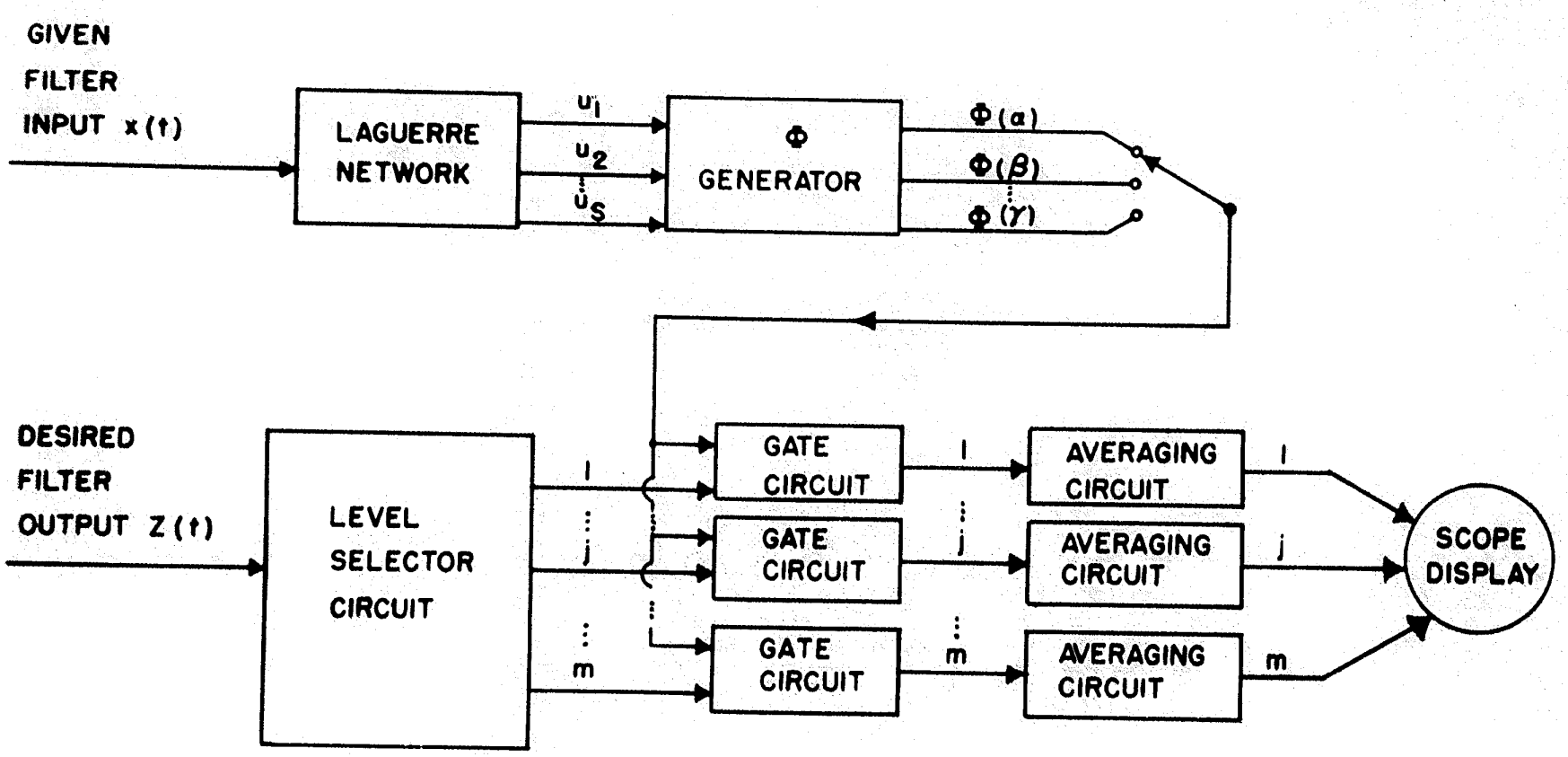

Fig. 16. Experimental procedure for determining optimum filters for a maximum probability criterion.

course assumes that m is large enough so that the amplitude interval associated with z_j is very small compared to the maximum amplitude of z(t).) Hence the optimum A_a for the maximum probability criterion is equal to z_k.

For convenience in rapidly determining which output of the averaging circuits is largest, the outputs can be displayed on an oscilloscope as indicated in Fig. 16. Once the A's are determined the filter can be synthesized in the standard form shown in Fig. 14.

3.9 Improving the Performance of a Given Filter

As we increase the complexity of the filter (i.e., we use more Laguerre coefficients to characterize the past and more gate functions for each coefficient) the number of A_a's necessary to specify the filter grows very rapidly. In particular, if we use s Laguerre coefficients and n gate functions for each coefficient we have n^s A_a's to evaluate. After evaluating a large number of A_a's we should like to have some guarantee that our filter would perform at least as well as, say, a linear filter or a simple nonlinear filter that can be designed with less effort. Methods of obtaining this guarantee will now be described.

Let us first of all prove the existance of a property of our class of filters which will be used in one of the methods. We want to show that the class of filters employing s Laguerre coefficients includes the class of filters that only uses any one of the s Laguerre coefficients. Since we can always renumber the Laguerre coefficients it is sufficient to prove that the s-coefficient class includes the class that uses only the first Laguerre coefficient u_1. The series representation for the

general system of this one coefficient class is

$$y(t) = \sum_{i=1}^{n} a_i \phi_i(u_1) \tag{82}$$

We now make use of the fact that the sum of the n gate functions of any one coefficient is unity in order to express the series representation (82) in the form

$$y(t) = \sum_{i=1}^{n} a_i \phi_i(u_1)\left[\sum_{j=1}^{n} \phi_j(u_2)\right]\left[\sum_{k=1}^{n} \phi_k(u_3)\right] \cdots \left[\sum_{h=1}^{n} \phi_h(u_s)\right] \tag{83}$$

which is recognized to be a special case of the expansion (Eq. 51) for the general s-coefficient system. In a similar way it can be shown that the class of filters using s Laguerre coefficients includes all classes having less than s coefficients. Note that this property is independent of the nature of the u's; they may be Laguerre coefficients of the past of x(t) or they may be obtained from x(t) by any linear or nonlinear operation.

We now make use of this property to determine a filter whose performance is equal or superior, with respect to a weighted mean square criterion, to a given filter F. F may be linear or nonlinear. We augment the Laguerre coefficients with the output, u_o, of the given filter F, as shown in Fig. 17. Then, by the property demonstrated above, the filter whose output is expressible as

$$y(t) = \sum_{i=1}^{n} a_i \phi_i(u_o) \tag{84}$$

is a member of the class of filters which has s Laguerre coefficients

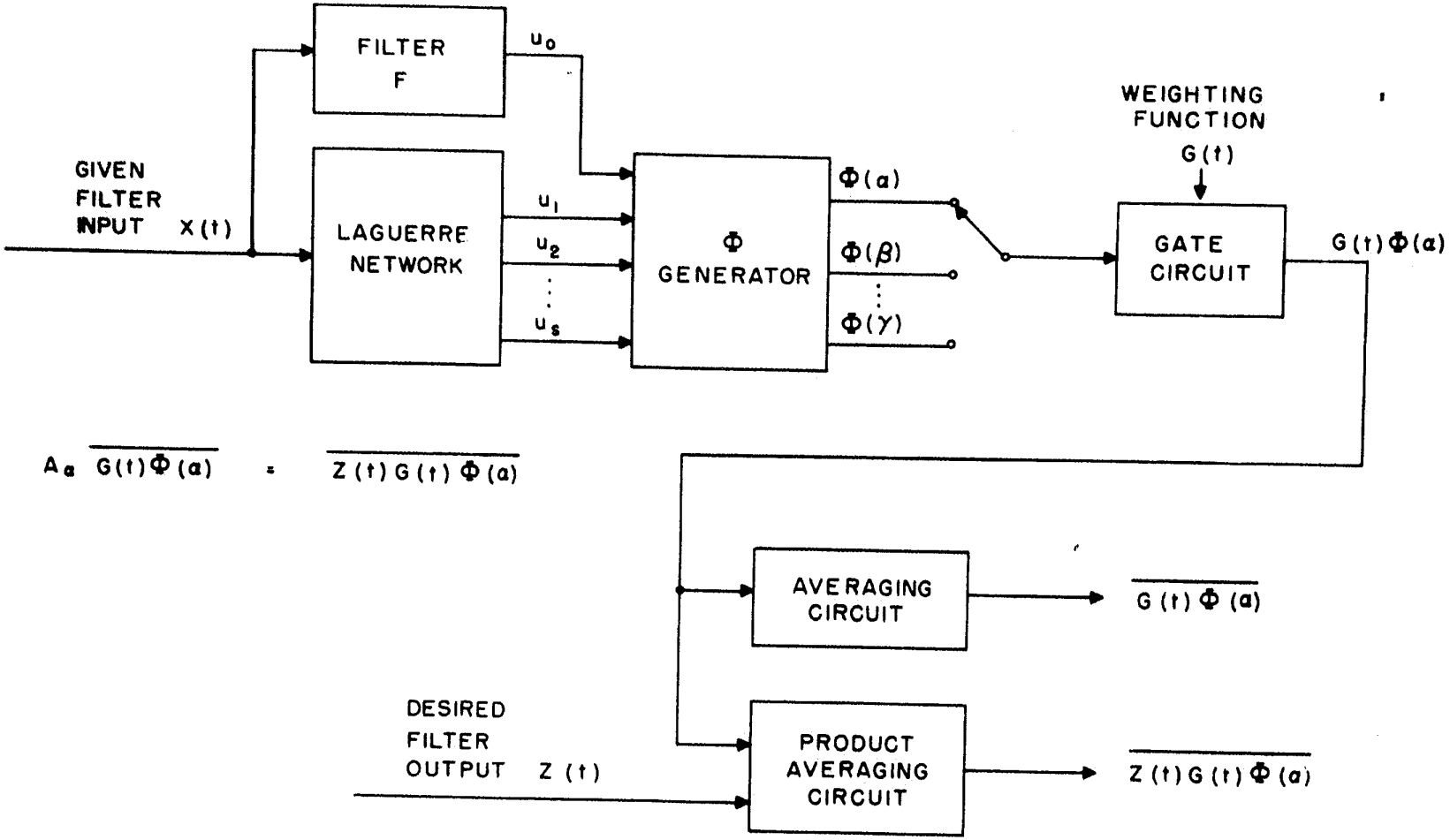

Fig. 17. Augmenting the Laguerre coefficients with the output of a given filter F to determine a filter having a better performance than that of F.

augmented by u_o. If the number n of gate functions associated with the variable u_o is sufficiently large, then to any degree of approximation Eq. (84) represents the class of filters shown in Fig. 18, consisting of F cascaded with a no-storage filter. Since the transfer characteristic of the no-storage system can be linear, the latter class certainly contains the filter F. Hence the filter determined by the procedure indicated in Fig. 17, for any s, performs at least as well as the given filter F and, in fact, at least as well as F cascaded with any no-storage filter.

Having determined the A_a's of the desired filter as indicated in Fig. 17 the filter synthesis is accomplished as shown in Fig. 19.

We now turn to another method of determining filters to improve the performance of given filters. Let the output of the given filter F be $u_o(t)$ when its input is x(t). Our object is to improve (with respect to a weighted mean square criterion) the performance of F by paralleling it with a filter which will be determined. The A_a's of the desired filter are those that minimize the error

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} G(t) \left\{ z(t) - \left[u_o(t) + \sum_a A_a \Phi(a) \right] \right\}^2 dt \qquad (85)$$

This expression is seen to be equivalent to

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} G(t) \left\{ [z(t) - u_o(t)] - \sum_a A_a \Phi(a) \right\}^2 dt \qquad (86)$$

Comparing Eq. (86) with Eq. (54) we see that the optimum A_a's are determined by an experimental procedure like that of Fig. 13 with z(t) replaced by $z(t) - u_o(t)$. The latter quantity is easily obtained by applying

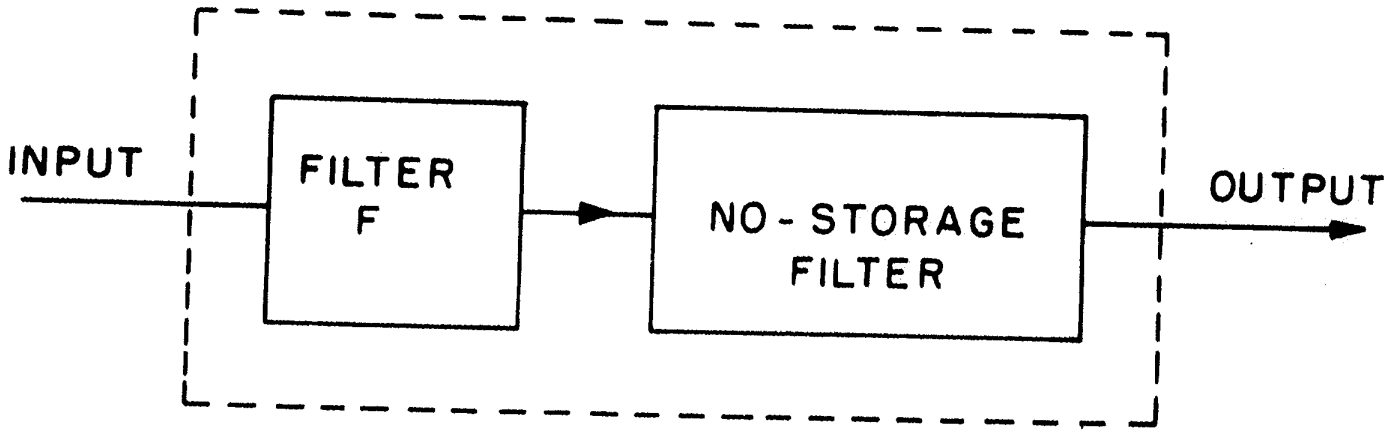

Fig. 18. The class of filters consisting of a given filter F cascaded with a no-storage filter.

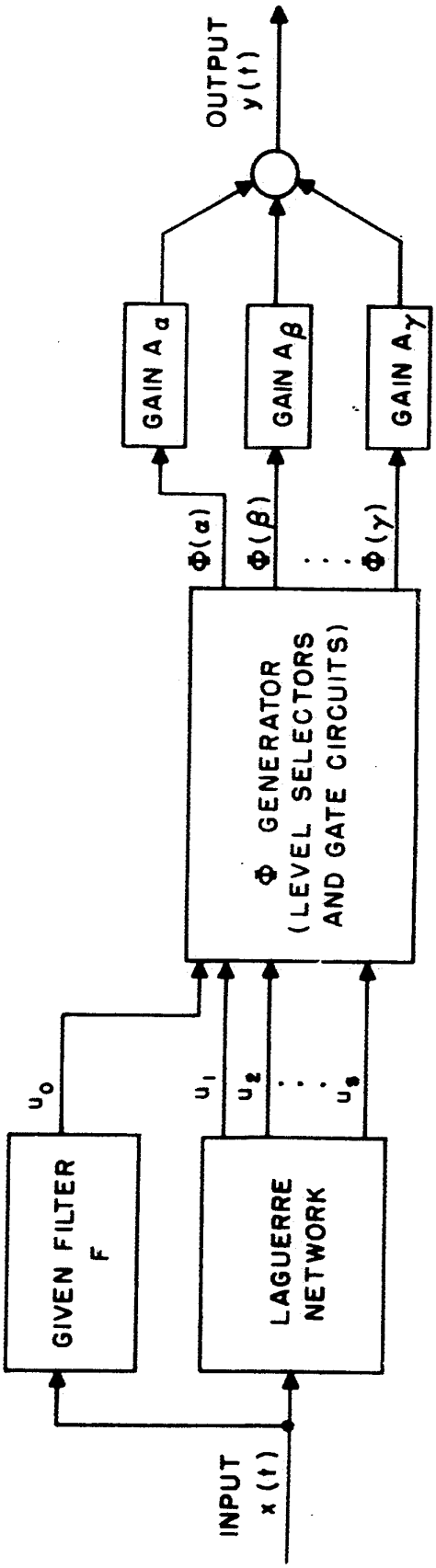

Fig. 19. Synthesis of the filter whose coefficients are determined according to Fig. 17.

x(t) to the given filter F and subtracting the output of F from z(t), as shown in Fig. 20. The parallel combination of F and the filter determined as we have just described will always perform at least as well as F, since that filter which has no transmission from input to output is a member of the class of filters considered in our theory. In other words, the solution in which all the A_a's are equal to zero is a possible solution of Eqs. (84) and (85).

The second method described for improving the performance of given filters offers the advantage of not having gate functions associated with the output time function of the given filter; therefore improvements can be made on the performance of F by very simple systems involving as little as one Laguerre coefficient and hence having a relatively small number of A_a's to evaluate. The first method does require a gate function expansion of the output of the given filter F but it has the advantage of ensuring that the performance of the resultant filter will always be at least as good as the performance of F cascaded with any no-storage system. In either method, the resultant over-all filter approaches the most general filter (of the class considered in this theory) as the number of Laguerre coefficients and gate functions is increased.

Still another design procedure involves the determination of that filter which when cascaded with F (with F as the first member of the combination) yields an over-all filter having a performance superior to that of F alone. In order to ensure that the resultant over-all filter performs at least as well as F we could augment the Laguerre coefficients of the cascaded filter by a variable u_o that is equal to the input of the Laguerre network. While this procedure gives a filter that is at least as good as F we have

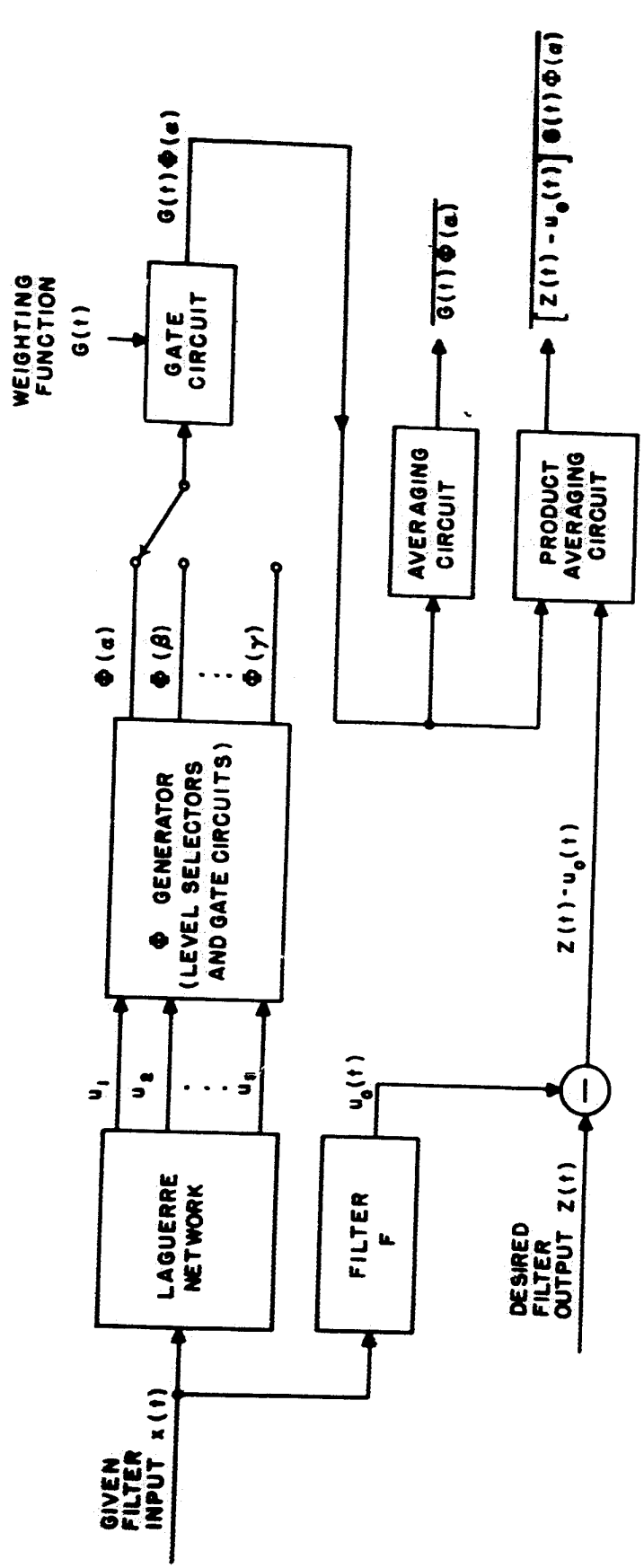

Fig. 20. Determining a filter which when connected in parallel with F will improve the performance of F.

no assurance that as the number of Laguerre coefficients and Hermite functions is increased the over-all filter will approach the most general filter.

3.10 Multiple Nonlinear Prediction

The problem of multiple prediction is that of predicting a time series from a knowledge of related time series. An example, cited by Wiener, is the prediction of weather at one location from a knowledge of the past of the weather at that and other surrounding locations.

The filter theory developed in the preceding sections is easily extended to the problem of multiple prediction. Let $z(t+\alpha)$ be the function that we desire to predict and let $x_1(t)$ through $x_p(t)$ be the input functions on whose past we operate to form our prediction. The set of functions $x_1(t)$ through $x_p(t)$ may, indeed, include $z(t)$. We shall characterize the past of each input by a set of Laguerre coefficients. Let $u_{1_j}, u_{2_j}, \ldots, u_{s_j}$ be the Laguerre coefficients of $x_j(t)$. Now let us think of a function space that encompasses the past of all p of the input time functions. That is, the basis of this space is formed by the Laguerre functions associated with each input. A point in this space then represents the past of all the inputs and hence the multiple prediction problem is just the problem of mapping this space onto a line (corresponding to the amplitude of the system output) in a manner that is optimum with respect to some error criterion. But this is the same problem encountered in the single input filter problem and we recognize that the only difference between the single and multiple input problem is in the number of dimensions of the function space. The solution of the multiple prediction problem directly parallels that

of the filter problem given in section 3.4.

We let y(t) be a gate function expansion of the Laguerre coefficients of all the inputs $x_1(t)$ through $x_p(t)$. The expansion is

$$y(t) = \sum_{i_1} \sum_{j_1} \cdots \sum_{h_1} \cdots \sum_{i_p} \sum_{j_p} \cdots \sum_{h_p} a_{i_1, j_1, \ldots, h_1, \ldots, i_p, j_p, \ldots, h_p}$$

$$\times \phi_{i_1}(u_{11}) \phi_{j_1}(u_{21}) \cdots \phi_{h_1}(u_{s1}) \cdots \phi_{i_p}(u_{1p}) \phi_{j_p}(u_{2p}) \cdots \phi_{h_p}(u_{sp}) \tag{87}$$

If we associate a $\Omega(\alpha)$ with each product of ϕ's in Eq. (87) and let A_α be the corresponding coefficient $a_{i_1, j_1, \ldots, h_1, \ldots, i_p, j_p, \ldots, h_p}$, Eq. (87) takes the simplified form

$$y(t) = \sum_{\alpha} A_\alpha \, \Omega(\alpha) \tag{88}$$

Just as in the filter problem (section 3.4) we adopt a weighted mean square error criterion and minimize the error

$$\mathcal{E} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} G(t) \left\{ z(t+\alpha) - \sum_{\alpha} A_\alpha \, \Omega(\alpha) \right\}^2 dt \tag{89}$$

This equation is seen to have the same form as Eq. (54) for the filter problem. The solution for the optimum A's proceeds exactly as in section 3.4 with result

$$A_\alpha \, \overline{G(t) \, \Omega(\alpha)} = \overline{z(t+\alpha) \, G(t) \, \Omega(\alpha)} \tag{90}$$

The apparatus for experimentally determining the A's according to this equation is shown in Fig. 21. Having determined the coefficients, the predictor can be synthesized in accordance with Eq. (88) as shown in Fig. 22.

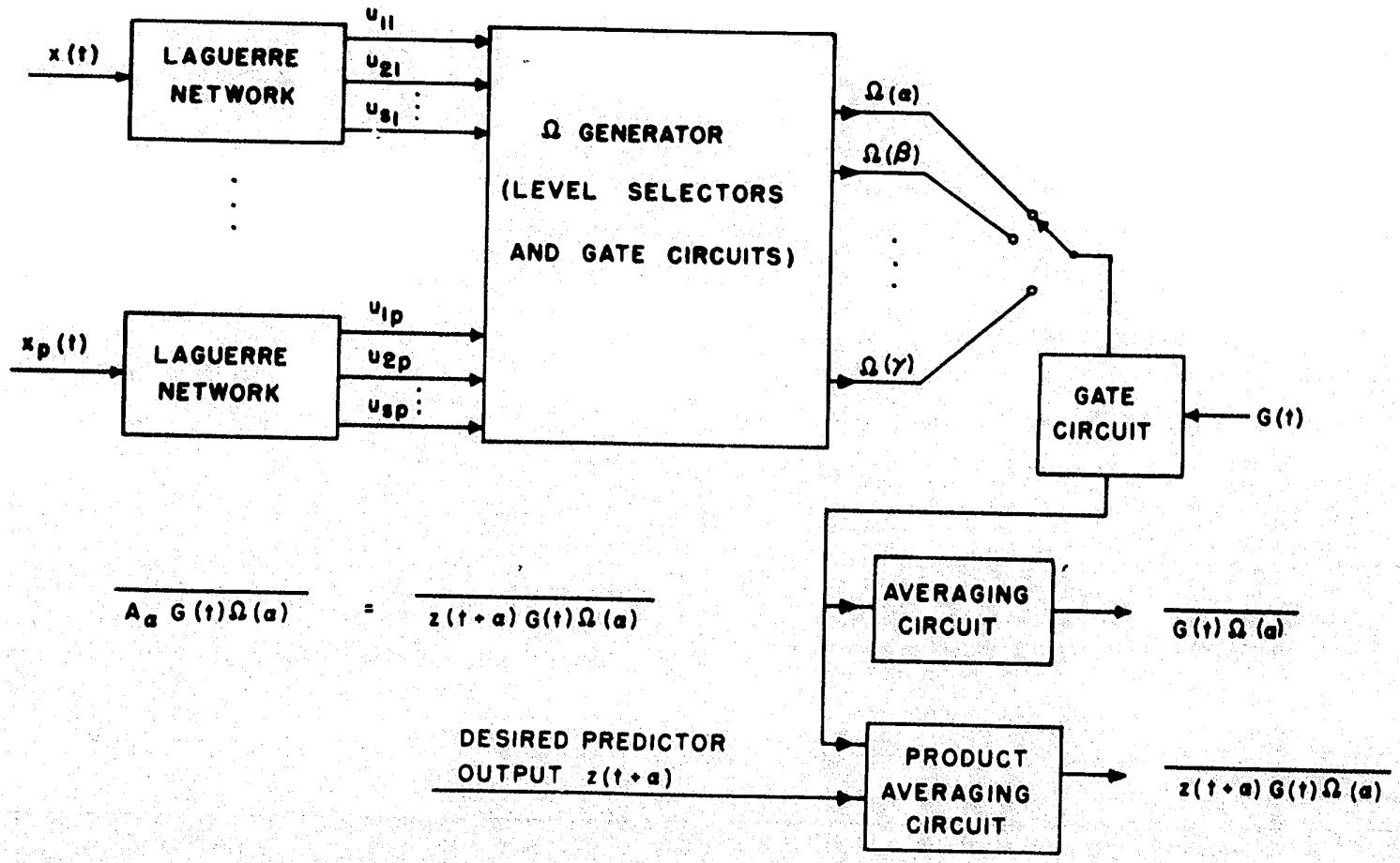

Fig. 21. Experimental procedure for the determination of the optimum multiple predictor.

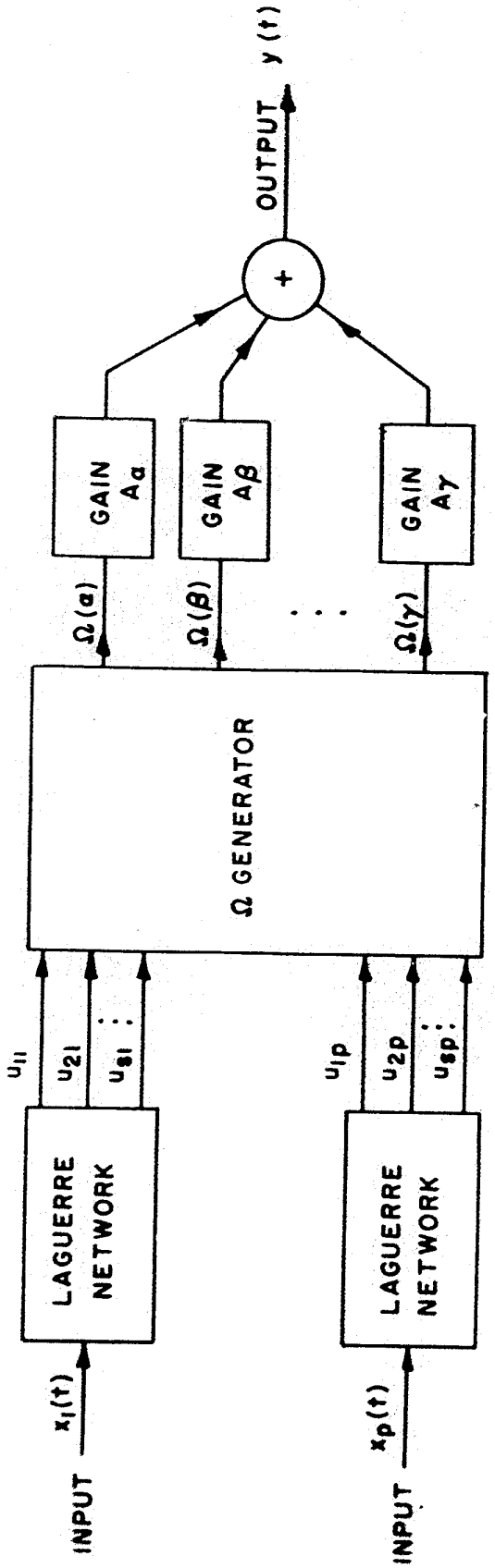

Fig. 22. Synthesis of the optimum multiple predictor.

IV. Linear and Simple Nonlinear Systems

4.1 A Test for the Coefficients

Once the coefficients for an optimum filter have been determined, the filter can, as we have seen, be synthesized formally as shown in Fig. 14. Much simpler synthesis procedures (apparatus wise) exist however if the filter is linear or belongs to a particular sub-class of nonlinear systems. Hence it is desirable to have a means of detecting linear and simple nonlinear systems directly from a knowledge of their characterizing coefficients. In this section a simple procedure for testing the coefficients (A_a's) is developed that detects such systems and directly yields a convenient synthesis of them.

The class of simple nonlinear systems that we shall consider is shown in Fig. 23. It consists of a Laguerre network and no-storage nonlinear two-poles (no-storage nonlinear systems with one input and one output terminal). Each output of the Laguerre network is fed into one no-storage two-pole circuit and the outputs of these circuits are added to form the system output y(t). In this class of systems the nonlinear circuits introduce no cross-talk among the Laguerre coefficients (i.e., there are no cross products of Laguerre coefficients introduced). This class of systems is clearly a sub-class of the general class considered in Section III. When the transfer characteristics of all the two-poles are straight lines the system is linear. In particular, it is an sth order Laguerre network in which the gain factors associated with the Laguerre network outputs are equal to the slopes of the respective linear two-pole transfer characteristics.

The synthesis of nonlinear systems belonging to this sub-class is relatively simple, apparatus wise. The nonlinear two-poles may be

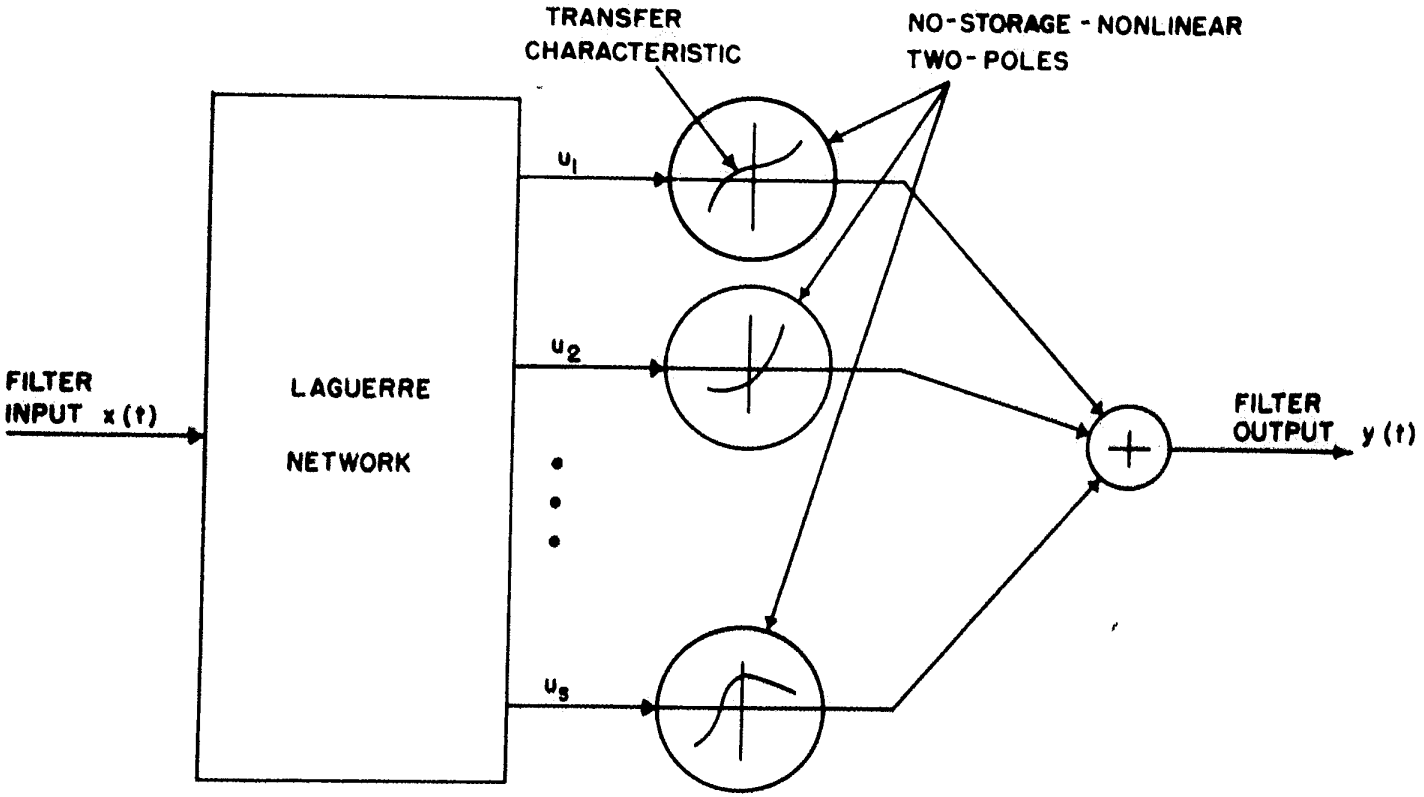

Fig. 23. The class of nonlinear systems having no cross products of the Laguerre coefficients.

synthesized by piecewise linear approximations using diodes and resistors. If the system is linear the synthesis takes the form of a Laguerre network whose outputs are properly amplified or attenuated before being added to form the system output. In this linear case it may be desirable to measure the transfer function of the optimum system and then use available synthesis techniques to obtain alternate realizations of this transfer function using linear passive circuits.

We now investigate how we can determine, from the characterizing coefficients A_a of a nonlinear system, if the system belongs to the class shown in Fig. 23 and, if it does, how we can determine the transfer characteristics of the nonlinear two-poles.

From the function space point of view the sub-class of nonlinear systems shown in Fig. 23 consists of all those systems for which the system output corresponding to each cell in function space is equal to the sum of the outputs corresponding to the coordinates (the Laguerre coefficients) of the cell. That is, since there is no cross-talk, each coordinate contributes to the output an amount that is independent of the other coordinates and hence the system output corresponding to any cell is the sum of the outputs corresponding to the coordinates of the cell. (We speak here of cells rather than points in function space because we represent our nonlinear system by a gate function expansion which quantizes the function space. In this connection we should also realize that we obtain a stepwise approximation to the two-pole characteristics of Fig. 23 rather than the continuous curves.) Hence Eq. (51), which represents a general gate function expansion of the Laguerre coefficients, takes on the form

$$y(t) = \sum_{i=1}^{n} b_{i1}\, \phi_i(u_1) + \sum_{j=1}^{n} b_{j2}\, \phi_j(u_2) + \ldots + \sum_{h=1}^{n} b_{hs}\, \phi_h(u_s) \qquad (91)$$

for the sub-class of systems of Fig. 23. Each sum in the equation yields the transfer characteristic of the nonlinear two-pole that is associated with the Laguerre coefficient indicated in that sum. For convenience in comparing this expansion with that of Eq. (51) the latter is repeated below.

$$y(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \cdots \sum_{h=1}^{n} a_{i,j,\ldots,h}\, \phi_i(u_1)\, \phi_j(u_2) \cdots \phi_h(u_s) \qquad (51)$$

In the experimental procedure described in Section III for finding optimum filters (Fig. 13) we determine the A_α or equivalently the $a_{i,j,\ldots,h}$ of the filter. If and only if these a's are such that Eq. (51) can be expressed in the form of Eq. (91), the system can be synthesized according to Eq. (91) in the form shown in Fig. 23. Let us see how the a's must be related to the b's if these two equations are to be equivalent. By the two equations being equivalent we mean that they yield the same value for every cell in function space; hence we must have

$$a_{i,j,\ldots,h} = b_{i1} + b_{j2} + \cdots + b_{hs} \qquad \text{for all } i,j,\ldots,h \qquad (92)$$

This relation represents a set of n^s simultaneous equations that the a's must satisfy. We shall now develop a simple way of finding whether, for any given set of a's, this set of equations is satisfied.

It is convenient to establish an order for the evaluation of the a's (and thus the A_α's). It is assumed henceforth that these coefficients are evaluated as follows. The first coefficient we evaluate is that for which $i = j = \ldots = h = 1$. The next $n - 1$ coefficients are obtained by letting h run from 2 to n while holding all other indicies equal to unity. To obtain the (n+1)th coefficient we set the index preceding h to 2 and let all the other indices be unity. The following $n - 1$ coefficients are obtained by

again letting h range from 2 to n. We continue this procedure until the index i has gone through all its n values, at which point all the a's will have been evaluated. This order of evaluating the coefficients is best illustrated by a simple example. Consider the coefficients (a's) of a nonlinear system having three Laguerre coefficients and two gate functions for each coefficient. That is, s = 3 and n = 2. There are $n^s = 8$ coefficients $a_{i,j,k}$ to evaluate. According to the above procedure these coefficients are evaluated in the following order:

1.	a_{111}	5.	a_{211}
2.	a_{112}	6.	a_{212}
3.	a_{121}	7.	a_{221}
4.	a_{122}	8.	a_{222}

A study of the order in this simple example is sufficient to establish the order of evaluating the a's for any s and n.

Now think of the coefficients $a_{i,j,\ldots,h}$, arranged in the order of evaluation, as components of a vector A and the corresponding coefficients $b_{11}, b_{21}, \ldots, b_{n1}, b_{12}, b_{22}, \ldots, b_{n2}, \ldots, b_{1s}, b_{2s}, \ldots, b_{ns}$, arranged as shown, as components of a vector B. Then the set of equations represented by Eq. (92) can be written in matrix form as follows

$$A] = [M]\,B] \tag{93}$$

where [M] is the matrix that operates on the vector B to give the vector A. Let us determine the form of the matrix [M]. In order to illustrate the form of this matrix we shall consider a nonlinear system for which s = n = 3 and for which Eq. (92) is assumed to hold. From the results of this example the form of [M] can be visualized for any s and n. The equations indicated

in Eq. (92) are written, for this example, in Table 1. The b's are written at the top of the columns so that the form of the matrix [M] is evident. The actual equations are obtained by dropping the b's down beside the unity coefficients. (All the blank spaces in the matrix represent zero matrix coefficients.) Thus the first equation reads

$$a_{111} = b_{11} + b_{12} + b_{13} \tag{94}$$

We see that the matrix [M] is composed entirely of zeros and ones. We also note the very regular pattern of the unity coefficients. A study of this pattern will enable the reader to visualize its form for any s and n.

We now state a test for the a's that enables us to directly find if the set of equations (92) or equivalently Eq. (93) is satisfied. The test was developed from a study of [M] and the reader can check its validity (for s = n = 3) by analyzing it in terms of the matrix of Table 1.

1. Starting with $a_{1,1,\ldots,1}$, plot the a's (consecutively in the order in which they are evaluated) in groups of n at unit intervals along a linear scale. That is, form a set of $n^{(s-1)}$ graphs, each of which contains n a's plotted at equal intervals along a line. We call this set of graphs set 1.

2. Take the first "a" of each graph above, starting with $a_{1,1,\ldots,1}$, and plot these consecutively in groups of n at unit intervals on a linear scale. Call this set of graphs set 2.

3. Repeat the procedure of step 2 until s sets of graphs are obtained. The sth set will consist of only one graph.

From a study of the general form of [M] it can be seen that if for a given set of a's there exists a set of b's such that Eq. (93) is satisfied then all

Table 1

		b_{11}	b_{21}	b_{31}	b_{12}	b_{22}	b_{32}	b_{13}	b_{23}	b_{33}
a_{111}	=	1			1			1		
a_{112}	=	1			1				1	
a_{113}	=	1			1					1
a_{121}	=	1				1		1		
a_{122}	=	1				1			1	
a_{123}	=	1				1				1
a_{131}	=	1					1	1		
a_{132}	=	1					1		1	
a_{133}	=	1					1			1
a_{211}	=		1		1			1		
a_{212}	=		1		1				1	
a_{213}	=		1		1					1
a_{221}	=		1			1		1		
a_{222}	=		1			1			1	
a_{223}	=		1			1				1
a_{231}	=		1				1	1		
a_{232}	=		1				1		1	
a_{233}	=		1				1			1
a_{311}	=			1	1			1		
a_{312}	=			1	1				1	
a_{313}	=			1	1					1
a_{321}	=			1		1		1		
a_{322}	=			1		1			1	
a_{323}	=			1		1				1
a_{331}	=			1			1	1		
a_{332}	=			1			1		1	
a_{333}	=			1			1			1

the graphs of these a's within each set of graphs will be identical except for a possible vertical translation. Further, the converse holds; that is, if all the graphs of the a's within each set of graphs are identical, within a vertical translation, then there exists a set of b's such that Eq. (93) is satisfied.

Hence, if there exists a system governed by Eq. (91) that is equivalent to a system governed by Eq. (51) for a given set of a's, then all the graphs of the a's, within each set of graphs, must be the same except for a possible vertical translation. And, conversely, if we apply the above test to the a's of a system and find that all the graphs within each set are the same within vertical translation we know that the system having these a's can be synthesized in the form shown in Fig. 23.

4.2 Synthesis Procedure

In this section we assume that the a's have been tested as described above and that they correspond to a system of the type shown in Fig. 23. We are now concerned with the synthesis of this system. In particular, we want to find the transfer characteristics of the no-storage two poles. One way to do this is to solve Eq. (93) for the b's, which, by Eq. (91), directly determine the no-storage transfer characteristics. As a consequence of the special form of [M] this solution is readily accomplished. However, a simpler method of synthesizing the desired system makes direct use of the graphs that are drawn when the a's are tested. Referring to the previous section let us examine, relative to the a's in Table 1, the sets of graphs defined in the test procedure. From inspection of Table 1 it is readily seen that all the graphs of set 1 are identical to

(except for a possible vertical translation) the graph that would be obtained if b_{13}, b_{23}, and b_{33} were plotted, in this order, at unit intervals on the same linear scale. Similarly it is seen that the graphs of set 2 are the same (within a vertical translation) as the graph that would be obtained by plotting b_{12}, b_{22}, and b_{32}. And finally the graphs of set 3 correspond, in a similar manner, to the graph that would be obtained by plotting b_{11}, b_{21}, and b_{31}. But notice (Eq. 91 for $n = s = 3$) that b_{13}, b_{23}, and b_{33} are just the heights of the steps in the stepwise approximation to the two-pole characteristic associated with u_3. Similarly b_{12}, b_{22}, and b_{32} determine the transfer characteristic of the nonlinear two-pole whose input is u_2. And finally b_{11}, b_{21}, and b_{31} determine the two-pole characteristic associated with u_1. Hence the graphs that are made when the a's are tested directly determine the desired two-pole transfer characteristics, within a vertical translation. In the synthesized nonlinear system (Fig. 23) the vertical displacement of the two-pole transfer characteristics affects only the dc level of the output. When this level is of interest it can be re-established by a series battery at the system output as will be shown in examples that follow.

From the above discussion we can readily generalize to the case of arbitrary s and n as follows: Except for a vertical translation, the graph of set 1 determines the transfer characteristic of the two-pole associated with u_s. The graph of set 2 determines the transfer characteristic of the two-pole associated with u_{s-1} and so on down to the graph of set s which determines the transfer characteristic of the two-pole associated with u_1

Example 4. 1

Suppose that we have determined the coefficients of an optimum filter of the class s = n = 3 and that they have the values given below.

$a_{111} = 1$	$a_{211} = 0$	$a_{311} = 2$
$a_{112} = 4$	$a_{212} = 3$	$a_{312} = 5$
$a_{113} = 3$	$a_{213} = 2$	$a_{313} = 4$
$a_{121} = 2$	$a_{221} = 1$	$a_{321} = 3$
$a_{122} = 5$	$a_{222} = 4$	$a_{322} = 6$
$a_{123} = 4$	$a_{223} = 3$	$a_{323} = 5$
$a_{131} = 5$	$a_{231} = 4$	$a_{331} = 6$
$a_{132} = 8$	$a_{232} = 7$	$a_{332} = 9$
$a_{133} = 7$	$a_{233} = 6$	$a_{333} = 8$

We shall test these coefficients to see if the corresponding system can be synthesized in the form shown in Fig. 23. The coefficients, plotted according to the test procedure described in section 4. 1, are shown in Fig. 24, In order to make the form of each graph stand out, consecutive points corresponding to the a's have been joined by straight lines. We see that all the graphs within each set of graphs are identical within a vertical translation. Hence the system can be synthesized in the form shown in Fig. 23.

The synthesized system is shown in Fig. 25. The transfer characteristic of the no-storage two-poles are obtained directly from the graphs of Fig. 24. The graph of set 3 is the transfer characteristic of the two-pole associated with u_1. That is, a_{111} is the coefficient of the first gate function $\phi_1(u_1)$, a_{211} is the coefficient of $\phi_2(u_1)$ and a_{311} is the coefficient of $\phi_3(u_1)$. Similarly, any graph in set 2 can be taken as the transfer

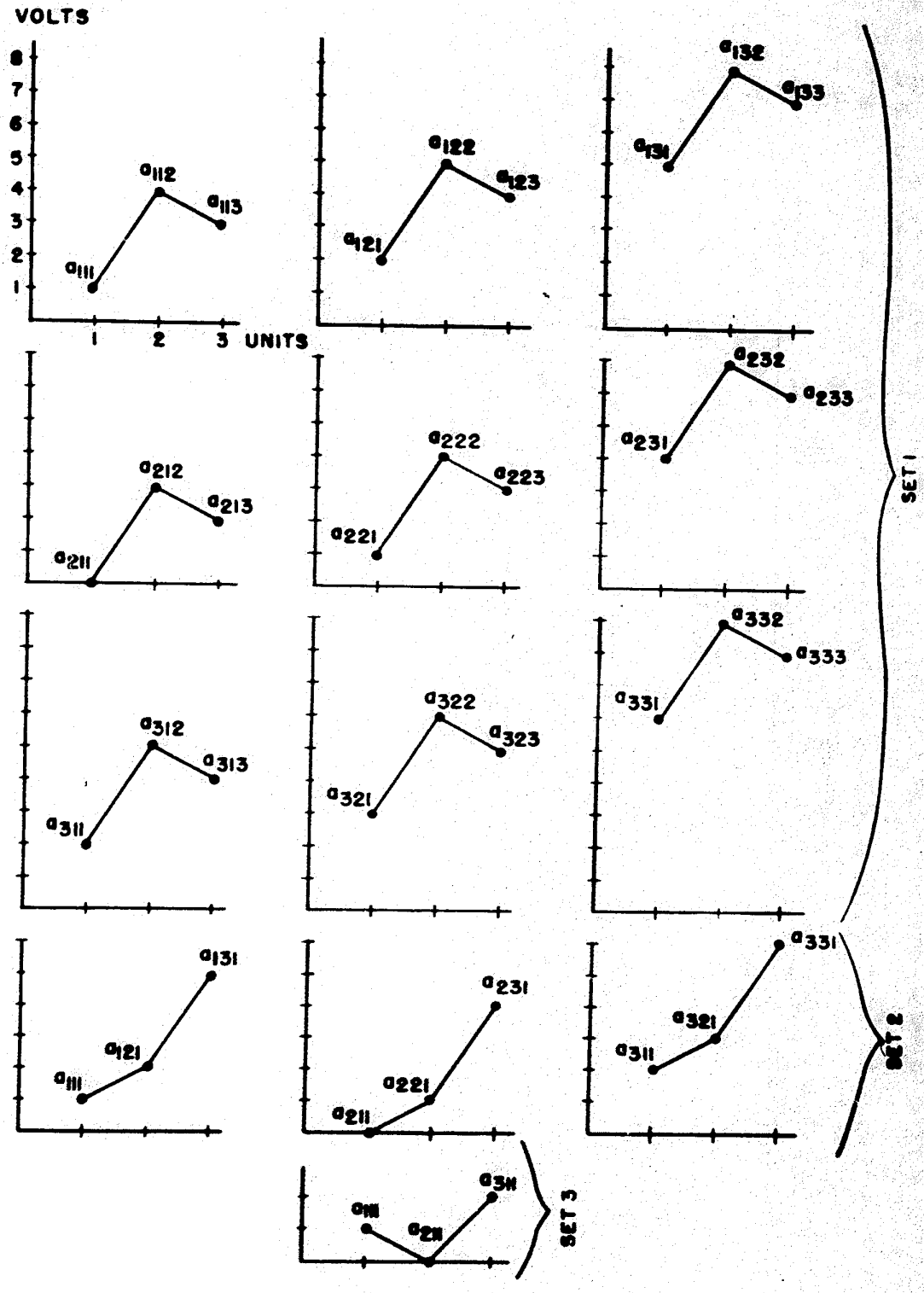

Fig. 24. Graphs of the coefficients in Example 4.1.

characteristic associated with u_2. And finally, any graph in set 1 can be taken as the transfer characteristic associated with u_3. As discussed earlier, the vertical translation of these transfer characteristics is unimportant since it can be compensated for by a battery in series with the system output as shown in Fig. 25. To find the value of the battery we simply find the output of the system of Fig. 25 for any cell, say cell 1, 1, 1, and choose the battery so that this output is equal to that given by the a's. That is, for cell 1, 1, 1 the system output (Fig. 25) without the battery is 2 volts (the scales for the transfer characteristics are assumed here to be in volts). But the system output for this cell should be equal to 1 volt since $a_{111} = 1$. Hence we insert a one volt battery, with the proper polarity, in series with the output of the system in Fig. 25. Note that in any system of the form shown in Fig. 23 or Fig. 25 the battery can always be absorbed in the no-storage transfer characteristics by simply translating one or more of them vertically.

In Fig. 24 we arbitrarily drew straight lines between the plots of the a's. This amounts to forming a transfer characteristic by a linear interpolation between the coefficients that specify the characteristic, as shown by the solid lines in Fig. 25. However, as we have seen, a gate function expansion yields a stepwise approximation to the transfer characteristic in which the height of each step is equal to the corresponding coefficient in the expansion. This step curve is shown by dotted lines in Fig. 25.

Example 4.2

Again we consider a filter of the class s = n = 3. Let the coefficients for this filter be

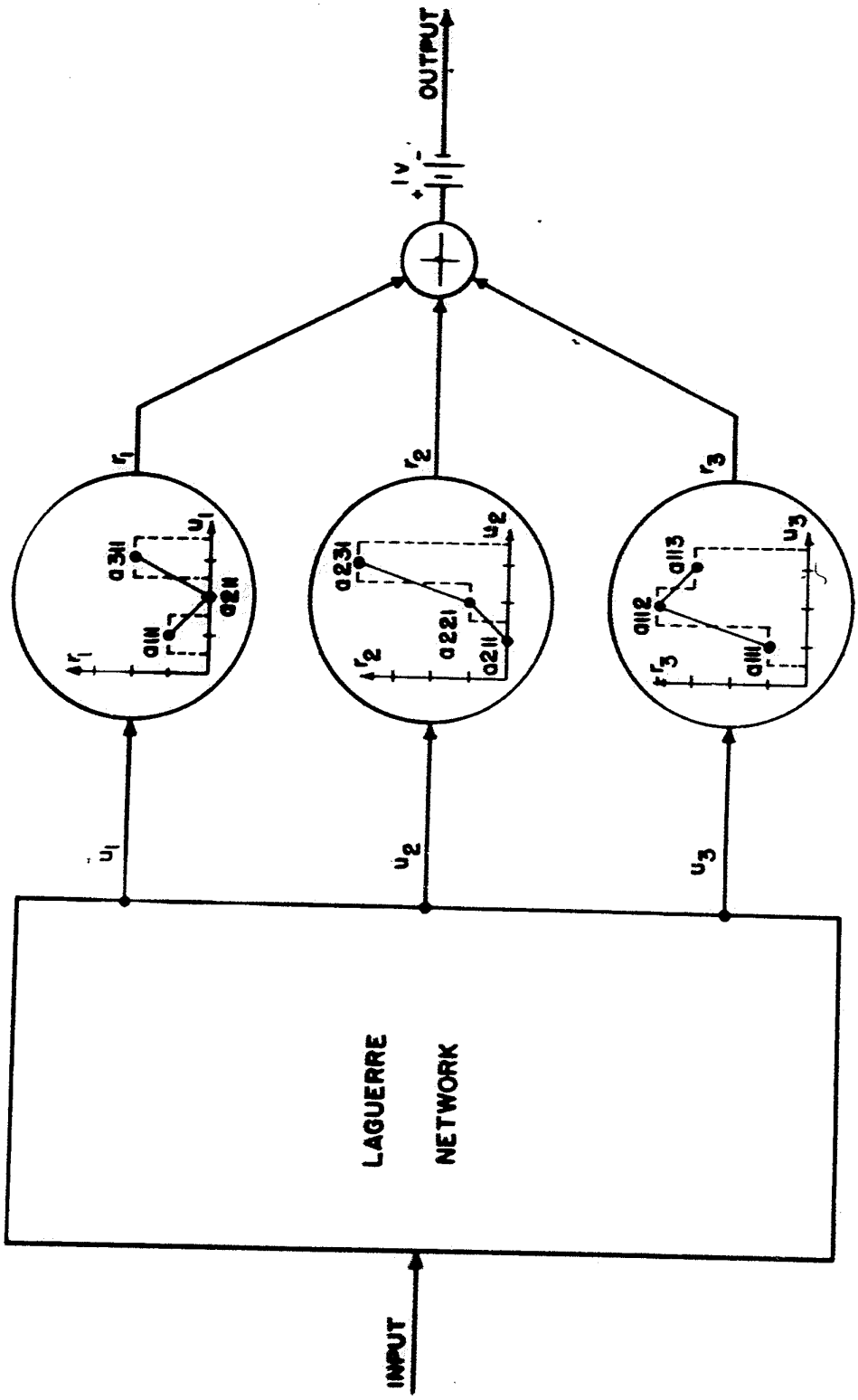

Fig. 25. The nonlinear filter of Example 4.1.

$a_{111} = 1$	$a_{211} = 0$	$a_{311} = -1$
$a_{112} = 2$	$a_{212} = 1$	$a_{312} = 0$
$a_{113} = 3$	$a_{213} = 2$	$a_{313} = 1$
$a_{121} = 3$	$a_{221} = 2$	$a_{321} = 1$
$a_{122} = 4$	$a_{222} = 3$	$a_{322} = 2$
$a_{123} = 5$	$a_{223} = 4$	$a_{323} = 3$
$a_{131} = 5$	$a_{231} = 4$	$a_{331} = 3$
$a_{132} = 6$	$a_{232} = 5$	$a_{332} = 4$
$a_{133} = 7$	$a_{233} = 6$	$a_{333} = 5$

These coefficients, plotted according to the test procedure described in section 4.1, are shown in Fig. 26. We see that all the graphs within each set of graphs are identical within a vertical translation, and further, we see that they are all linear. Hence the system can be synthesized as shown in Fig. 27. The transfer characteristics of the no-storage two-poles are found directly from the graphs of Fig. 26 as discussed in Example 4.1. The solid lines in the transfer characteristics of Fig. 27 indicate a linear interpolation between the coefficients that specify these characteristics. Since these solid line transfer characteristics are all linear they may be replaced by ampliers whose gains are equal to the slope of the lines.

4.3 Approximating Filters by Linear and Simple Nonlinear Filters

We have seen that if the graphs of the characterizing coefficients (the a's) of a system satisfy certain conditions the system can be synthesized in the relatively simple form shown in Fig. 23 and that under certain additional conditions the system is linear and the

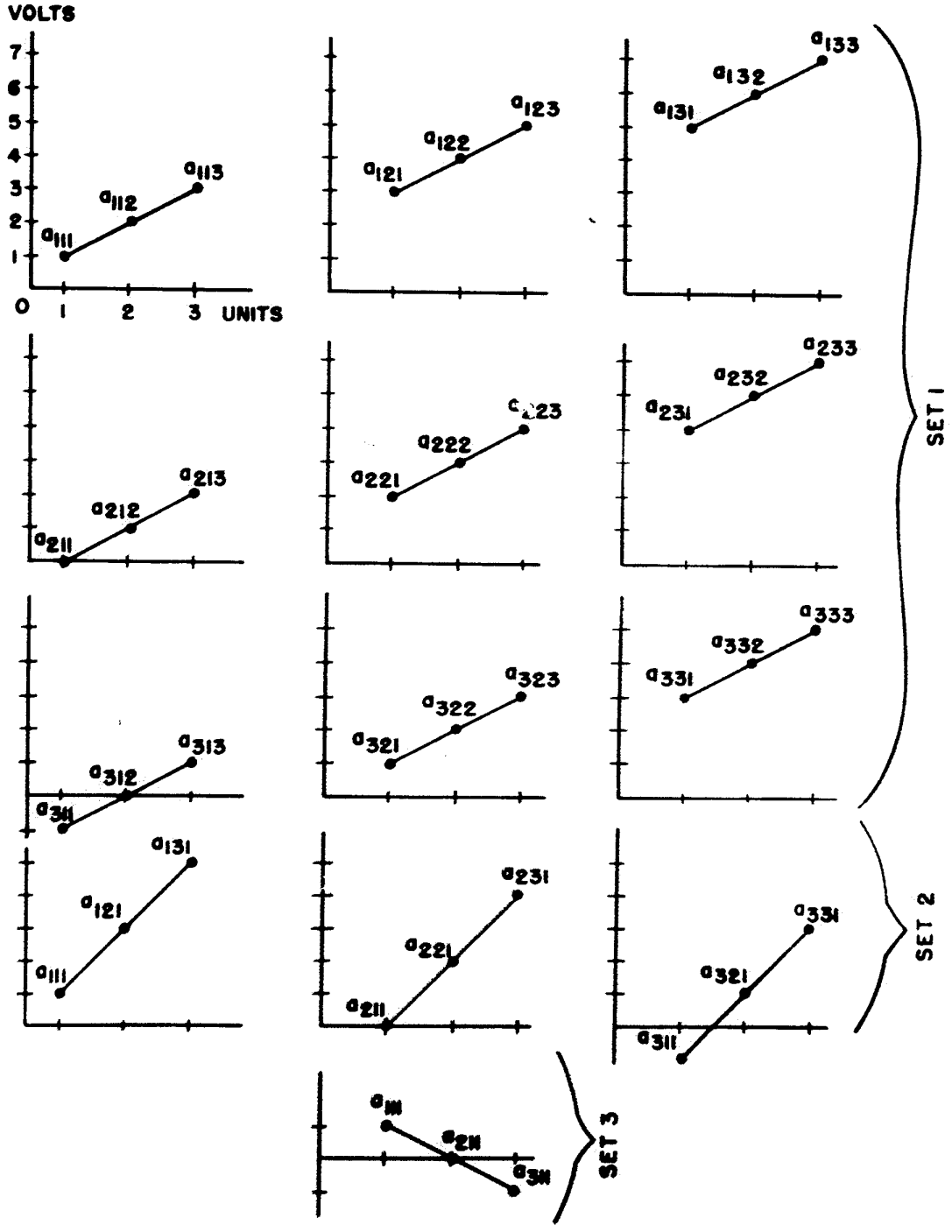

Fig. 26. Graphs of the coefficients in Example 4.2.

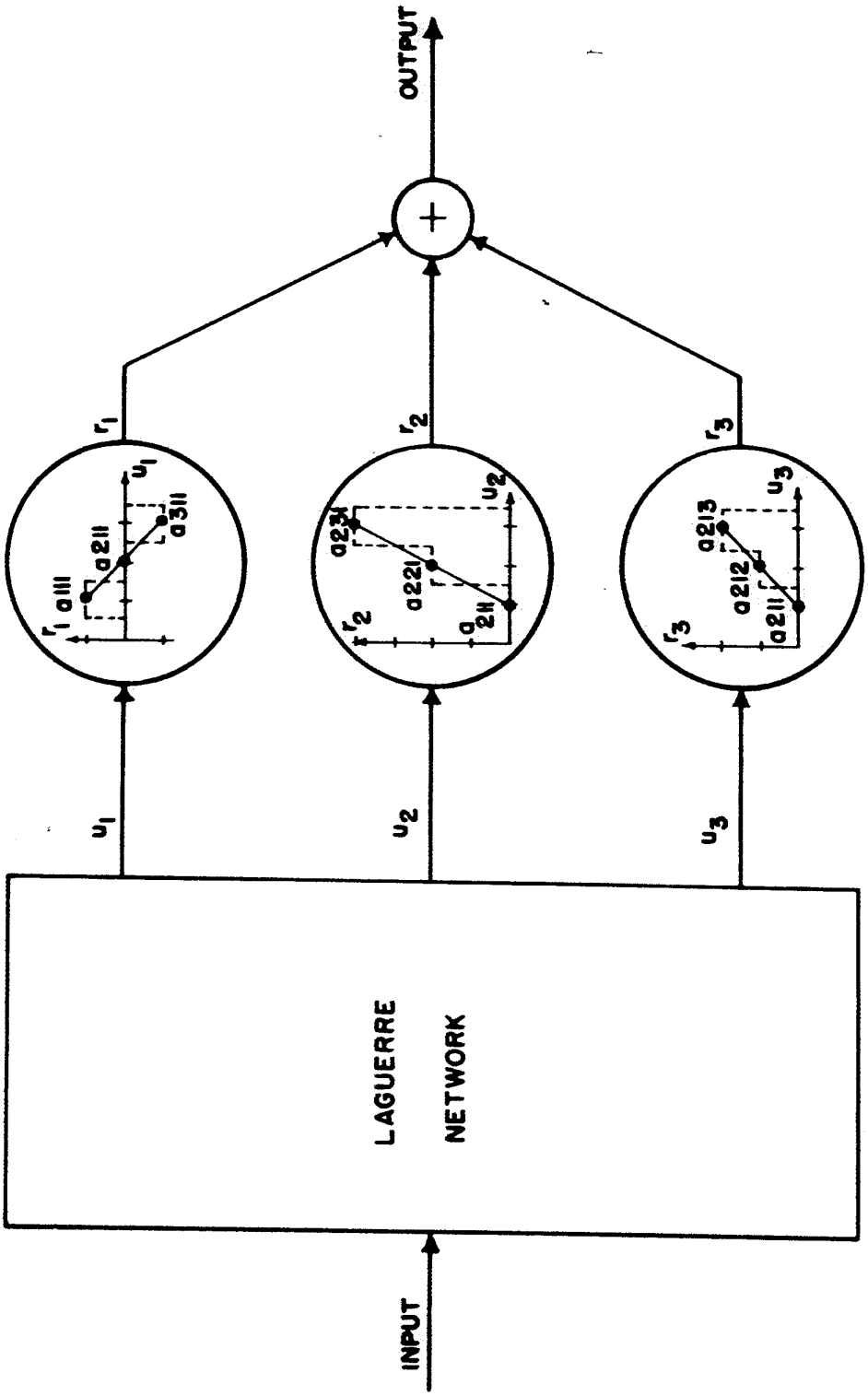

Fig. 27. The filter of Example 4.2.

synthesis even simpler. In practical problems it is unlikely that the system coefficients will exactly satisfy these conditions. However, the relative simplicity of the systems of Fig. 23 makes it worthwhile for us to determine when a more complicated system can be approximated by one having this simple form.

The concept of approximating one system by another is meaningful only when we specify a criterion for the approximation and specify the degree of approximation relative to this criterion. In this section we consider the approximation of one system by another from the point of view of two different error criteria.

We first consider the weighted mean square error criterion defined by Eq. (54). Relative to this criterion we ask how much error is introduced if we change the coefficients of the system from their optimum values A_a to the values A'_a.

According to Eq. (54), the error for a system whose coefficients are A'_a is given by

$$\mathcal{E} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} G(t) \left\{ z(t) - \sum_a A'_a \Phi(a) \right\}^2 dt \tag{95}$$

Expanding this equation we have

$$\mathcal{E} = \overline{z^2(t)\, G(t)} - \overline{2G(t)\, z(t) \sum_a A'_a \Phi(a)} + \overline{G(t) \sum_a \sum_\beta A'_a\, A'_\beta\, \Phi(a)\, \Phi(\beta)} \tag{96}$$

Taking advantage of the time domain orthogonality of the $\Phi(a)$, Eq. (96) can be written

$$\mathcal{E} = \overline{z^2(t)\, G(t)} - 2 \sum_a A'_a \overline{G(t)\, z(t)\, \Phi(a)} + \sum_a A'^2_a \overline{G(t)\, \Phi(a)} \qquad (97)$$

in which $\Phi^2(a)$ has been replaced by its equivalent, $\Phi(a)$.

For the optimum filter coefficients A_a, $\mathcal{E}$ takes on its minimum value $\mathcal{E}_{min}$, given by Eq. (62) as follows

$$\mathcal{E}_{min} = \overline{z^2(t)\, G(t)} - \sum_a \frac{\overline{G(t)\, z(t)\, \Phi(a)}^2}{\overline{G(t)\, \Phi(a)}} \qquad (62)$$

Using Eq. (58) for the optimum filter coefficients, Eq. (62) can be written

$$\mathcal{E}_{min} = \overline{z^2(t)\, G(t)} - \sum_a A_a \overline{G(t)\, z(t)\, \Phi(a)} \qquad (98)$$

From Eqs. (97) and (98) we obtain the relation

$$\mathcal{E} - \mathcal{E}_{min} = \sum_a (A_a - 2A'_a) \overline{G(t)\, z(t)\, \Phi(a)} + \sum_a A'^2_a \overline{G(t)\, \Phi(a)} \qquad (99)$$

Again using Eq. (58), we can write Eq. (99) as follows

$$\mathcal{E} - \mathcal{E}_{min} = \sum_a (A^2_a - 2A_a A'_a + A'^2_a) \overline{G(t)\, \Phi(a)} \qquad (100)$$

which is equivalent to

$$\mathcal{E} - \mathcal{E}_{min} = \sum_a (A_a - A'_a)^2 \overline{G(t)\, \Phi(a)} \qquad (101)$$

This equation is the desired expression for the error that is introduced when the system coefficients are changed from their optimum values A_a

to the values A'_α.

There are several interesting points to notice about Eq. (101) and the equations that lead to it. First, we notice that $\mathcal{E} - \mathcal{E}_{min}$ is always positive since G(t) and $\Phi(\alpha)$ are non-negative functions. Thus, Eq. (101) shows that the optimum coefficients (the A_α) determined by Eq. (58) actually render the error a minimum. Next, we notice that as a consequence of the time domain orthogonality of the $\Phi(\alpha)$ each cell contributes independently to the expression for the system error (Eq. 97). This very convenient property of the gate function representation of a nonlinear system enables us to directly and independently relate changes in any system coefficient to changes in the error $\mathcal{E}$. From Eq. (101) we see that the increase in error due to a change in the βth coefficient from its optimum value A_β to A'_β is just

$$\mathcal{E} - \mathcal{E}_{min} = (A_\beta - A'_\beta)^2 \overline{G(t)\,\Phi(\alpha)} \tag{102}$$

But recall (Fig. 13) that $\overline{G(t)\,\Phi(\alpha)}$ is a quantity that we must evaluate in determining the optimum system. Hence, if any coefficient is changed from its optimum value (as it may be for purposes of approximating a system by a simpler system as we shall see) we can immediately write down the corresponding increase in error. Finally we notice the interesting fact that for $G(t) \equiv 1$ (i.e., the mean square error criterion) the increase in error introduced by a given change in a coefficient is proportional to the probability that the corresponding cell in function space is occupied (recall that $\overline{\Phi(\alpha)}$ is equal to the probability that the αth cell is occupied).

We now introduce another error criterion and relative to it we examine the effect of changing the system coefficients. As the criterion

we specify an amplitude tolerance band T for a system output and regard two systems as approximately equivalent if their outputs (for any input that is common to both systems) coincide within this tolerance band. T can be chosen to have a constant width for all amplitudes of the system output or its width can be chosen as a function of the amplitude of the system output (examples of these choices are illustrated later). Now recall that in the gate function representation of a nonlinear system the output of the system at any instant is equal to the value of the coefficient A_α that corresponds to the occupied cell in the function space. Hence, if we change the βth coefficient from A_β to A'_β, then the system output will change from A_β to A'_β when the βth cell is occupied. At all other times (i.e., when the βth cell is not occupied) the system output will be the same as that before the change was made. Notice that this result holds regardless of the system input. As a consequence of this simple relation between changes in the coefficients and changes in the amplitude of the system output we can directly transfer the tolerance band on the system output to the coefficients. That is, for example, if we choose a two volt wide tolerance band about the output waveform then we can alter any and all of the coefficients by as much as ± 1 volt amplitude and the system output will remain within this tolerance band.

In the optimum filter problem (for the case $G(t) \equiv 1$) if we alter any coefficient from its optimum value A_β to A'_β, in addition to knowing that the system output will change from A_β to A'_β when the βth cell is occupied, we know the probability of the occurrence of this error. This probability is just $\overline{\Phi(\beta)}$ which was measured in the process of determining A_β.

In summary, when any filter coefficient is changed from its optimum

value A_β to A'_β we can immediately determine the following quantities:

1. The change in the weighted mean square error.
2. The tolerance band relating the optimum and the altered systems.
3. (For the case $G(t) \equiv 1$) The probability that the output of the altered system will differ from that of the original optimum system.

Conversely, if we specify upper limits on any of these quantities we can find the permissible variation in the A_α's. In this respect it is most convenient to specify a tolerance band, for, as we have seen, the same tolerance band can be applied to the system coefficients, thus directly determining their maximum permissible variation. Furthermore, of the three quantities listed above, the tolerance band is the only one whose specification determines the permissible range of the A_α's independent of the system input. Hence, this criterion is truly characteristic of the system itself. (More precisely, it is characteristic of the gate function representation of the system.) Henceforth when we speak of two systems as being nearly equivalent it is understood that this "equivalence" holds with respect to some suitably chosen tolerance band. To illustrate these concepts we consider an example of a nearly linear system.

Example 4.3

Let the coefficients of an optimum mean square filter be the same as those of Example 4.2 with the one difference that $a_{222} = 3.5$ instead of 3.0. The graphs of these coefficients are shown in Fig. 28. It is recognized that these graphs do not satisfy the conditions (section 4.1) for synthesis in the simple form of Fig. 23. However, let us establish a tolerance band of ± 0.5 volts about the system output and ask if there is a linear system that is nearly equivalent to this system. In Fig. 28 the dashed lines in each

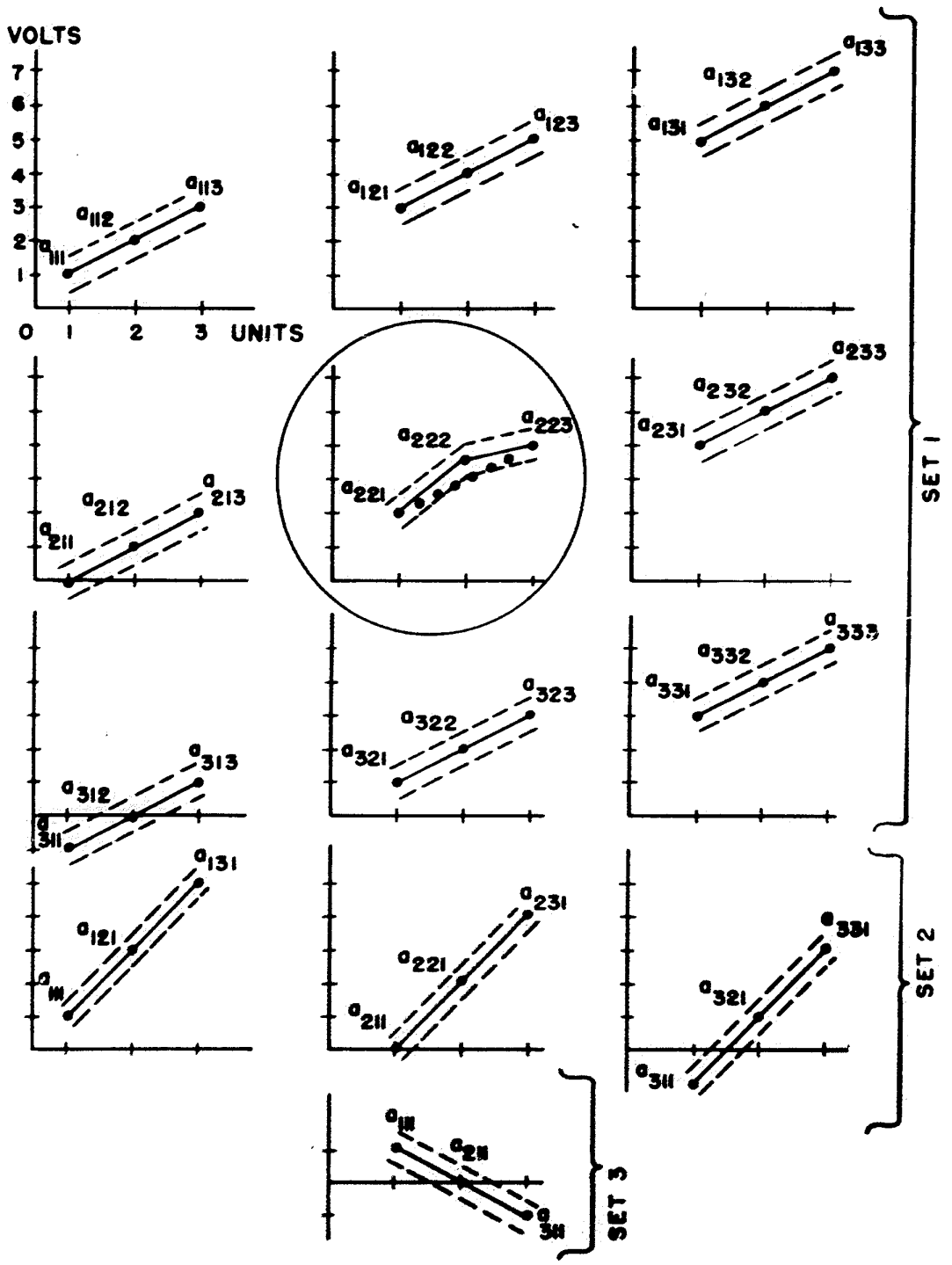

Fig. 28. Graphs of the coefficients of a nearly linear system.

graph indicate the ± 0.5 volt tolerance band. Inspection of this figure shows immediately that the system is nearly linear with respect to this tolerance band. All the graphs are linear except one (shown circled) and this one can be made linear, with the same slope as the other graphs of set 1, without exceeding the tolerance band, as shown by the dotted line. Hence this system is nearly equivalent to that of Example 4.2 and can thus be approximated by the simple linear system of Fig. 27. Furthermore, by Eq. (102), the increase in the mean square error when this optimum system is approximated by the linear system is

$$\mathcal{E} - \mathcal{E}_{\min} = (3.5 - 3.0)^2 \overline{\Phi(2, 2, 2)} \tag{103}$$

where $\overline{\Phi(2, 2, 2)}$ is one of the measurements that was made in determining a_{222}. Finally we see that the output of the linear system of Fig. 27 will differ from that of the optimum system only when the 2, 2, 2 cell is occupied and then this output will be 3.0 volts instead of 3.5 volts. The probability of this error is the known quantity $\overline{\Phi(2, 2, 2)}$.

In this simple example only one graph is nonlinear. However, the general procedure for finding a linear or simple nonlinear system of the class shown in Fig. 23 (if it exists) to approximate any system is clear. We draw tolerance limits on the graphs of the coefficients and then see if we can fit into these limits curves that satisfy the conditions of the class of systems of Fig. 23.

In the preceding example a tolerance band was chosen that had a constant width for all amplitudes of the signal output. In some applications it may be desirable to choose a band whose width is a function of the amplitude of the system output. For example suppose we choose a band whose width is proportional to the system output. This corresponds to a percentage

error criterion. That is, a system that approximates a given system with respect to this criterion has an output that is within a fixed percentage of the output of the given system at all times. An illustration of a 10 per cent tolerance band applied to a graph of the coefficients is shown in Fig. 29.

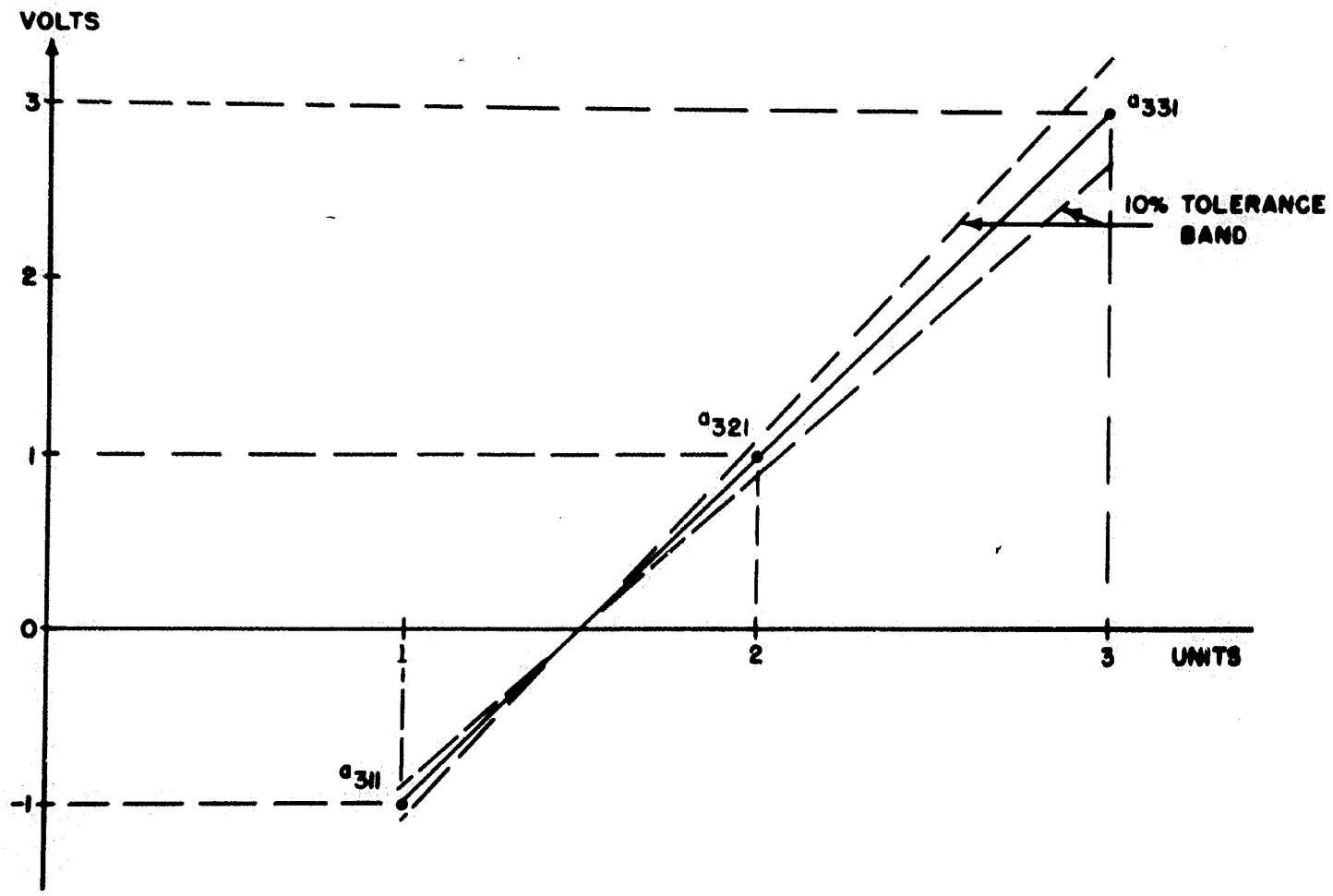

Fig. 29. Example of a 10 per cent tolerance band on a graph of system coefficients.

V. Notes on the Application of the Theory

5.1 Reducing the Complexity of Systems by Extracting Simple Systems

In section 4.3 we considered the approximation of nonlinear filters by linear and simple nonlinear filters. The object of this approximation is to reduce the complexity of the filter. In many cases we may find that, for a given tolerance band, there exists no system of the form shown in Fig. 23 that approximates a given system. It may, however, still be possible to simplify the synthesis of this system by dividing it into two or more parts, each of which can be synthesized in a relatively simple form. The basis for the division of the system into separate parts is provided by the gate function representation of nonlinear systems given in Eq. (53). This representation lends itself conveniently to the decomposition of systems into parallel connected component systems. For any cell of the given system, the coefficient A_a is just the sum of the coefficients of the corresponding cell in each of the parallel connected component systems. Hence we can extract a simple system from a given system by subtracting the coefficients of the simple system from the corresponding coefficients of the given system. If the extraction simplifies the original system then we have succeeded in breaking one complex system into two less complex parts. To illustrate this let us consider Example 4.3. Instead of approximating the system of this example by a linear one let us synthesize it as the parallel combination of two simple systems. We note (Fig. 28) that, except for the coefficient $a_{2,2,2}$, all the graphs of the coefficients satisfy the conditions for a linear system. Let us subtract the set of coefficients corresponding to this

linear system from the coefficients given in Fig. 28. The result is that all the coefficients of the remaining system are zero except the one corresponding to the 2, 2, 2 cell, which is 0.5. The coefficients of the extracted linear system are those of Example 4.2. Hence, the nonlinear system of Example 4.3 can be synthesized as the parallel combination of the linear system of Fig. 27 and a very simple nonlinear system having only one coefficient, $a_{2,2,2} = 0.5$. This approach of extracting simple systems to reduce the complexity of a system is effective only when, as in this example, the extraction causes many of the coefficients of the remaining system to be zero, thereby simplifying its synthesis.

5.2 Optimization of the Laguerre Function Scale Factor

It has been assumed throughout that the scale factor associated with the argument of the Laguerre functions is unity. By substituting kx for x in Eq. (1) we obtain the Laguerre functions having the scale factor k. Unfortunately, we have no convenient analytical method for determining the optimum value of k in the filter problem. Recall that the impulse response at the nth output terminal of the Laguerre network is $h_n(t)$, or $h_n(kt)$ with the scale factor k. We see that k effects a time scaling of the impulse response of the Laguerre network and hence a frequency scaling of the transfer function of this network. In determining optimum filters by the methods described we can make use of our knowledge of the frequency band of x(t) to judiciously choose k. It would be convenient however to have a method for obtaining the optimum k rather than just a good guess for it. An experimental procedure for accomplishing this

result is now described.

Consider the effect of changing the time scale of x(t) and z(t) in the experimental procedure for determining optimum filters. That is, consider that these time functions are replaced by x(t/k) and z(t/k) respectively. In the laboratory this could be achieved by recording the ensemble members of x(t) and z(t) and then playing the recording back at a speed of 1/k times the recording speed. First, consider the effect of this time scaling on the determination of no-storage filters according to Fig. 9. It is clear that since the apparatus for determining the filters has no memory, its operation is independent of the time scaling indicated above. That is, the same filter will be determined regardless of the value of k. Now consider the procedure (Fig. 13) for determining optimum filters involving memory. We notice that the only portion of the apparatus that involves memory is the Laguerre network. As a consequence if we replace x(t) and z(t) by x(t/k) and z(t/k) respectively, we obtain the same filter as if we had used x(t) and z(t) but changed the scale factor of the Laguerre network from unity to k. Now recall (section 3.6) that apparatus can be constructed which indicates the minimum error for the optimum filter (of the class of filters having a given Laguerre network and Φ generator). This apparatus, like that of Fig. 13, consists of no-memory circuits and a Laguerre network. By varying the speed of the recordings of x(t) and z(t) that are fed into this apparatus and observing its output ($\mathcal{E}_{min}$ of Eq. 62) we can directly determine the optimum k. In practice we would build the Laguerre network in the minimum error detecting apparatus to have a scale factor corresponding to our judicious

guess of the optimum k. Then we could vary the speeds of the recordings of x(t) and z(t) in order to explore scale factors in the neighborhood of our guess. We should note that in general the amplitude of the signals at the outputs of the Laguerre network will change with the change in speed of the recordings of x(t). The maximum amplitude of these signals must be kept at a constant level corresponding to the range a-b (Eq. 37) over which the gate functions are defined. If desired, circuitry can be constructed to accomplish this adjustment automatically.

5.3 Choosing the Functions that Characterize the Past of the Filter Input

For the theoretical discussions of the preceding sections it was convenient to characterize the past of the filter input by Laguerre functions since they form a complete set of functions on this past. In practice, however, we will only use a small number of Laguerre functions and hence the fact that they form a complete set is no longer of prime importance to us. The important question in the practical case is, "How well can we do with a small number of functions of the past?"

In many filter problems involving the separation of signals from noise the immediate past of the filter input contains more information about the present value of the desired signal than does the more remote past. Since the Laguerre functions decay exponentially (Eq. 1) they weight the immediate past of the input considerably more than the more remote past and hence we expect that they will form a convenient set of functions for use in many filter problems.

For some applications it may be convenient to derive the functions of the past from taps on a delay line rather than from a Laguerre network

(even though the outputs of a delay line do not form a complete set of functions on the past of the input). For example, suppose we are dealing with messages in which consecutive symbols are coded in a sequence of pulses of different amplitudes. Then it would be appropriate to replace the Laguerre network by a delay line with taps so spaced that consecutive pulses appear at adjacent taps. Then at any instant the s outputs of the delay line would specify the amplitudes of the s preceding pulses. Another example for which it is appropriate to replace the Laguerre network by a delay line is the problem of detecting the presence of a train of pulses that are equally spaced in time. If the delay between the taps is equal to the time interval between the pulses then when the train of pulses is present it will affect all outputs of the delay line simultaneously. These are examples in which it is desirable to give equal weight to different portions of a finite interval of the past and hence a delay line is more appropriate than a Laguerre network.

It is clear that there are many choices we can make for the functions that characterize the past of the filter input. Each choice implies a restriction to a class of filters from which the experimental procedure, similar to that of Fig. 13, will pick the optimum for the particular problem. We can make use of the apparatus that determines the minimum error for a given class of filters (mentioned in sections 3.6 and 5.2) in order to decide whether to use a Laguerre network or a delay line etc. in a given problem. We insert the different networks into this apparatus in place of the Laguerre network and choose that one which yields the smallest error.

5.4 Choosing the Number of Gate Functions

By increasing the number of gate functions associated with each Laguerre coefficient we quantize the function space of the input into smaller cells. While in general this is desirable from the point of view of reducing the filtering error, it leads to a very large number of coefficients to evaluate. In application of the theory we should, when possible, make use of any information about the particular filter problem that will enable us to reduce the number of gate functions. For example, consider the problem of pure (noiseless) prediction of a stationary time function consisting of equally spaced pulses having amplitudes of zero or unity with a certain probability distribution. As discussed in the previous section it is appropriate, for a problem of this type, to replace the Laguerre network by a delay line. The output from each tap on the delay line will then take on only the values zero and unity. Hence we need only two gate functions for each output of the delay line.

In the previous section we have also mentioned that in many filter problems the immediate past of the input contains more information about the present value of the desired output than does the more remote past. A study of the Laguerre functions shows that the higher order functions weight the remote past more strongly than do the lower order functions. Hence, we expect that, for purposes of filtering, it might not be as important to distinguish small changes in the amplitude of the higher order Laguerre coefficients as in the lower order ones (or, in the case of delay lines it might not be as important to distinguish small changes in those outputs that represent samples of the more remote past). Thus we might choose a smaller number of gate functions for the higher order Laguerre coefficients

than for the lower order ones.

In summary, the choices of the scale factor, the functions that characterize the past of the input, and the number of gate functions, are all choices that determine the class of filters, from which the procedure similar to that of Fig. 13 determines the optimum filter. We should use any information about the particular filter problem that will enable us to judiciously choose the class of filters so that the number of coefficients is reasonable in the light of the task that the filter is to perform.

Biographical Note

Mr. Amar G. Bose was born in Philadelphia on November 2, 1929. He entered M. I. T. in September 1947. From 1949 to 1952 he participated in the co-operative course in Electrical Engineering with the Philco Corporation in Philadelphia. In June 1952 he received his S. B. and S. M. degrees in Electrical Engineering from M. I. T. He then was awarded a General Electric – Gerard Swope Fellowship to continue graduate studies at M. I. T. during the academic year 1952-53. During the summer of 1953 he received an M. I. T. Overseas Fellowship for research with the Philips Company in Eindhoven, Holland. From October 1953 until June 1956 he was a research assistant in the Research Laboratory of Electronics at M. I. T. He received a Fulbright Fellowship to study mathematics at the Indian Statistical Institute in Calcutta, India, during the academic year 1956-57.

He presented a paper as joint author with Dr. S. D. Pezaris at the National Convention of the Institute of Radio Engineers in March 1955 entitled, "A Theorem Concerning Noise Figures." At the March 1956 Convention he presented a paper entitled, "A Theory for the Experimental Determination of Optimum Nonlinear Systems." Mr. Bose is a member of Sigma Xi.

References

1. Norbert Wiener, Mathematical Problems of Communication Theory, Summer Session lecture notes, M. I. T., 1953-1954, unpublished.

2. Y. W. Lee, "Synthesis of Electric Networks by Means of the Fourier Transforms of Laguerre's Functions," Journal of Mathematics and Physics, Vol. II, No. 2, June 1932, pp. 83-113.

3. E. C. Titchmarsh, Introduction to the Theory of Fourier Integrals (Oxford University Press, London, 1937) pp. 76-82.

4. Norbert Wiener, "The Fourier Integral and Certain of its Applications" (Dover Publications, Inc., 1933).

5. H. E. Singleton, "Theory of Nonlinear Transducers," Technical Report No. 160, Research Laboratory of Electronics, M. I. T., 1950, pp. 30-32.

6. Y. W. Lee, "Statistical Theory of Communication," Course 6.563, 1952 class notes, unpublished.

7. Norbert Wiener, "Extrapolation, Interpolation, and Smoothing of Stationary Time Series" (John Wiley and Sons, New York, 1949).

8. L. A Zadeh, "Optimum Nonlinear Filters for the Extraction and Detection of Signals," presented at the IRE National Convention, March 25, 1953.

9. T. E Stern, "Piecewise Linear Analysis and Synthesis," presented at the Symposium on Nonlinear Circuit Analysis, Polytechnic Institute of Brooklyn, April 26, 1956.