Representations of quantum algebras arising from non-compact quantum groups: Quantum orbit method and super-tensor products

by

Leonid I. Korogodsky

Submitted to the Department of Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY May 1996

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Abstract

In this thesis, I present a version of the quantum orbit method which provides a geometrical construction of the irreducible *-representations of the quantum universal enveloping algebras for the real Lie groups of equal rank. It is shown that they correspond to the symplectic leaves of the dual Poisson Lie group equipped with a special structure called quantum orbit lattice.

At the same time, the thesis is dealing with the important problems in the foundations of the theory of quantum non-compact groups. By introducing a canonical super-tensor product in the situation where the usual concept of tensor product fails to exist, it open a new direction in the search for the axiomatic theory to embrace the quantum non-compact groups.

Thesis Supervisor: Bertram Kostant
Title: Professor
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Now, looking back at those years, I can hardly imagine them without many hours of time fruitfully spent in numerous discussions with those who expressed interest and helped me in my research. Professors Kac, Kazhdan, Lusztig, Soibelman, Vogan only open the list of the names to whom I want to bring my words of thankfulness.

And, of course, all the time I was feeling warmth and support from my family. And to my daughter Anna I want to dedicate this work.

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Chapter 1

Introduction

This work is based on my research in the field of quantum groups, especially in the interesting and less developed area of non-compact quantum groups.

The theory of quantum groups arose from the inverse scattering method developed by L.D. Faddeev and his school. The concept of quantum group was introduced by V.G. Drinfeld, as well as independently by M. Jimbo, Lusztig, Y. Manin, and S.L. Woronowicz, each of them coming up with their own approach and point of view on quantum groups.

It turned out that the variety of approaches was reflected happily by a variety of relations the quantum groups have with many areas of mathematics and mathematical physics, most notably, with the knot theory, the theory of Lie groups and algebras, infinite-dimensional Lie algebras, quantum functional analysis, string theory, and quantum field theory.

Unfortunately, while the theory of compact quantum groups is enjoying wide popularity and fast development, little or nothing is known regarding non-compact quantum groups. I am happy to say that I achieved some results that shed light on them in important special cases, and I consider them as a foundation of my research program for the future.
1.1 Quantum Orbit Method

Two different directions in the theory of non-compact quantum groups are presented in the thesis. One of the chapters is devoted to a version of the quantum orbit method developed by me in [9]. The method provides a geometrical realization of irreducible *-representations of the quantum universal enveloping algebras for real Lie groups of equal rank in the spirit of the classical orbit method. It provides a correspondence between the unitary equivalence classes of irreducible *-representations of the quantum enveloping algebra and the symplectic leaves in the dual Poisson Lie group equipped with some additional structure which I call quantum orbit lattice.

The main idea is to use the Drinfeld’s duality which suggests that the quantum universal enveloping algebra can be considered as a quantum algebra of functions on the dual Poisson Lie group. I construct a family of quantum $G$-spaces, which are quantum analogs of the orbits of the dressing action in the dual Poisson Lie group (or more precisely, certain unions of symplectic leaves in the dual Poisson Lie group, as the dressing action does not necessarily extends to a global action). The algebra of functions on that family of quantum $G$-spaces is shown to be canonically isomorphic to a quantum analog of the Heisenberg algebra. Then, the Drinfeld’s duality can be used to construct a quantum moment map as a quantum analog of the usual realization of the enveloping algebra by differential operators. The next step is to construct quantum polarizations which yield a geometric realization of irreducible *-representations of $U_{qg}$.

In fact, a construction is given which associates an irreducible *-representation of the quantum enveloping algebra $U_{qg0}$ to any $U_{qg0}$ module *-algebra $F$ equipped with a surjective quantum moment map $J : U_{qg0} \rightarrow F$, a quantum polarization, and a character of the commutative subalgebra of $U_{qh}$-invariant elements. The concept of module *-algebra is used to describe quantum $G$-spaces. It is outlined and discussed in Appendix A.

Roughly, the quasi-classical analog of the $U_{qg0}$-module *-algebra $F$ is a symplectic leaf in the dual Poisson Lie group or a union of such symplectic leaves in some cases.
The quantum polarization corresponds to the usual polarization in the cases when the corresponding $G$-space is isomorphic to a coadjoint orbit, while the character corresponds to the choice of a local system on it. Not any character can be used in general but only those which arise from the structure called by me quantum orbit lattice.

It is interesting to note that in the case of $U_q\mathfrak{su}(1,1)$ the above construction yields all irreducible $*$-representations, while the classical orbit method does not yield the complimentary series representations. These turn out to correspond to the so-called quantum 'tunnel' hyperboloids. By drawing an intuitive analogy, those can be very informally described as two-sheet hyperboloids which are so close to a nilpotent cone that behave as connected manifolds in the quantum case, as the quantum Haar measure (being discretely supported) can be extended from one sheet to another. As a simple illustration, consider the $*$-algebra generated by $x = x^*, y, y^*$ with the relations

$$yx = q^2 xy, \quad y^*x = q^{-2} xy^*, \quad y y^* = (qx - c)(qx - d), \quad y^*y = (q^{-1}x - c) (q^{-1}x - d),$$

where $c$ and $d$ are some positive real numbers ($c \leq d$). (It is a quantum analog of the algebra of regular functions on a hyperboloid $|y|^2 = (x - c)(x - d)$.) Any geometric progression of the form $\{q^{2k+\beta}\}$ supports the restriction of a quantum Haar measure to the space of spherical functions:

$$\int f(x)d\mu = (q^{-1} - q) \sum_k q^{2k+\beta} f(q^{2k+\beta}),$$

provided that no point of the progression lies inside the forbidden interval $(c, d)$ separating the sheets of the hyperboloid (we allow it to be truncated only if one of its points coincides with $c$ or $d$). It is easy to see that if $q^2 < \frac{c}{d} \leq 1$, then there are progressions that can 'jump' from one sheet to another. Those precisely correspond to the complementary series representations in that special case.
1.2 Super-Tensor Products

Another part of my research has been prompted by the problems in finding a good axiomatic ground under the concept of non-compact quantum group. For, even though some results have been obtained about quantum non-compact groups before, so far the researchers used only intuitive vision of the concept, since the numerous attempts to axiomatize the theory led only to negative results. And even though I have not constructed an axiomatic theory yet, an interesting approach has been opened in the totally unexpected direction, as an amazing observation brings about the concept of super-tensor product.

The notion of quantum topological real group is generally believed to involve some topological Hopf $*$-algebras which could become the analogues of algebras of continuous functions. Thus, a successful axiomatics for quantum compact groups was described in [29] in the language of Hopf $C^*$-algebras.

As to the quantum non-compact groups, no axiomatics does still exist, although some positive results have been obtained, those which do not require necessarily any topological algebra of functions. One can mention, for instance, harmonic analysis on the quantum group $SU(1, 1)$ developed in [13, 14, 28].

Nevertheless, an attempt undertaken by S.L.Woronowicz in [30] to construct the quantum group $SU(1, 1)$ on the $C^*$-algebra level has led him to a result of inexistence. Actually, this negative result is based on the fact that there is no correctly defined tensor product in the category of $*$-representations of the quantum algebra $C[SU(1, 1)]_q$ of regular functions.

I have tried to look at the picture from a bit different point of view. As a result I observed an amazing quantum effect explained in [10] which shows that, although a quantum group $SU(1, 1)$ does not exist (on the $C^*$-algebra level), one can possibly consider the quantum group $SU(1, 1) \rtimes \mathbb{Z}_2$ in a certain sense. Let me explain it a general idea.

An intuitive reason why we would be looking for something like tensor product in the category of $*$-representations of the quantum algebra of regular functions is a de-
sire to have a quantum analogue of the group multiplication (as far as representations of the quantum algebra of functions correspond to symplectic leaves).

As to quantum compact groups, the tensor product is obviously correctly defined, since the quantum algebra of regular functions is represented always by bounded operators. In the case of the quantum group $M(2)$ of the motions of the Euclidean plane (cf. [12, 30]), the tensor product also can be correctly defined (because of another reason, though), and this very fact made it possible to construct in [30] the quantum group on the $C^*$-algebra level.

When we consider quantum non-compact groups, the problem becomes quite untrivial because the quantum algebra of regular functions is represented usually by unbounded operators.

While in the case of the quantum group of motions of the Euclidean plane this problem can be overcome, it is impossible to define correctly tensor product of two infinite-dimensional irreducible $*$-representations of $\mathbb{C}[SU(1, 1)]_q$, since a certain symmetric operator cannot be extended to a self-adjoint one in accordance with the other operators of representations, as was shown in [30] by the methods of the operator theory and here in terms of a geometric realization.

However, there is a certain extension of the point of view. It involves two principal steps. First, one should go onto the quantum group $SU(1, 1) \rtimes \mathbb{Z}_2$, the normalizer of $SU(1, 1)$ in $SL_2(\mathbb{C})$. Second, one should abandon, at least for a while, the $C^*$-algebra level and look for some already existing structure on the category of $*$-representations of $\mathbb{C}[SU(1, 1) \rtimes \mathbb{Z}_2]_q$.

Is there no natural monoidal category structure? Well, but there is another, a bit surprising, structure which is worth to be considered – that of super-tensor products. Indeed, an amazing symmetry is present in the picture.

The Poisson Lie group $N_{SL_2(\mathbb{C})}(SU(1, 1)) \simeq SU(1, 1) \rtimes \mathbb{Z}_2$, the normalizer of $SU(1, 1)$ in $SL_2(\mathbb{C})$, is the union of two connected components $SU(1, 1) \cup SU(1, 1) \cdot w$ of $SL_2(\mathbb{C})$ where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The irreducible $*$-representations of the corre-
sponding quantum algebra of functions correspond to the local systems on the symplectic leaves in it. Let us call those corresponding to the symplectic leaves in the component of the unit element 'even' representations, and those which correspond to the symplectic leaves in the other component 'odd'.

For any $\beta \in \mathbb{R}/\mathbb{Z}$, one can consider a certain subcategory $C_\beta$ of the category of $\ast$-representations of the quantum algebra of functions, which corresponds roughly to choosing a local system for each symplectic leaf. Then one can define canonically the following super-tensor products:

$$
\pi_1^+ \otimes \pi_2^+ \oplus \pi_1^- \otimes \pi_2^-,
$$

(1.1)

$$
\pi_1^+ \otimes \pi_1^- \oplus \pi_2^- \otimes \pi_2^+,
$$

(1.2)

where $\pi_i^+$ are even and $\pi_i^-$ odd infinite-dimensional irreducible representations from the category $C_\beta$. Note that no summand in (1.1),(1.2) can be correctly defined.

The arising structure might be treated, for instance, as follows. Given a pair $\sigma_1, \sigma_2$ of $\ast$-representations from the category $C_\beta$, one can consider a set $P(\sigma_1, \sigma_2)$ which parameterizes 'different' (in the sense of a certain equivalence relation) tensor products of $\sigma_1$ and $\sigma_2$:

$$
\sigma_1 \otimes_p \sigma_2, \ p \in P(\sigma_1, \sigma_2)
$$

For instance, $P(\pi_1^+ \oplus \pi_1^- , \pi_2^+ \oplus \pi_2^-)$, as is easy to see, consists of just one element. If $P(\sigma_1, \sigma_2) = \emptyset$, this means that there is no tensor product of $\sigma_1$ and $\sigma_2$. In general, the different tensor products are parameterized by different ways to assign a term of the form $\pi^+ \otimes \pi^\pm$ to each term of the form $\pi^- \otimes \pi^\mp$.

The super-tensor products (1.1) and (1.2) have interesting quasi-classical analogs. For an important special case of (1.1), the obtained geometrical realization leads to the global right dressing action of the dual Poisson Lie group on the biggest Schubert cell. It is isomorphic to the complex plane, but, unlike the Poisson structure induced from the quantum group $SU(2)$, now the corresponding Poisson manifold is not a symplectic leaf anymore. Both the interior and the exterior of the unit disc are symplectic leaves, together with each point of the unit circle.
The dual Poisson Lie group acts on the complex plane by dilations and translation. The fact that the summands in (1.1) cannot be defined separately corresponds to the fact that the local action when restricted to either of the interior or the exterior of the unit disc cannot be extended to the global action. And the fact that there is a canonical way to define the super-tensor product corresponds to the fact that there is a canonical extension of such local action to the global action in the whole Schubert cell.
Chapter 2

Quantum Orbit Method

2.1 Quantum Heisenberg Algebras

Throughout the thesis we suppose that \( q \) is real, \( 0 < |q| < 1 \).

Suppose that \( G \) is a finite-dimensional complex simple Poisson Lie group, \( \mathfrak{g} \) its Lie bialgebra, and \( U_q\mathfrak{g} \) the corresponding quantized universal enveloping algebra (cf. [1, 5]). Let \( V \) be a finite-dimensional simple \( U_q\mathfrak{g} \)-module, and \( V^\ast \) the dual \( U_q\mathfrak{g} \)-module given by

\[
\langle \xi, \varphi \rangle = (\varphi, S(\xi)v),
\]

where \( \xi \in U_q\mathfrak{g}, \varphi \in V, \varphi \in V^\ast \), and \( S \) is the antipode in \( U_q\mathfrak{g} \). Below we describe a construction of what we call quantum Heisenberg algebra \( \mathcal{H}_\mathfrak{g}(V) \).

Let \( R \) be the quantum \( R \)-matrix acting in \( (V \oplus V^\ast)^{\otimes 2} \), and \( \tilde{R} = PR \), where \( P : (V \oplus V^\ast)^{\otimes 2} \to (V \oplus V^\ast)^{\otimes 2} \) is the usual permutation operator \( a \otimes b \mapsto b \otimes a \). As is well known, the operator \( \tilde{R} \) is invertible and diagonalizable, has real spectrum and commutes with the action of \( U_q\mathfrak{g} \).

Consider the algebra \( \mathbb{C}[VR_q] \), which is the quotient of the tensor algebra

\[
\mathcal{T}(V \oplus V^\ast) = \mathbb{C} \oplus (V \oplus V^\ast) \oplus (V \oplus V^\ast)^{\otimes 2} \oplus ... 
\]

over the two-sided ideal \( J(W) \) generated by the span \( W \subset (V \oplus V^\ast)^{\otimes 2} \) of all eigen-
vectors of $\hat{R}$ with negative eigen-values.

The tensor algebra $\mathcal{T}(V \oplus V^*)$ has a canonical $U_q\mathfrak{g}$-module algebra structure, which means that the canonical $U_q\mathfrak{g}$-module structure defined by the action of $U_q\mathfrak{g}$ on $V \oplus V^*$ is compatible with the algebra structure in the sense that the multiplication map $\mathcal{T}(V \oplus V^*) \otimes \mathcal{T}(V \oplus V^*) \to \mathcal{T}(V \oplus V^*)$ is a morphism of $U_q\mathfrak{g}$-modules.

Since $\hat{R}$ commutes with the $U_q\mathfrak{g}$-action, the two-sided ideal $J$ is a $U_q\mathfrak{g}$-submodule. It follows that $C[V_{\mathfrak{R}}]_q$ has the quotient $U_q\mathfrak{g}$-module algebra structure. The algebra $C[V_{\mathfrak{R}}]_q$ can be thought of as a quantum analog of the algebra of polynomial functions on the $G$-space $V \oplus V^*$, the subalgebra generated by $V^*$ (resp. $V$) playing the role of the algebra of holomorphic (resp. anti-holomorphic) polynomials.

In the classical case the algebra of polynomials has a canonical central extension given by

$$ab - ba = \langle a, b \rangle C,$$

where $C$ is the central element, and $\langle , \rangle$ is the canonical bilinear form on $V \oplus V^*$ given by the natural pairing between $V$ and $V^*$ and such that both $V$ and $V^*$ are isotropic subspaces. The algebra obtained this way is known as the Heisenberg algebra. The quantum analog is described below.

Consider the subspace $I$ of $U_q\mathfrak{g}$-invariant elements in $V \otimes V^* \oplus V^* \otimes V$ (that is, the elements $v$ such that $\xi v = \varepsilon(\xi)v$ for any $\xi \in U_q\mathfrak{g}$, where $\varepsilon$ is the counit). It is two-dimensional, with one generator from $V \otimes V^*$ and another one from $V^* \otimes V$. Since $\hat{R}$ commutes with the action of $U_q\mathfrak{g}$, $I$ is invariant with respect to $\hat{R}$. Since $\hat{R}$ permutes $V \otimes V^*$ and $V^* \otimes V$, it has two distinct eigen-values in $I$. On the other hand, it is easy to see that $\hat{R}^2 |_I$ ia a constant, so that the eigen-values are of opposite sign. Thus, we have proved the following proposition.

**Proposition 1.1.** The vector space $I_0 = I \cap W$ is one-dimensional.

Note that the $U_q\mathfrak{g}$-module $W$ is completely reducible, so that there is a unique $U_q\mathfrak{g}$-submodule $W_0 \subset W$ such that $W = W_0 \oplus I_0$. Let $J(W_0) \subset \mathcal{T}(V \oplus V^*)$ be the two-sided ideal generated by $W_0$. 

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Definition 1.1. The $U_q\mathfrak{g}$-module algebra $\mathcal{H}_g(V) = \mathcal{T}(V \oplus V^*)/J(W_0)$ is called the quantum Heisenberg algebra corresponding to $\mathfrak{g}$ and $V$. The following short sequence is exact, where $C$ is the image of a generator of $I_0$ and $p$ is the quotient map:

$$0 \to C[C] \hookrightarrow \mathcal{H}_g(V) \xrightarrow{p} C[V_R]_q \to 0.$$

Example 1.1. Consider $\mathfrak{g} = \mathfrak{sl}(n+1)$ equipped with the so-called standard Lie bialgebra structure. It is defined by the Manin triple $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}, \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-)$, where $\mathfrak{g}$ is embedded into $\mathfrak{g} \oplus \mathfrak{g}$ as the diagonal, $\mathfrak{n}_+$ as $(\mathfrak{n}_+, 0)$, $\mathfrak{n}_-$ as $(0, \mathfrak{n}_-)$, $\mathfrak{h}$ as $\{(a, -a) \mid a \in \mathfrak{h}\}$. Here $\mathfrak{h}$ is the Cartan subalgebra of diagonal matrices, and $\mathfrak{b}_+$ (resp. $\mathfrak{b}_-$) the nilpotent subalgebra of nilpotent upper- (resp. lower-) tripoidal matrices.

Recall that $U_q\mathfrak{sl}(n + 1)$ is generated by $E_i, F_i, K_i, K_i^{-1}, i = 1, 2, \ldots, n$, subject to the relations

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q^{-1} - q}, \quad K_i K_j = K_j K_i,$$

$$K_i E_j = q^{-a_{ij}} E_j K_i, \quad K_i F_j = q^{a_{ij}} F_j K_i,$$

$$E_i E_j = E_j E_i, (|i - j| > 1), \quad F_i F_j = F_j F_i, (|i - j| > 1),$$

$$E_i^2 E_{i+1} - (q + q^{-1}) E_i E_{i+1} E_i + E_{i+1} E_i^2 = 0,$$

$$F_i^2 F_{i+1} - (q + q^{-1}) F_i F_{i+1} F_i + F_{i+1} F_i^2 = 0,$$

where $a_{ii} = 2, a_{i,i+1} = -1$, and $a_{ij} = 0$ otherwise.

The Hopf algebra structure on $U_q\mathfrak{sl}(n + 1)$ is given by

$$\Delta(K_i) = K_i \otimes K_i,$$

$$\Delta(E_i) = E_i \otimes 1 + K_i^{-1} \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i + 1 \otimes F_i,$$

$$S(E_i) = -K_i E_i, \quad S(F_i) = -F_i K_i^{-1}, \quad S(K_i) = K_i^{-1},$$

$$\varepsilon(E_i) = 0, \quad \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1,$$
where $\Delta$ is the comultiplication, $S$ the antipode, and $\varepsilon$ the counit.

Let $V$ be the finite-dimensional simple $U_{q}\mathfrak{sl}(n+1)$-module corresponding to the first fundamental weight $\omega_1$ (that is, the $n+1$-dimensional defining representation). The algebra $C[V_{R}]_{q}$ in this case is generated by $z_0, z_1, \ldots, z_n$ and $\hat{z}_0, \hat{z}_1, \ldots, \hat{z}_n$ with the relations (cf. [4])

$$
\begin{align*}
    z_iz_j &= qz_jz_i, \quad (i < j), \\
    \hat{z}_i\hat{z}_j &= q^{-1}\hat{z}_j\hat{z}_i, \quad (i < j), \\
    z_iz_j &= q\hat{z}_jz_i, \quad (i \neq j), \\
    z_i\hat{z}_i - \hat{z}_iz_i &= (q^{-2} - 1) \sum_{k>i} z_k\hat{z}_k.
\end{align*}
$$

Here $z_0, z_1, \ldots, z_n$ and $\hat{z}_0, \hat{z}_1, \ldots, \hat{z}_n$ are the projections of the vectors of the mutually dual canonical bases in $V$ and $V^*$ respectively.

The action of $U_{q}\mathfrak{sl}(n+1)$ on $C[V_{R}]_{q}$ is given by

$$
\begin{align*}
    E_i : z_j &\mapsto \delta_{ij}z_{j-1}, \quad \hat{z}_j &\mapsto -\delta_{i-1,j}q^{-1}\hat{z}_{j+1}, \\
    F_i : z_j &\mapsto \delta_{i-1,j}z_{j+1}, \quad \hat{z}_j &\mapsto -\delta_{ij}q\hat{z}_{j-1}, \\
    K_i : z_j &\mapsto \begin{cases} 
        q^{-1}z_{i-1} & \text{if } j = i - 1 \\
        qz_i & \text{if } j = i \\
        z_j & \text{if otherwise}
    \end{cases}, \quad \hat{z}_j &\mapsto \begin{cases} 
        q\hat{z}_{i-1} & \text{if } j = i - 1 \\
        q^{-1}\hat{z}_i & \text{if } j = i \\
        \hat{z}_j & \text{if otherwise}
    \end{cases}
\end{align*}
$$

In this case the subspace $I_0$ is spanned by the element

$$
\sum_{k=0}^{n} z_k \otimes \hat{z}_k - \sum_{k=0}^{n} q^{-2k}\hat{z}_k \otimes z_k.
$$

The quantum Heisenberg algebra $\mathcal{H}_g(V)$ is generated by $z_0, z_1, \ldots, z_n, \hat{z}_0, \hat{z}_1, \ldots, \hat{z}_n$ and $C$ with the relations (cf. [11])

$$
\begin{align*}
    z_iz_j &= qz_jz_i, \quad (i < j), \\
    \hat{z}_i\hat{z}_j &= q^{-1}\hat{z}_j\hat{z}_i, \quad (i < j), \\
    z_iz_j &= q\hat{z}_jz_i, \quad (i \neq j), \\
    z_i\hat{z}_i - \hat{z}_iz_i &= C + (q^{-2} - 1) \sum_{k>i} z_k\hat{z}_k, \\
    z_tC &= q^2Cz_i, \\
    \hat{z}_tC &= q^{-2}\hat{z}_tC_i.
\end{align*}
$$
The action of $U_q\mathfrak{sl}(n + 1)$ on $\mathcal{H}_g(V)$ is given by (2.1)-(2.3) and

$$\xi C = \varepsilon(\xi) C$$

for any $\xi \in U_q\mathfrak{sl}(n + 1)$ (i.e., $C$ is a $U_q\mathfrak{sl}(n + 1)$-invariant element).

Now suppose that $g_c$ is a compact real form of $g$ (unique up to an inner automorphism). Then, there is an antilinear involutive automorphism $\omega$ of $g$ such that

$$g_c = \{a \in g \mid \omega(a) = a\}.$$  

As is well known, there is a quantization of $g_c$ - a Hopf *-algebra $U_qg_c = (U_qg, b)$, where $b$ is an antilinear involutive algebra anti-automorphism and coalgebra automorphism such that

$$\omega_c : \xi \mapsto (S(\xi))^b$$

is an involution (thus, an antilinear involutive algebra automorphism and coalgebra anti-automorphism). Then, any finite-dimensional $U_qg$-module – in particular, our module $V$ – has a Hilbert space structure which makes the action of $U_qg_c$ into a *-representation. Let $\iota : V \rightarrow V^*$ be the antilinear isomorphism of vector spaces induced by the scalar product on $V$. The following proposition is rather obvious.

**Proposition 1.2.** The map $\iota : V \rightarrow V^*$ can be uniquely extended to an antilinear involutive anti-automorphism $\sharp$ of $\mathcal{H}_g(V)$ such that $C^\sharp = C$. Then the *-algebra $\mathcal{H}_{g_c}(V) = (\mathcal{H}_g(V), \sharp)$ is a $U_qg_c$-module *-algebra, which means that for any $\xi \in U_qg_c$, $f \in \mathcal{H}_{g_c}(V)$ one has that

$$(\xi f)^\sharp = \omega_c(\xi) f^\sharp.$$  

Now suppose that $U_qg_0$ is a Hopf *-algebra $(U_qg, b)$ which is a quantization of a non-split real form $g_0$ of $g$ equipped with a standard Lie bialgebra structure. Recall
that such structure is (cf. [11]) given by the Manin triple
\[(g, g_0, n_+ \oplus i\mathfrak{h}_0),\]
where \(b_0 = \mathfrak{h} \cap g_0\) (it depends, of course, on the choice of the maximal nilpotent subalgebra \(n_+\)). Let
\[\omega_0 : \xi \mapsto (S(\xi))^3\]
be the corresponding antilinear involutive algebra automorphism and coalgebra anti-automorphism on \(U_{qg_0}\).

\textbf{Proposition 1.3.} There exists a unique antilinear involutive anti-automorphism \(\ast\) of \(\mathcal{H}_g(V)\) such that \(\mathcal{H}_g(V) = (\mathcal{H}_g(V), \ast)\) is a \(U_{qg_0}\)-module \(*\)-algebra, which means that for any \(\xi \in U_{qg_0}, f \in \mathcal{H}_g(V)\), one has that
\[(\xi f)^\ast = \omega_0(\xi) f^\ast.\]

\textit{Proof.} The composition \(\tau = \omega_0 \omega_c\) is a linear Hopf algebra automorphism of \(U_{qg}\). Then, there exists an operator \(t : V \to V\) such that
\[(\tau \xi)(f) = (t^{-1} \xi t)(f)\] (2.4)
for any \(\xi \in U_{qg}, f \in V\). The \(U_{qg_0}\)-module \(*\)-algebra structure on \(\mathcal{H}_g(V)\) is given by
\[f^\ast = t(f^t)\text{ for any } f \in V, \ C^\ast = C.\] (2.5)
The uniqueness is obvious.

\textit{Example 1.2.} In the setting of Example 1.1 we have that the compact real form of
$U_q\mathfrak{sl}(n+1)$ is $U_q\mathfrak{su}(n+1) = (U_q\mathfrak{sl}(n+1), \phi)$, where $\phi$ is given by

$$E_i^\phi = K_i^{-2}F_i, \quad F_i^\phi = E_iK_i^2, \quad K_i^\phi = K_i.$$ 

The $U_q\mathfrak{su}(n+1)$-module $\ast$-algebra structure on $\mathcal{H}_{g_e}(V) = (\mathcal{H}_g(V), \#)$ is given by

$$z_i^\ast = \tilde{z}_i, \quad C^\ast = C.$$

Consider the real forms $U_q\mathfrak{su}(\bar{\iota}) = (U_q\mathfrak{sl}(n+1), \bar{\iota})$ of $U_q\mathfrak{sl}(n+1)$ parameterized by a sequence $\bar{\iota} = (\iota_0, \iota_1, \ldots, \iota_n)$, where $\iota_i = \pm 1$. These are given by

$$E_i^{\bar{\iota}} = \iota_{i-1}\iota_iE_i, \quad F_i^{\bar{\iota}} = \iota_{i-1}\iota_iF_i, \quad K_i^{\bar{\iota}} = K_i.$$  \hspace{1cm} (2.6)

They are quantizations of different Lie bilagebra structures on $\mathfrak{g}_0 = \mathfrak{su}(m, n+1 - m)$, where $m$ is the number of instances when $\iota_i = 1$ and $n+1 - m$ is the number of instances when $\iota_i = -1$.

Then the $U_q\mathfrak{su}(\bar{\iota})$-module $\ast$-algebra structure on $\mathcal{H}_{\mathfrak{g}_0}(V) = (\mathcal{H}_g(V), \ast)$ is given by

$$z_i^\ast = \iota_i\tilde{z}_i, \quad C^\ast = C.$$ \hspace{1cm} (2.7)

\textit{Remark 1.1.} We will be particularly interested in the case when $\bar{\iota} = (-1, 1)$. We denote $U_q\mathfrak{su}(-1, 1)$ by $U_q\mathfrak{su}(1, 1)$, $\mathcal{H}_{\mathfrak{g}_0}(V)$ by $\mathcal{H}$, $E_1$ by $E$, $F_1$ by $F$, and $K_1$ by $K$.

\section{2.2 Quantum Generalized Flag Manifolds}

In this section we establish a correspondence between the quantum Heisenberg algebras and certain classes of quantum $G$-spaces. From now on we use a shorter notation $\mathcal{H}$ to stand for $\mathcal{H}_g(V)$.

Recall the following well-known construction. Given a finite-dimensional simple
$U_q\mathfrak{g}$-module $V = L(\Lambda)$ with the highest weight $\Lambda$, we define a multiplication on

$$C[\mathcal{O}_V]^+_q = \bigoplus_{k=0}^{\infty} L(k\Lambda)$$

(2.8)

as follows. For any $a \in L(k\Lambda)$ and $b \in L(m\Lambda)$, we define their product as the projection of $a \otimes b$ onto $L((k + m)\Lambda) \subset L(k\Lambda) \otimes L(m\Lambda)$ (note that it is correctly defined, as the multiplicity of $L((k + m)\Lambda)$ in $L(k\Lambda) \otimes L(m\Lambda)$ is equal to 1). It is easy to see that it defines, in fact, a $U_q\mathfrak{g}$-module algebra structure on $C[\mathcal{O}_V]^+_q$.

Similarly, we define a $U_q\mathfrak{g}$-module algebra by applying the same construction to

$$C[\mathcal{O}_V]^-_q = \bigoplus_{k=0}^{\infty} L(k\Lambda)^*.$$ 

(2.9)

The following result follows immediately from the construction. It is an analog of the Poincaré-Birkhoff-Witt theorem for the $U_q\mathfrak{g}$-module algebras $C[V_R]_q$ and $\mathcal{H}_q(V)$.

**Theorem 1.1.** The multiplication maps

$$C[V_R]^+_q \otimes C[V_R]^-_q \rightarrow C[V_R]_q,$$

$$\mathcal{H}^+ \otimes \mathcal{H}_q(V)^0 \otimes \mathcal{H}_q(V)^- \rightarrow \mathcal{H}_q(V)$$

are isomorphisms of $U_q\mathfrak{g}$-module algebras. Here $C[V_R]^+_q = \mathcal{H}^+$ is the subalgebra generated by $V \subset V \oplus V^*$, while $C[V_R]^-_q = \mathcal{H}^-$ is the subalgebra generated by $V^* \subset V \oplus V^*$, and $\mathcal{H}^0$ the subalgebra generated by $C$.

Note that $C[\mathcal{O}_V]^+_q$ (resp. $C[\mathcal{O}_V]^-_q$) is a quantum analog of the algebra of holomorphic (resp. anti-holomorphic) polynomial functions on the $G$-orbit of $Cv_\Lambda$ where $v_\Lambda \subset L(\Lambda)$ is a highest weight vector. At the same time, $\mathcal{H}^+ = C[V_R]^+_q$ (resp. $\mathcal{H}^- = C[V_R]^-_q$) is a quantum analog of the algebra of holomorphic (resp. anti-holomorphic) polynomial functions on $V = L(\Lambda)$.

In the classical case (when $q = 1$) $C[\mathcal{O}_V]_1$ is a quotient of $C[V]_1$ over the ideal generated by the Plücker relations. A similar situation takes place in the quantum
case. Namely, it was shown in [25] that $C[O_V]^+_q$ is a quotient of $H^+$ by an ideal $J_+$ generated by the subspace

$$E_\Lambda^+ = \left(q^Z - q^{4(\Lambda + \rho, \Lambda)}\right)(L(\Lambda) \otimes L(\Lambda))$$

(2.10)

of quadratic relations called (holomorphic) quantum Plücker relations. Here $q^Z$ is the canonical central element of $U_qg$ defined in [2], and $\rho$ is the half of the sum of all positive roots of $g$. Similarly, we get that $C[O_V]^+_q$ is a quotient of $H^-$ over the ideal $J^-$ generated by the subspace

$$E_\Lambda^- = \left(q^Z - q^{4(\Lambda + \rho, \Lambda)}\right)(L(\Lambda)^* \otimes L(\Lambda)^*)$$

(2.11)

of what may be called anti-holomorphic quantum Plücker relations.

Define a $U_qg$-module algebra $C[O_V]^+_q$ as the quotient of $H$ over the ideal $J$ generated by all the quantum Plücker relations in both $J^+$ and $J^-$. Instead of $C$ introduce a new generator

$$c = \frac{1}{q-1}C.$$  

(2.12)

Now, if we take the quasi-classical limit $q \to 1$, keeping $c$, not $C$ constant, we will get a commutative Poisson algebra – the algebra of homogeneous polynomials on a family of projective Poisson $G$-spaces with a homogeneous parameter $c$. Homogeneous, because $z_i$'s, $\tilde{z}_i$'s and $c$ are defined by up to the group of automorphisms

$$\kappa_\alpha: z_i \mapsto \alpha z_i, \quad \tilde{\kappa}_\alpha: \tilde{z}_i \mapsto \tilde{\alpha}\tilde{z}_i, \quad \kappa_\alpha: c \mapsto |\alpha|^2 c,$$

(2.13)

for any $\alpha \in \mathbb{C}$. Note that the same formulas define a group of automorphisms of the $U_qg$-module algebra $C[O_V]^+_q$ in the quantum case.

The above-mentioned projective $G$-spaces are the projectivizations of the $G$-orbits of the space $Cu_\Lambda$ of highest weight vectors. They are called generalized flag manifolds. They are of the form $G/P$, where $P$ is the parabolic subgroup of $G$ whose Lie algebra is generated by $b_+ = n_+ \oplus h$ and the root vectors $E_\Lambda^-$ such that $(\lambda, \Lambda) = 0$. In
particular, if $\Lambda = \rho$, we get flag manifolds themselves. If $\Lambda$ is a fundamental weight, we get Grassmanians. This observation prompts the following definition.

**Definition 1.2** The $U_q\mathfrak{g}$-module algebra $C[\mathcal{O}_V]_q$ is called the algebra of homogeneous polynomial functions on a family of quantum generalized flag manifolds.

Let $G_c$ be the compact real form of $G$ whose quantization is isomorphic to $U_q\mathfrak{g}_c$, $G_0$ the non-compact real form whose quantization is isomorphic to $U_q\mathfrak{g}_0$. It is easy to see that

$$J^* = J, \quad J^* = J.$$ 

Therefore, $C[\mathcal{O}_V]_q$ has canonical $U_q\mathfrak{g}_c$- and $U_q\mathfrak{g}_0$-module $*$-algebra structures. The $U_q\mathfrak{g}_c$-module $*$-algebra $(C[\mathcal{O}_V]_q, *)$ can be thought of as a quantum algebra of homogeneous polynomials on a family of generalized flag manifolds of the form $G/P$ considered as Poisson $G_c$-spaces.

The $U_q\mathfrak{g}_0$-module $*$-algebra $(C[\mathcal{O}_V]_q, *)$ is a quantum analog of the algebra of homogeneous polynomials on a family of corresponding symmetric Poisson $G_0$-spaces of non-compact type. As $G_0$-spaces each of them is isomorphic to the $G_0$-orbit of the image of $P \subset G$ in $G/P$ with respect to the quotient map, where $P$ is the same parabolic group as described above.

**Example 1.3.** Return to Examples 1.1 and 1.2. In this case there are no Plücker relations, so that the relations in $C[\mathcal{O}_V]_q$ look as follows:

\[
\begin{align*}
z_i z_j &= q z_j z_i, \quad (i < j), \\
z_i \hat{z}_j &= q^{-1} \hat{z}_j z_i, \quad (i < j), \\
z_i \hat{z}_j &= q \hat{z}_j z_i, \quad (i \neq j), \\
z_{i} \hat{z}_{i} - \hat{z}_{i} z_{i} &= (q^{-2} - 1) \left( \sum_{k > i} z_k \hat{z}_k + qc \right), \\
z_{c} z_{c} &= q^2 c z_{c}, \\
\hat{z}_{c} &= q^{-2} c \hat{z}_{c}.
\end{align*}
\]

The $U_q\mathfrak{su}(\bar{v})$-module $*$-algebra $(C[\mathcal{O}_V]_q, *)$ is the quantum algebra of homogeneous polynomials on a family of quantum $\mathbb{C}P^n$, while the $U_q\mathfrak{su}(\bar{v})$-module $*$-algebra $(C[\mathcal{O}_V]_q, *)$ is the quantum algebra of homogeneous polynomials on a family of quantum hyperboloids which possess a complex manifold structure (inherited from $G/P$).
Remark 1.2. When \( G_c = SU(2) \), the family of quantum \( CP^1 \) is nothing but the family of quantum Podleś 2-spheres introduced in [22].

It is always good to have a large commutative subalgebra. Let \( U_q^H \) be the Hopf subalgebra of \( U_q \text{sl}(n+1) \) generated by \( K_i, K_i^{-1}, i = 1, 2, ..., n \). Consider the subalgebra \( C[O_V]_q^H \subset C[O_V]_q \) of the \( U_q^H \)-invariant elements. Denote by \( C[O_V]_q^{inv} \subset C[O_V]_q \) the subalgebra of \( U_q^H \)-invariant elements.

**Proposition 1.4.** (1) The algebra \( C[O_V]_q^H \) is commutative and generated by

\[
x_i = \sum_{k \geq i} z_k \Hat{z}_k + q c, \quad i = 0, ..., n + 1.
\]

Moreover, the following relations hold:

\[
\begin{align*}
z_i x_j &= q^2 x_j z_i, \quad (i < j), \\
\Hat{z}_i x_j &= q^{-2} x_j \Hat{z}_i, \quad (i < j), \\
z_i x_j &= x_j z_i, \quad (i \geq j), \\
\Hat{z}_i x_j &= x_j \Hat{z}_i, \quad (i \geq j).
\end{align*}
\]

(2) The algebra \( C[O_V]_q^{inv} \) is generated by

\[
c = q^{-1} x_{n+1} \quad \text{and} \quad d = q x_0.
\]

Moreover, \( d \) belongs to the center of \( C[O_V]_q \).

The proof is straightforward.

The formulas (2.14)-(2.15) let us consider an extension of \( C[O_V]_q \) by adding functions of \( \Bar{x} = (x_0, x_1, ..., x_{n+1}) \) so that the following relations hold:

\[
\begin{align*}
z_i f(\Bar{x}) &= f(x_0, ..., x_i, q^2 x_{i+1}, ..., q^2 x_{n+1}) z_i, \\
\Hat{z}_i f(\Bar{x}) &= f(x_0, ..., x_i, q^{-2} x_{i+1}, ..., q^{-2} x_{n+1}) \Hat{z}_i.
\end{align*}
\]
We denote the extended algebra by $\text{Func}(O_V)_q$.

**Proposition 1.5.** The $U_{qg^c}$- and $U_{qg^0}$-module *-algebra structures can be uniquely extended from $C[O_V]_q$ on $\text{Func}(O_V)_q$. The action of $U_qg$ is given by

$$E_i : f(\tilde{x}) \mapsto \frac{f(\tilde{x}) - T_i f(\tilde{x})}{x_i - q^2 x_i} z_{i-1} \tilde{z}_i,$$

(2.17)

$$F_i : f(\tilde{x}) \mapsto \frac{T_i^{-1} f(\tilde{x}) - f(\tilde{x})}{q^{-2} x_i - x_i} z_i \tilde{z}_{i-1},$$

(2.18)

$$K_i : f(\tilde{x}) \mapsto f(\tilde{x}),$$

(2.19)

where

$$T_i : f(\tilde{x}) \mapsto f(x_0, \ldots, x_{i-1}, q^2 x_i, x_{i+1}, \ldots, x_{n+1}),$$

with the involutions given by

$$f(\tilde{x})^h = f(\tilde{x}), \quad f(\tilde{x})^* = f(\tilde{x}).$$

**Example 1.4.** When $G_0 = SU(1,1)$, one can modify our construction, so that we get more quantum $SU(1,1)$-spaces. Note that $z_0, z_1, \hat{z}_0, \hat{z}_1$ are the homogeneous co-ordinates and $c$ a homogeneous parameter on a family of quantum $SU(1,1)$-spaces, defined up to the automorphisms $\kappa_\alpha$ given by (2.13).

Consider the subalgebra $\text{Func}(X)_q$ of $\kappa_\alpha$-invariant elements in $\text{Func}(O_V)_q$. Denote by $\widetilde{\text{Func}}(X)_q$ the subalgebra in $\text{Func}(O_V)_q$ of the elements which are invariant with respect to $\kappa_\alpha$ with $|\alpha| = 1$.

**Proposition 1.6** The $U_{q\text{sl}(2)}$-module algebra $\widetilde{\text{Func}}(X)_q$ is generated by the elements

$$x = z_1 \hat{z}_1 + qc, \quad y = z_0 \hat{z}_1, \quad \hat{y} = z_1 \hat{z}_0$$
with the relations

\[ yf(x) = f(q^2 x)y, \quad \hat{y}f(x) = f(q^{-2} x)\hat{y}, \quad (2.20) \]

\[ \hat{y}y = -(q^{-1} x - c)(q^{-1} x - d), \quad y\hat{y} = -(qx - c)(qx - d), \quad (2.21) \]

while \( c \) and \( d \) belong to the center of \( \widehat{\text{Func}}(X)_q \).

The proof is a straightforward computation.

It turns out that, besides the involution \( * \) given in (2.7), there exists yet another one which makes \( \text{Func}(X)_q \) into a \( U_q\text{su}(1,1) \)-module \(*\)-algebra. To keep the notation shorter, we will use the somewhat larger algebra \( \widehat{\text{Func}}(X)_q \).

**Proposition 1.7.** There are two non-equivalent \( U_q\text{su}(1,1) \)-module \(*\)-algebra structures on \( \text{Func}(X)_q \), one of them given by

\[ y^* = -\hat{y}, \quad x^* = x, \quad c^* = c, \quad d^* = d, \quad (2.22) \]

and the other one given by

\[ y^* = -y, \quad x^* = x, \quad c^* = d. \quad (2.23) \]

The proof is a straightforward computation.

In both cases we can define a \( U_q\text{su}(1,1) \)-module \(*\)-algebra \( \text{Func}(X_{c_0, d_0})_q \) as the quotient of \( \widehat{\text{Func}}(X)_q \) over the ideal generated by \( c - c_0 \) and \( d - d_0 \), where \( c_0, d_0 \in \mathbb{R} \) in the first case and \( c_0 \in \mathbb{C}, c_0 = d_0 \) in the second case.

It is clear that if \( c_0, d_0 \in \mathbb{R} \) and \( c_0 \neq d_0 \), \( \text{Func}(X_{c_0, d_0})_q \) is a quantum algebra of functions on the two-sheet hyperboloid \( |y|^2 = (x - c_0)(x - d_0) \). If \( c_0 = d_0 \), \( \text{Func}(X_{c_0, d_0})_q \) is a quantum algebra of functions on the cone given by the same equation. Finally, if \( c_0 = d_0, c_0 \neq d_0 \), \( \text{Func}(X_{c_0, d_0})_q \) is a quantum algebra of functions on
the corresponding one-sheet hyperboloid.

2.3 Quantum Moment Map

Recall the definition of the classical moment map, generalized to the case when $G$ is a Poisson Lie group with a non-trivial Poisson structure. Consider the corresponding Lie bialgebra $\mathfrak{g}$ and the dual Poisson Lie group $G^*$ defined as the connected and simply connected Poisson Lie group with the Lie bialgebra $\mathfrak{g}^*$. For any $\xi \in \mathfrak{g}$, let $\alpha_\xi$ be the left invariant differential 1-form on $G^*$ with $\alpha_\xi(e) = \xi$. The Poisson bivector field $\pi_{G^*}$ on $G$ defines a map $\tilde{\pi}_{G^*} : \Omega^1(G^*) \to \text{Vect}(G^*)$.

**Definition 1.3.** The vector field $\rho_\xi = \tilde{\pi}_{G^*}(\alpha_\xi)$ is called the left dressing vector field on $G^*$ corresponding to $\xi \in \mathfrak{g}$. The left dressing vector fields define a local action of $G$ on $G^*$ which is called the left dressing action.

In some cases, for example, when $G$ is compact, the local dressing action can be extended to a global one. But in general, this need not be the case, as the example of $G = SU(1, 1)$ readily shows.

Suppose now that $M$ is a left Poisson $G$-manifold, that is, $M$ is a Poisson manifold with the action of $G$ on $M$ such that the corresponding map $G \times M \to M$ is a Poisson map. Let $\sigma_\xi$ be the vector field corresponding to the infinitesimal action of $\xi \in \mathfrak{g}$. We use a slightly different definition of generalized moment map that the one given in [17] to serve the purposes of such situations when there is no global dressing action.

**Definition 1.4.** Let $M'$ be a union of symplectic leaves in $M$ such that $M'$ is a dense subset in $M$. A map $J : M' \to G^*$ is called moment map for $M$ if

$$\sigma_\xi = \tilde{\pi}_M(J^*(\alpha_\xi)). \quad (2.24)$$

Definition 1.4 means that $J : M' \to G^*$ intertwines locally the $G$-action on $M$ with the dressing action of $G$ on $G^*$. When $G$ is a Poisson Lie group with the trivial Poisson structure, the dual Poisson Lie group $G^*$ is isomorphic to $\mathfrak{g}^*$ as a Poisson manifold.
and is abelian as a group. The corresponding dressing action always extends to a global one which is nothing but the usual coadjoint action of $G$ on $\mathfrak{g}^*$. Thus, in this case for any Hamiltonian $G$-space $M$, there exists a moment map onto a coadjoint orbit in $\mathfrak{g}^*$.

It looks that in order to quantize the above definition, we should look for a quantum moment map in the form $\text{Func}(G^*)_q \rightarrow \text{Func}(M)_q$. However, the Drinfeld’s duality tells us that the quantum enveloping algebra $U_q\mathfrak{g}$ can be thought of as a quantum algebra of functions on $G^*$. Below we obtain a quantum moment map in the form $U_q\mathfrak{g} \mapsto \text{Func}(M)_q$.

As is well known, the quasi-classical analog of the quantum adjoint action of $U_q\mathfrak{g}$ on itself given by

$ad_q(a) : b \mapsto \sum_k a_k^{(1)} b S \left( a_k^{(2)} \right)$, whenever $\Delta(a) = \sum_k a_k^{(1)} \otimes b_k^{(2)}$ (2.25)

is nothing but the left dressing action of $U\mathfrak{g}$ on $\text{Func}(G^*)$. Also, it is well known that for any Hopf algebra $A$, the quantum adjoint action of $A$ on itself equips $A$ with an $A$-module algebra structure, or an $A$-module $*$-algebra structure if $A$ is a Hopf $*$-algebra. This inspires the following definition.

**Definition 1.5** (1) Given a $U_q\mathfrak{g}$-module algebra $\mathcal{F}$, a homomorphism $J : U_q\mathfrak{g} \rightarrow \mathcal{F}$ is called quantum moment map if $J$ is a morphism of $U_q\mathfrak{g}$-module algebras, with $U_q\mathfrak{g}$ acting on itself by the quantum adjoint action (2.25).

(2) Given a $U_q\mathfrak{g}_0$-module $*$-algebra $\mathcal{F}_0$, a $*$-homomorphism $J_0 : U_q\mathfrak{g}_0 \rightarrow \mathcal{F}_0$ is called quantum moment map if $J_0$ is a morphism of $U_q\mathfrak{g}_0$-module $*$-algebras, with $U_q\mathfrak{g}_0$ acting on itself by the quantum adjoint action (2.25).

We see that the quantum Heisenberg algebra $\mathcal{H}$ contains the subalgebras $\mathcal{H}^+$ and $\mathcal{H}^-$ generated by $V$ and $V^*$ respectively. Of course, both are $U_q\mathfrak{g}$-module subalgebras of $\mathcal{H}$. Consider the subalgebra $\mathcal{H}_0^-$ generated by $\mathcal{H}^-$ and $C$. It has a one-dimensional
representation $\chi$ in $C_x$ given by

$$\chi(V) = 0, \quad \chi(C) = 1.$$

Consider the corresponding induced $\mathcal{H}$-module

$$W = Ind_{\mathcal{H}_{\mathfrak{c}}}^{\mathcal{H}}(C_x).$$

It is spanned by monomials of the form

$$a_i^{m_i}a_2^{m_2}...a_k^{m_k}\bar{1}_x,$$  \hspace{1cm} (2.26)

where $a_i \in V \subset \mathcal{H}^+$ and $\bar{1}_x$ is a generator of $C_x$. Thus, we see that $W$ is isomorphic to $\mathcal{H}^+$ as a vector space, with a $\mathbb{Z}^{\text{dim}V}$-grading defined by (2.26). This equips $W$ with a $U_q\mathfrak{g}$-module structure so that $W$ is isomorphic to $\mathcal{H}^+$ as a $U_q\mathfrak{g}$-module.

**Proposition 1.8** (1) The $\mathcal{H}$-module $W$ is simple and faithful.

(2) The subalgebra $\mathcal{H}^{\text{inv}}$ of the $U_q\mathfrak{h}$-invariant elements in $\mathcal{H}$ is commutative. Moreover, any homogeneous monomial of the form (2.26) in $W$ is an eigen-vector for the action of $\mathcal{H}^{\text{inv}}$.

**Proof.** The first statement is obvious. To prove the second one, consider a basis $v_0, v_1, ..., v_n$ of $V$ and the dual basis $\hat{v}_0, \hat{v}_1, ..., \hat{v}_n$ of $V^*$. Of course, one can choose the bases so that the vectors of the bases would be eigen-vectors for the action of $U_q\mathfrak{h}$.

Denote by $\hat{m} = (m_0, m_1, ..., m_n)$ the degree of a monomial

$$f = v_0^{m_0}v_1^{m_1}...v_n^{m_n}\bar{1}_x \in W.$$  \hspace{1cm} (2.27)

By definition, the degree of $v_j f$ is equal to $(m_0, ..., m_{j-1}, m_j + 1, m_{j+1}, ..., m_n)$, while the degree of $\hat{v}_j f$ is equal to $(m_0, ..., m_{j-1}, m_j - 1, m_{j+1}, ..., m_n)$. Of course, the action of $C$ preserves the grading.

It is easy to see that $\mathcal{H}^{\text{inv}}$ is generated by $v_j\hat{v}^j$ $(j = 0, 1, ..., n)$ and $C$. Therefore,
the action of $\mathcal{H}^{inv}$ preserves the grading. This means that any monomial of the form (2.27) is an eigen-vector for the action of $\mathcal{H}^{inv}$. This immediately follows that $\mathcal{H}^{inv}$ is commutative.

The statement (1) of Proposition 1.8 shows that $\mathcal{H}$ is isomorphic to its image in $End W$. Also, that there exists a basis in $W$ (spanned by the monomials of the form (2.27)) which diagonalizes the action of $\mathcal{H}^{inv}$. This allows us to extend the algebra $\mathcal{H}$ by the functions on the spectrum of $\mathcal{H}^{inv}$ in $W$. Denote the extended algebra by $\mathcal{H}$. One can show that the $U_q\mathfrak{g}$-module algebra structure can also be extended from $\mathcal{H}$ to $\mathcal{H}$.

Obviously, $\mathcal{H}$ is isomorphic to $End W$ as an algebra. On the other hand, $U_q\mathfrak{g}$ acts in $W$. This induces a homomorphism $J : U_q\mathfrak{g} \rightarrow \mathcal{H}$. It is clear that the image of $U_q\mathfrak{g}$ lies in $Func(X)_q \subset \mathcal{H}$ — the subalgebra of $\kappa_\alpha$-invariant elements in $\mathcal{H}$.

**Theorem 1.2.** (1) There exists a unique homomorphism

$$J : U_q\mathfrak{g} \rightarrow Func(X)_q$$

such that the composition of $J$ with the action of $Func(X)_q \subset \mathcal{H}$ in $W$ coincides with the action of $U_q\mathfrak{g}$ in $W$.

(2) $J$ is a morphism of $U_q\mathfrak{g}$-module algebras, with $U_q\mathfrak{g}$ acting on itself via the quantum adjoint action. In other words, $J$ is a quantum moment map for $Func(X)_q$.

**Proof.** The first statement has been already proved above. To show that (2) holds, note that the image of $U_q\mathfrak{g}$ in $\mathcal{H}$ preserves the scalar degree $m = |\vec{m}| = m_0 + m_1 + \ldots + m_n$ of any monomial of the form (2.27). Therefore, $U_q\mathfrak{g}$ maps into the subalgebra generated by the elements of the form $\varphi \hat{\psi}$, where $\varphi$ (resp. $\hat{\psi}$) belongs to the subalgebra of $\kappa_\alpha$-invariant elements in the extension $\mathcal{H}^+$ (resp. $\mathcal{H}^-$) of $\mathcal{H}^+$ (resp. $\mathcal{H}^-$) by $\mathcal{H}(V)^{inv}$. In particular, $U_q\mathfrak{h}$ can be shown to map into the subalgebra of $\kappa_\alpha$-invariant elements in $\mathcal{H}^{inv}$.
For any \( a \in U_q g \) with \( \Delta(a) = \sum_k a_k^{(1)} \otimes a_k^{(2)} \), we get that
\[
a(\varphi \hat{\psi}) = \sum_k a_k^{(1)}(\varphi) \left( S\left(a_k^{(2)}\right) \psi \right) = \sum_k J\left(a_k^{(1)}\right) \varphi(J\left(S\left(a_k^{(2)}\right)\right)) \psi.
\]

Recalling that \( \hat{\psi} = \psi^{-1}f \), where \( f \in \hat{H}^{inv} \), one can prove (2) after some straightforward computations.

Note that the moment map \( J \) for \( Func(X)_q \) was constructed originally as a homomorphism from \( U_q g \) into the quantum Heisenberg algebra. Thus, we see yet another manifestation of the Drinfeld's duality. In this case, the same map has two quasi-classical analogs. One of them is the homomorphism from the classical universal enveloping algebra to the Heisenberg algebra, corresponding to the realization of \( U_g \) by differential operators on a \( G \)-space. Another one is a moment map for a family of generalized flag manifolds. Let us look at a few examples.

**Example 1.5.** In the context of Example 1.1 (that is, when \( g = sl(n + 1) \) and \( V \) being the first fundamental representation), the map \( J \) of Theorem 1.2 is given by
\[
\begin{align*}
J : E_i & \mapsto \frac{(q^{-1} - q)^{\frac{1}{2}}}{(x_{i-1} x_{i+1})^{\frac{1}{2}}} z_{i-1} \hat{z}_i, & (2.28) \\
J : F_i & \mapsto \frac{(q^{-1} - q)^{\frac{1}{2}}}{x_i x_i} z_i \hat{z}_{i-1}, & (2.29) \\
J : K_i & \mapsto \frac{x_i}{(x_{i-1} x_{i+1})^{\frac{1}{2}}}. & (2.30)
\end{align*}
\]

Moreover, given a Hopf *-algebra structure \( U_q su(i) \) on \( U_q sl(n + 1) \) and the involution (2.7) on \( Func(X)_q \), we see that \( J \) is in fact a morphism of \( U_q su(i) \)-module *-algebras, thus defining a quantum moment map for the \( U_q su(i) \)-module *-algebra \( Func(X)_q \).

Note that the subalgebra \( Func(X)_q \) of the \( \kappa_\alpha \)-invariant elements in \( \hat{H} \) is isomorphic to the quantum Weyl algebra constructed in [6]. Also, the quantum moment map \( J \) is equivalent to the quantum oscillator map from \( U_q sl(n + 1, \mathbb{C}) \) into the quantum
Weyl algebra constructed there. However, now we give a different interpretation of it in the light of the quantum moment map.

**Remark 1.3.** For any automorphism $I$ of the $U_q\mathfrak{g}$-module algebra structure on $U_q\mathfrak{g}$ (defined by the quantum adjoint action), the map $J \circ I : U_q\mathfrak{g} \to \text{Func}(X)_q$ defines another quantum moment map. In particular, in the above example the group of such automorphisms is generated modulo the center by the automorphisms given by

$$I_i : E_j \mapsto (-1)^i E_j, \quad I_i : F_j \mapsto F_j, \quad I_i : K_j \mapsto (-1)^i K_j.$$

It is easy to see that the corresponding moment maps $J_i = J \circ I_i$ are given by the same formulas (2.28)-(2.30) except that we take another value of $(x_i x_{i+1})^{\frac{1}{2}}$.

**Example 1.6.** Consider the case $\mathfrak{g} = \mathfrak{sl}(2)$. Then $\text{Func}(X)_q$ can be described in terms of the generators $x, y, \hat{y}$ as defined in Proposition 1.6. We have that

$$J : E \mapsto \frac{(q^{-1} - q)^{\frac{1}{2}} y}{(cd)^{\frac{1}{2}}},$$

$$J : F \mapsto \frac{(q^{-1} - q)^{\frac{1}{2}} \hat{y}}{x},$$

$$J : K \mapsto \frac{x}{(cd)^{\frac{1}{2}}}.$$

Moreover, $J$ is a morphism of $U_q\mathfrak{su}(1,1)$-module *-algebras for any of the involutions * and $\ast$ on $\text{Func}(X)_q$ defined by (2.22) and (2.23) respectively. Suppose that we fix $c = c_0$ and $d = d_0$ so that $c_0 d_0 > 0$. Then $J$ is a quantum moment map for any of the quantum hyperboloids (or a quantum cone) $X_{c_0,d_0}$ defined in the previous section. It is interesting to note that the quantum quadratic Casimir element

$$C_q = \frac{1}{2} (EF + FE) + \frac{q^{-1} + q}{2 (q^{-1} - q)^2} (K - 2 + K^{-1})$$

has the following image:

$$J : C \mapsto \frac{1}{(q^{-1} - q)^2} \left( \frac{c_0}{d_0} + \frac{d_0}{c_0} - q^{-1} - q \right).$$
The quasi-classical analog \( J_0 \) of \( J \) imbeds a dense subset of the hyperboloid (or cone) \(|y|^2 = (x - c_0)(x - d_0)\) (precisely, the one defined by \( x \neq 0 \)) into the dual Poisson Lie group \( SU(1,1)^* \). The picture is as follows. One can show that \( SU(1,1)^* \) is isomorphic as a Lie group to the group of translations and dilations of a plane, or to the group of the matrices of the form \(
abla \begin{pmatrix} t & z \\ 0 & t^{-1} \end{pmatrix} \), where \( t > 0 \) and \(z \in \mathbb{C} \). We can assume without loss of generality that \( c_0d_0 = 1 \). Then, the subset of the manifold \(|y|^2 = (x - c_0)(x - d_0)\) defined by \( x > 0 \) has the image \(|z|^2 = (t - c_0)(t - d_0)\), while the subset defined by \( x < 0 \) has the image \(|z|^2 = (t + c_0)(t + d_0)\), which is nothing but the reflection \( t \mapsto -t \) of \(|z|^2 = (t - c_0)(t - d_0)\) with \( t < 0 \).

Of course, these imbeddings preserve the symplectic leaves. Indeed, for the one-sheet hyperboloids, both subsets \( x > 0 \) and \( x < 0 \) are two-dimensional symplectic leaves, while any point of the circle \( x = 0 \) is a zero-dimensional symplectic leaf. For a two-sheet hyperboloid with \( 0 < c_0 < d_0 \), the whole sheet \( x \geq d_0 \) is a symplectic leaf, and the two-dimensional pieces \( x < 0 \) and \( 0 < x \leq c_0 \) of the other sheet are symplectic leaves as well, while any point of the circle \( x = 0 \) is a zero-dimensional symplectic leaf. Finally, for the cone with \( c_0 = d_0 = 1 \), the subsets defined by \( x < 0 \), \( 0 < x < 1 \), and \( x > 1 \) respectively all are symplectic leaves, as are the points on the circle \( x = 0 \) and the vertex of the cone – the unit element of the group.

2.4 Quantum Polarizations

In the previous section we constructed a quantum moment map \( J : U_{qg} \to Func(X)_q \). Now we want to find a way to construct irreducible *-representations of \( U_{qg} \) as compositions of the form \( \pi \circ J \), where \( \pi \) is an irreducible *-representation of \( Funq(X) \). Recall that the classical orbit method provides a construction of an irreducible representation of an algebra of functions on a Hamiltonian manifold in sections of a certain linear bundle with connection (whose curvature is equal to the symplectic form), which are constant along a given polarization. Our construction in the quantum case draws the ideas from that classical picture.
We have constructed $Func(X)_q$ as a subalgebra in $\tilde{H}$ which consists of the $\kappa_\alpha$-invariant elements. Recall that $\kappa_\alpha$ is a family of automorphisms parameterized by a non-zero complex number $\alpha$ (see (2.13)). If we think of $Func(X)_q$ as an algebra of functions on a quantum space $X$, then $\tilde{H}$ can be thought of as an algebra of functions on the total space of a linear bundle over $X$.

Further we will present a construction of irreducible $*$-representations of $U_{q\mathfrak{g}_0}$ using what we call here quantum polarization.

**Definition 1.6.** Suppose that $\mathcal{F}$ is a $U_{q\mathfrak{g}_0}$-module $*$-algebra.

1. A $U_{q\mathfrak{g}}$-module subalgebra $\mathcal{P}$ is called quantum polarization of $\mathcal{F}$ if $\mathcal{F}$ is generated by $\mathcal{P}$ and the subalgebra $\mathcal{F}^{inv}$ of $U_{q\mathfrak{h}}$-invariant elements in $\mathcal{F}$, and if $\mathcal{P} \cap \mathcal{F}^{inv} = C$.

2. A quantum polarization $\mathcal{P}$ of $\mathcal{F}$ is called complex if $\mathcal{P} \cap \mathcal{P}^* = C$. A quantum polarization $\mathcal{P}$ of $\mathcal{F}$ is called real if $\mathcal{P} = \mathcal{P}^*$.

Consider the subalgebra $Pol(X)_q^+$ (resp. $Pol(X)_q^-$) in $Func(X)_q$ generated by $v_kv^{-1}_m$ (resp. $\hat{v}_k\hat{v}^{-1}_m$), where $v_k$ are vectors of a $U_{q\mathfrak{h}}$-invariant basis in $V$. We will see in examples that it plays the role of the algebra of holomorphic (resp. anti-holomorphic) functions in the case of a complex polarization. The following proposition is an immediate consequence of the definitions.

**Proposition 1.9.** $Pol(X)_q^\pm$ is a $U_{q\mathfrak{g}}$-module subalgebra in $Func(X)_q$.

**Example 1.7.** Suppose that, as in Example 1.1, $\mathfrak{g} = \mathfrak{sl}(n + 1, \mathbb{C})$ (equipped with the standard Lie bialgebra structure), and $V$ is the highest weight $U_{q\mathfrak{g}}$-module corresponding to the first fundamental weight $\omega_1$. We keep the same notation as before.

Then $Pol(X)_q^+$ is generated by

$$\zeta_i = z_i^{-1}z_{i-1}, \quad (i = 1, 2, ..., n),$$

subject to the relations

$$\zeta_i\zeta_j = \zeta_j\zeta_i, \text{ if } |i - j| \neq 1,$$
\[ \zeta_1 \zeta_{i+1} = q \zeta_{i+1} \zeta_i. \]

Respectively, \( Pol(X)_q^- \) is generated by

\[ \hat{\zeta}_i = \hat{z}_{i-1} \hat{z}_i^{-1}, \quad (i = 1, 2, \ldots, n), \]

subject to the relations

\[ \hat{\zeta}_i \hat{\zeta}_j = \hat{\zeta}_j \hat{\zeta}_i, \quad \text{if } |i - j| \neq 1, \]

\[ \hat{\zeta}_i \hat{\zeta}_{i+1} = q^{-1} \hat{\zeta}_{i+1} \hat{\zeta}_i. \]

It is easy to write down explicit formulas for the action of \( U_q\mathfrak{sl}(n+1, \mathbb{C}) \) in \( Pol(X)^\pm_q \), but we will do it only in the special case of \( n = 1 \) (see the following example).

**Example 1.8.** When \( n = 1 \), \( Pol(X)^+_q \) is generated by a single element

\[ \zeta = z^{-1}_1 z_0 = (qx - c_0)^{-1} y, \quad (2.36) \]

while \( Pol(X)^-_q \) is generated by

\[ \hat{\zeta} = \hat{z}_0 \hat{z}_1^{-1} = \hat{y}(qx - c_0)^{-1}. \]

The action of \( U_q\mathfrak{sl}(2, \mathbb{C}) \) in \( Pol(X)^+_q \) is given by

\[ E : f(\zeta) \mapsto -q^2 f(\zeta) - f(q^2 \zeta), \]

\[ F : f(\zeta) \mapsto f(q^{-2}\zeta) - f(\zeta), \]

\[ K : f(\zeta) \mapsto f(q^{-2}). \]

The action of \( U_q\mathfrak{sl}(2, \mathbb{C}) \) in \( Pol(X)^-_q \) is given by similar formulas. One can check that the center of \( U_q\mathfrak{sl}(2, \mathbb{C}) \) acts trivially in \( Pol(X)^\pm_q \).

**Proposition 1.10.** The \( U_q\mathfrak{sl}(n + 1, \mathbb{C}) \)-module algebra \( Func(X)_q \) is generated by \( \zeta_i \),
\( i = 1, 2, \ldots, n \), and the functions \( f(x_1, x_2, \ldots, x_n) \), subject to the relations

\[
\zeta_i f(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, q^2 x_i, x_{i+1}, \ldots, x_n) \zeta_i,
\]

\[
\check{\zeta}_i f(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, q^{-2} x_i, x_{i+1}, \ldots, x_n) \check{\zeta}_i,
\]

\[
\zeta_i \check{\zeta}_i = \frac{x_{i-1} - x_i}{x_i - q^{-2} x_{i+1}}, \quad (2.37)
\]

\[
\check{\zeta}_i \zeta_i = \frac{q^2 x_{i-1} - x_i}{x_i - x_{i+1}}. \quad (2.38)
\]

For any real form \( U_q su(\ell) \) of \( U_q sl(n+1, \mathbb{C}) \), the involution (2.7) equips \( Func(X)_q \) with a \( U_q su(\ell) \)-module \(*\)-algebra structure. The involution (2.7) is given by

\[
\zeta_i^* = i \check{\zeta}_i, \quad x_i^* = x_i.
\]

In the next section we will consider first the simplest case of \( U_q su(1, 1) \) to illustrate the basic ideas. We will use them later to construct some irreducible \(*\)-representations of \( U_q su(\ell) \) which correspond to the dressing orbits of the minimal dimension. Let us describe, therefore, the relations in the former, more special, case more explicitly.

**Corollary 1.1.** (1) For \( c_0 d_0 = 1, c_0, d_0 \in \mathbb{R} \), the \( U_q su(1, 1) \)-module \(*\)-algebra \( Func(X_{c_0,d_0})_q \) is generated by \( \zeta, \zeta^* \), and the functions \( f(x) \), subject to the relations

\[
\zeta f(x) = f(q^2 x) \zeta, \quad \zeta^* f(x) = f(q^{-2} x) \zeta^*, \quad (2.39)
\]

\[
\zeta \zeta^* = \frac{x - q^{-1} d_0}{x - q^{-1} c_0}, \quad \zeta^* \zeta = \frac{z - c_0}{z - q c_0}. \quad (2.40)
\]

In particular, the following relation holds:

\[
\Phi(\zeta^*) = q^2 \Phi(\zeta^* \zeta),
\]

where

\[
\Phi(t) = \frac{1 - \gamma t}{1 - t}, \quad \gamma = \frac{c_0}{d_0}. \quad (2.41)
\]

(2) For \( c_0 d_0 = 1, c_0, d_0 \not\in \mathbb{R} \), the \( U_q su(1, 1) \)-module \(*\)-algebra \( Func(X_{c_0,d_0})_q \) is
generated by $\zeta$, $\hat{\zeta}$, and the functions $f(x)$ with the relations

\begin{align}
\zeta f(x) &= f(q^2 x) \zeta, \quad \hat{\zeta} f(x) = f(q^{-2} x) \hat{\zeta}, \\
\hat{\zeta} \zeta &= \frac{x - q^{-1} d_0}{x - q^{-1} c_0}, \quad \hat{\zeta} \hat{\zeta} = \frac{z - q d_0}{z - q c_0}, \quad (2.42) \\
\zeta \zeta^* &= \zeta^* \zeta = 1, \quad \hat{\zeta} \hat{\zeta}^* = \hat{\zeta}^* \hat{\zeta} = 1. \quad (2.43)
\end{align}

In particular, the following relation holds:

$$
\Phi (\zeta \hat{\zeta}) = q^2 \Phi (\hat{\zeta} \zeta),
$$

where $\Phi(t)$ is given by (2.41).

**Proposition 1.11.** (1) The $U_{q^*}su(\tilde{\ell})$-module subalgebra $Pol(X)^q_\pm$ is a quantum polarization of $Func(X)_q$.

(2) The quantum polarization $Pol(Xq_{c_0,d_0})^\pm$ of the $U_{q^*}su(1, 1)$-module $\ast$-algebra $Func(Xq_{c_0,d_0})^\pm_q$ is complex when $c_0$ and $d_0$ are distinct real numbers and real when $c_0 = d_0$.

**Proof.** This is an immediate consequences of Definition 1.6, Propositions 1.9 and 1.10, and Corollary 1.1. Thus, by Proposition 1.9, $Pol(X)^q_\pm$ is a $U_{q^*}sl(n + 1, C)$-module subalgebra in $Func(X)_q$. By Proposition 1.10, $Pol(X)^q_\pm$ and $(Pol(X)^q_\pm)^\ast$ generate $Func(X)_q$. But, as follows from (2.38), $(Pol(X)^q_\pm)^\ast$ is generated by $Pol(X)^q_\pm$ and the functions $f(x_0, \ldots, x_{n+1})$ which are $U_{q^*}\hbar$-invariant. The fact that the are no $U_{q^*}\hbar$-invariant elements in $Pol(X)^q_\pm$ besides constants follows from the explicit construction of $Pol(X)^q_\pm$. This proves the first statement of the proposition. The second statement follows from Corollary 1.1 and the explicit formulas (2.22) and (2.23) for the involutions $\ast$ and $\hat{\ast}$.

We will see later that $Pol(X)^q_\pm$ plays the role of the algebra of functions constant along a certain polarization. In particular, when $n > 1$ and $c_0, d_0 \in \mathbb{R}$ ($n = 1$), the corresponding polarization is complex, so that they are quantum analogs of the algebras of holomorphic and anti-holomorphic functions. For the quantum one-sheet
hyperboloids $c_0, d_0 \notin \mathbb{R} (n = 1)$, however, the polarization is real, and $Pol(X_{c_0, d_0})^\pm$ are quantum analogs of the algebras of functions which are constant along the two families of straight lines on the corresponding one-sheet hyperboloids.

### 2.5 Irreducible $\ast$-Representations Of $U_{qsu}(1, 1)$

In this section we will construct irreducible $\ast$-representations of $U_{qsu}(1, 1)$ associated with the quantum spaces $X_{c_0, d_0}$. We assume that $c_0 d_0 = 1$.

As follows from (2.40) and (2.43), $\text{Func}(X_{c_0, d_0})_q$ is generated as an algebra by $\zeta, \zeta^{-1}$, and the functions $f(x)$ (resp. by $\zeta^*, (\zeta^*)^{-1}$, and the functions $f(x)$, or by $\hat{\zeta}, \hat{\zeta}^{-1}$, and $f(x)$), since we see that, for instance, $\zeta^* = \frac{z - ndo}{z - q_{co}} \zeta^{-1}$. In particular, we see that $\text{Func}(X_{c_0, d_0})_q$ is generated by two distinguished subalgebras – $Pol(X_{c_0, d_0})^\pm$ and $\text{Func}(X_{c_0, d_0})^{\text{inv}}_q$.

In the classical case the orbit method provides a geometric realization of the representations in sections of a linear bundle with connection whose curvature coincides with the symplectic form on the corresponding coadjoint orbit. It is a well-known fact that for any symplectic form, such linear bundles with connection are parameterized by the local systems on the manifold. The construction of representations of $U_{qsu}(1, 1)$ which we give below can be considered as its quantum analog. We will see that it is based on the quantum moment map and a choice of character of the commutative subalgebra $\text{Func}(X_{c_0, d_0})^{\text{inv}}_q$ – the one generated by the functions $f(x)$ – which plays the role of a local system on the corresponding symplectic leaf.

Consider a $\ast$-homomorphism $\nu : \text{Func}(X_{c_0, d_0})^{\text{inv}}_q \to \mathbb{C}$ of the form

$$\nu : f(x) \mapsto f(\nu_0), \text{ where } \nu_0 \in \mathbb{R} \setminus \{0\}.$$ 

It defines a one-dimensional $\text{Func}(X_{c_0, d_0})^{\text{inv}}_q$-module $\mathbb{C}_\nu$. Consider the induced $\text{Func}(X_{c_0, d_0})_q$-module

$$\Pi_\nu = \text{Ind}_{\text{Func}}^F \mathbb{C}_\nu,$$

where we use a shortened notation $F = \text{Func}(X_{c_0, d_0})_q$. 

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Proposition 1.12 The $\text{Func}(X_{c_0,d_0})_q$-module $\Pi_\nu$ is isomorphic to $\text{Pol}(X_{c_0,d_0})_q^\pm$ as a $\text{Pol}(X_{c_0,d_0})_q^\pm$-module (with respect to the left multiplications). Moreover, the action of $\text{Func}(X_{c_0,d_0})_q$ in $\Pi_\nu$ is given by

\begin{align}
\zeta : f(\zeta) \bar{I}_\nu & \rightarrow \zeta f(\zeta) \bar{I}_\nu, \quad (2.45) \\
x : f(\zeta) \bar{I}_\nu & \rightarrow \nu_0 f(\zeta q^{-2}) \bar{I}_\nu, \quad (2.46) \\
y : f(\zeta) \bar{I}_\nu & \rightarrow \zeta^2 \nu_0 f(\zeta q^{-2}) - c_0 f(\zeta) \bar{I}_\nu, \quad (2.47) \\
\hat{y} : f(\zeta) \bar{I}_\nu & \rightarrow -\frac{q \nu_0 f(\zeta q^{-2}) - d_0 f(\zeta)}{\zeta} \bar{I}_\nu, \quad (2.48)
\end{align}

where $\Pi_\nu$ is realized as the span of monomials of the form $\zeta^k \bar{I}_\nu$, where $\bar{I}_\nu$ generates $C_\nu$. Similar formulas hold if we realize $\Pi_\nu$ as the span of monomials of the form $(\zeta^*)^k \bar{I}_\nu$ or $\hat{\zeta}^k \bar{I}_\nu$.

Proof. The first part of the proposition follows immediately from the construction. The explicit formulas for the action of $\text{Func}(X_{c_0,d_0})_q$ can be easily obtained by a straightforward computation.

Note that by (2.2), the set of eigen-values for the action of $x$ in $\Pi_\nu$ is a part of the geometric progression

$$m_0 = \{\nu_0 q^{2k} \}_{k \in \mathbb{Z}}.$$

Definition 1.7. Suppose that $F$ is a $\ast$-algebra.

(1) We call a $F$-module $\Pi$ unitarizable if there exists a positive definite Hermitian scalar product $(\ , \ )$ in $\Pi$ such that

$$(av_1, v_2) = (v_1, a^* v_2)$$

for any $a \in F$ and $v_1, v_2 \in \Pi$.

(2) Suppose that $\Pi$ is a unitarizable $F$-module. Consider the Hilbert space $H$ which is the completion of $\Pi$. We say that the action of $F$ in $\Pi$ defines a $\ast$-representation $\pi$ of $F$ in $H$ if the action of any element $a \in F$ in $\Pi$ can be extended to a closed
Theorem 1.3. (1) Let $0 < c_0 \leq d_0$ (the case of the quantum two-sheet hyperboloids and the quantum cone). Suppose that neither $x_2 = q c_0$ nor $x_0 = q^{-1} d_0$ belongs to $\mathcal{M}_{v_0}$. Then there exists a scalar product $(\ , \ )$ in $\Pi_\nu$ making it into a simple unitarizable $\text{Func}(X_{c_0,d_0})_q$-module if and only if no point of $\mathcal{M}_{v_0}$ lies in the interval $(q c_0, q d_0)$. The corresponding scalar product in $\Pi_\nu$ is given by

$$
(f(\zeta)I_\nu, g(\zeta)I_\nu) = \nu(g(\zeta) f(\zeta)),
$$

where $\nu$ is extended to $\text{Func}(X_{c_0,d_0})_q$ by $\nu(\zeta) = \nu(\zeta^*) = 0$. Moreover, the action of $\text{Func}(X_{c_0,d_0})_q$ in $\Pi_\nu$ defines an irreducible $*$-representation of $\text{Func}(X_{c_0,d_0})_q$. The spectrum of the linear operator representing the action of $x$ in the corresponding Hilbert space is equal to $\mathcal{M}_{v_0} \cup \{0\}$.

(2) Let $c_0 = d_0 \notin \mathbb{R}$ (the case of the quantum one-sheet hyperboloids). Then there exists a scalar product $(\ , \ )$ in $\Pi_\nu$ which makes it into a unitarizable $\text{Func}(X_{c_0,d_0})_q$-module. It is given by (2.49). Moreover, the action of $\text{Func}(X_{c_0,d_0})_q$ in $\Pi_\nu$ defines an irreducible $*$-representation of $\text{Func}(X_{c_0,d_0})_q$. The spectrum of the linear operator representing the action of $x$ in the corresponding Hilbert space is equal to $\mathcal{M}_{v_0} \cup \{0\}$.

(3) Let $0 < c_0 \leq d_0$. Suppose that $q c_0 \in \mathcal{M}_{v_0}$ (resp. $q^{-1} d_0 \in \mathcal{M}_{v_0}$). Then there exists a scalar product $(\ , \ )$ in $\Pi_\nu$ which makes it into a unitarizable $\text{Func}(X_{c_0,d_0})_q$-module. It is given by (2.49). Moreover, the action of $\text{Func}(X_{c_0,d_0})_q$ in $\Pi_\nu$ defines an irreducible $*$-representation of $\text{Func}(X_{c_0,d_0})_q$. The spectrum of the linear operator representing the action of $x$ in the corresponding Hilbert space is equal to

$$
\mathcal{M}_+ = \{c_0 q^{2k+1}\}_{k=0}^\infty \cup \{0\} \ (\text{resp. } \mathcal{M}_- = \{d_0 q^{-2k-1}\}_{k=0}^\infty).
$$

Proof. It is easy to see that any monomial of the form $\zeta^k I_\nu \in \Pi_\nu$ is an eigen-vector for the action of $x$ with the eigen-value $\nu_0 q^{-2k}$. At the same time (2.45)-(2.48) show
that the set of eigen-values of $x$, being a part of the geometric progression $m_{r_0}$ could possibly be truncated only if either $qc_0$ or $q^{-1}d_0$ belong to $m_{r_0}$. This follows also from (2.36). Therefore, we need to show only the unitarizability of $\Pi_\nu$. We need to remind some definitions.

**Definition 1.8.** Suppose that $A$ is a Hopf $*$-algebra, $F$ an $A$-module $*$-algebra. A linear functional $f \mapsto \int f d\mu$ defined on a linear subset $F_0$ of $F$ is called an **invariant integral** on $F$ if the following properties are satisfied:

$$\int a f d\mu = \varepsilon(a) \int f d\mu, \quad \int f^* d\mu = \overline{\int f d\mu},$$

$$f \mapsto \int f^* f d\mu$$

is a positive definite form on $F_0$,

for any $a \in A$ and $f \in F_0$, where $\varepsilon$ is the counit in $A$.

The following lemma is well known.

**Lemma 1.1.** Suppose that $A$ is a Hopf $*$-algebra, $F$ an $A$-module $*$-algebra with an invariant integral $\int d\mu : F_0 \to \mathbb{C}$. Consider the space $L_2(F,d\mu)$ consisting of all $f \in F$ such that $\int f^* f d\mu < \infty$. Then $L_2(F,d\mu)$ is a Hilbert space with the scalar product given by

$$(f, g) = \int g^* f d\mu, \quad (2.50)$$

and the action of $F$ in $L^2(F,d\mu)$ by left multiplication defines a $*$-representation of $F$.

**Proposition 1.13.** Under the assumptions of Theorem 1.3 (1)-(2), the linear functional

$$\int \zeta^k f(x) d\mu = \delta_{k,0} \left( q^{-1} - q \right) \sum_{x \in m_{r_0}} x f(x) \quad (2.51)$$

is an invariant integral on $\text{Func}(X_{c_0,d_0})_q$. Similarly, under the assumptions of Theo-
rem 1.3 (3), the linear functional

\[
\int \zeta^k f(x) d\mu = \delta_{k,0} (q^{-1} - q) \sum_{x \in \mathfrak{m}_\pm} x f(x)
\]  

(2.52)

is an invariant integral on \(\text{Func}(X_{c_0,d_0})_q\).

Proof. Follows from the fact that \(J(q^\rho) = J(K) = x\), where \(\rho \in U_q\mathfrak{g}\) is the half of the sum of all positive roots. But it can also be checked by a straightforward computation.

Proof of Theorem 1.3 (continued). Now we can finish the proof of Theorem 1.3. Realize \(\Pi_\nu\) as a subspace in \(\text{Func}(X_{c_0,d_0})_q\) by mapping \(\tilde{1}_\nu\) into the function \(\delta_{\nu_0}\) which takes the value 1 at \(\nu_0 \in \mathfrak{m}_{\nu_0}\) and 0 at any other point of \(\mathfrak{m}_{\nu_0}\). It is easy to see that this map intertwines the action of \(\text{Func}(X_{c_0,d_0})_q\) in \(\Pi_\nu\) and itself by left multiplications. By Lemma 1.1, (2.50) defines a \(*\)-representation of \(\text{Func}(X_{c_0,d_0})_q\) in \(\Pi_\nu\) when \(f \, d\mu\) is of the form (2.51)-(2.52). Obviously, this representation is irreducible, since the module \(\Pi_\nu\) is simple. On the other hand, (2.40) forbids any point from the spectrum of \(x\) to lie in the interval \((q_{c_0}, q_d)\).

The full list of irreducible \(*\)-representations of \(U_q\mathfrak{su}(1, 1)\) was described in a number of papers (cf. [21, 13, 14]). They are parameterized by the so-called spin parameter \(l \in \mathbb{C}\) and the parity \(|\epsilon| \leq \frac{1}{2}\). The corresponding representations \(T_{l,\epsilon}\) are subjects to the symmetries

\[
T_{l,\epsilon} \simeq T_{-l-1,\epsilon}, \quad T_{l,\epsilon} \simeq T_{l + \frac{2\pi}{\log q}, \epsilon}.
\]

They are divided into the following series:

1. principal continuous series: \(l = -\frac{1}{2} + i\rho\),

2. complimentary series: \(l \in (-\frac{1}{2}, 0), \ |\epsilon| < |l|\),

3. holomorphic discrete series: \(l - \epsilon \in \mathbb{Z}\),

4. anti-holomorphic discrete series: \(l + \epsilon \in \mathbb{Z}\),

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5. strange series: $\text{Im} \, l = \frac{\pi}{\log q}$

They are separated by the quantum quadratic Casimir element (2.34) which acts as 

$$\frac{(q^l-q^{-l})(q^{l+1}-q^{-l-1})}{(q^{-1}-q)^2}$$

and by the fact that any eigen-value of $K$ in $T_{l,e}$ is of the form $q^{k+\epsilon}$.

Note that the strange series does not survive in the classical limit. We will soon see how it manifests itself in the language of the symplectic leaves in $SU(1,1)^*$.

**Theorem 1.4.** For any irreducible $*$-representation $\pi$ of $\text{Func}(X_{c_0,d_0}), \pi \circ J$ is an irreducible $*$-representation of $U_qsu(1,1)$.

1. The irreducible $*$-representations described in Theorem 1.3 (1) give rise to the complimentary series representations if $\nu_0 > 0$ and the strange series representations if $\nu_0 < 0$.

2. The irreducible $*$-representations described in Theorem 1.3 (2) give rise to the principal continuous series representations.

3. The irreducible $*$-representations described in Theorem 1.3 (3) give rise to the holomorphic discrete series representations when they correspond to $\mathfrak{m}_+$ and the anti-holomorphic discrete series representations when they correspond to $\mathfrak{m}_-$.

**Proof.** The theorem follows immediately from (2.35) if we assume $c_0 = q^{l+\frac{1}{2}}$ and $d_0 = q^{-l-\frac{1}{2}}$ and from (2.33), since any eigen-value of $K$ will be an eigen-value of $x$.

Note that the principal continuous series representations correspond to the symplectic leaves that are the halves ($x > 0$ and $x < 0$) of the one-sheet hyperboloids $X_{c_0,d_0}$ as described in Example 1.6. The holomorphic discrete series representations and the strange series representations correspond to the halves ($x < 0$ and $0 < x \leq c_0$) of a sheet of the corresponding two-sheet hyperboloids $|y|^2 = (x-c_0)(x-d_0)$, while the anti-holomorphic series representations correspond to the other sheet of the two-sheet hyperboloid.

What is especially interesting is that the complimentary series representation appear in the case when a geometric progression $\mathfrak{m}_{\nu_0}$ can jump over the narrow interval $(qc_0, qc_0)$. It looks as if in the quantum case the invariant measure can be extended.
from one sheet of a quantum two-sheet hyperboloid onto another one, thus making it 'connected'. We will call such quantum hyperboloids quantum tunnel hyperboloids. Note that there is no classical construction that would realize the complimentary series representations in the spirit of the orbit method.

As we consider the quasi-classical limit, we see that the choice of a geometric progression reflects the value of the parity $\epsilon$ of the corresponding irreducible $\ast$-representation of $U_q\mathfrak{su}(1, 1)$. Therefore, we can think of the choice of $C_\nu$ as the choice of a local system on the corresponding symplectic leaf in $SU(1,1)^\ast$. However, with the 'tunnel effect' in mind, we see that certain choices of $C_\nu$ may not have any local system as their quasi-classical analogs—those that give rise to the complimentary series representations.

On the other hand, the observed correspondence between the symplectic leaves in $SU(1,1)^\ast$ and the representations of $U_q\mathfrak{su}(1, 1)$ depends on $q$, as $\frac{s_\alpha}{d_\alpha} = q^{2l+1}$. As we keep the spin $l$ of the representation fixed and take the limit $q \to 1$, the corresponding symplectic leaves face two options. Those which give rise to the strange series representations will go to the infinity (which reflects the fact that there are no strange series representations in the classical case). The rest will converge to the nilpotent cone $|y|^2 = (x - 1)^2$. If we consider them as points in the corresponding orbifold, we can look at the rate with which the corresponding curve in the orbifold converge to the cone. It will be an orbit of the coadjoint action in $\mathfrak{su}(1,1)^\ast$. This gives us the usual correspondence between the representations and coadjoint orbits described by the classical orbit method. Except that the 'tunnel effect' will disappear in this limit.

2.6 Quantum Orbit Lattices and Irreducible $\ast$-Representations

Generalizing the ideas of the previous section, we get the following construction of irreducible $\ast$-representations of $U_q\mathfrak{g}_0$. Suppose that we have a $U_q\mathfrak{g}_0$-module $\ast$-algebra $\mathcal{F}$ together with a surjective quantum moment map $J : U_q\mathfrak{g}_0 \to \mathcal{F}$ and a polarization
\[ P \subset F. \] Suppose also that the subalgebra \( F^{\text{inv}} \) of \( U_q \mathfrak{g} \)-invariant elements in \( F \) is commutative.

Consider a non-trivial one-dimensional \(*\)-representation \( \chi : F^{\text{inv}} \to C \) of \( F^{\text{inv}} \). Denote the corresponding one-dimensional \( F^{\text{inv}} \)-module by \( C_\chi \). Consider the representation of \( F \) in the induced module

\[ W_\chi = \text{Ind}_{\tilde{F}^{\text{inv}}}^F C_\chi. \]

**Proposition 1.14.** The \( F \)-module \( W_\chi \) is simple. It is isomorphic to \( P \) as a \( P \)-module with respect to the left multiplications.

**Proof.** Indeed, by definition of complex polarization (cf. Definition 1.6), \( F^{\text{inv}} \) and \( P \) generate \( F \). On the other hand, \( F^{\text{inv}} \) and \( P \) don't have common elements besides constants. Therefore, the map \( W_\chi \to P \) given by \( a.\tilde{1}_\chi \mapsto a \) for any \( a \in P \) is an isomorphism of \( P \)-modules, where \( \tilde{1}_\chi \) is a generator of \( C_\chi \). It immediately follows that \( W_\chi \) is a simple \( F \)-module.

Suppose that there is an extension \( \tilde{F} \) of \( F \) by the functions on the spectrum of the commutative subalgebra \( F^{\text{inv}} \). Suppose also that there exists an invariant integral \( \int d\mu \) on a subspace \( \tilde{F} \) in \( \tilde{F} \) such that \( \chi \) belongs to the spectrum of its restriction to the subspace \( \tilde{F}^{\text{inv}} \) of \( U_q \mathfrak{g} \)-invariant elements. Extend \( \chi \) to a linear map \( \tilde{\chi} : F \to C \) by \( \tilde{\chi}(af) = 0 \) if \( a \in P, f \in F^{\text{inv}} \) and \( a \) is not a constant. Suppose also that there is an extension \( \tilde{F} \) of \( F \) by the functions on the spectrum of the commutative subalgebra \( F^{\text{inv}} \). Then the following theorem holds.

**Theorem 1.5.** (1) Under the above assumptions, the induced \( F \)-module \( W_\chi \) is unitarizable and gives rise to an irreducible \(*\)-representation \( \pi_\chi \) of \( F \), with the scalar product on \( W_{\chi i} \) given by

\[ (a.\tilde{1}_\chi, b.\tilde{1}_\chi) = \tilde{\chi}(b^*a). \]

(2) Under the above assumptions, the composition \( J^\Phi \circ \pi_\chi \) is an irreducible \(*\)-representation of \( U_q \mathfrak{g}_0 \), where \( J^\Phi = J \circ \Phi \) is a quantum moment map obtained by

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twisting $J$ by an automorphism $\Phi$ of the $U_q\mathfrak{g}_0$-module $*$-algebra $U_q\mathfrak{g}_0$.

(3) The induced $\mathcal{F}$-module $W_\chi$ is not unitarizable if there is no invariant integral on $\text{Func}(X)_q$ such that $\hat{\chi}$ belongs to the spectrum of its restriction to $\text{Func}(X)^{\text{inv}}_q$.

Proof. The proof of the first part follows the ideas of the proof of Theorem 1.3. Consider the function $\delta_{\chi}$ on the spectrum of the restriction $\mu_0$ of the invariant integral $\int d\mu$ which takes the value 1 at $\chi$ and zero at any other point (it is easy to show that the spectrum of $\mu_0$ is always the completion of a discrete set, and that the only limit point it can have is zero). Then we can map $W_\chi$ isomorphically into a $\mathcal{P}$-submodule in $\hat{\mathcal{F}}$ so that $\delta_{\chi}$ is the image of $\hat{1}_\chi$. It is easy to see that this map intertwines the actions of the whole $\mathcal{F}$.

The invariant integral $\int d\mu$ defines a scalar product (2.50) on $\hat{\mathcal{F}}$ which makes it into a unitarizable $\mathcal{F}$-module. Since $\chi$ belong to the spectrum of $\mu_0$, the restriction of that scalar product to the isomorphic image of $W_\chi$ is non-degenerate. This proves the statement (1) of the theorem.

To prove the third part, assume the contrary. Map $W_\chi$ isomorphically into a $\mathcal{P}$-submodule in $\hat{\mathcal{F}}$ the same way as above. Define a linear functional on $\hat{\mathcal{F}}^{\text{inv}} \cap W_\chi$ by

$$f \mapsto (f.\delta_{\chi}, \delta_{\chi})$$

Since $\hat{\mathcal{F}}$ is completely reducible as a $U_q\mathfrak{g}_0$-module, we can extend that linear functional to the whole $\hat{\mathcal{F}}$. It is easy to see that it will give us an invariant integral. By construction, $\hat{\chi}$ belongs to its spectrum.

The second part is an immediate consequence of the first part of the theorem, since the quantum moment map $J$ is a surjective $*$-homomorphism.

Example 1.9. For any quantum polarization $\text{Pol}(X)^{\pm}_q$ of the $U_q\mathfrak{su}(\tilde{\mathfrak{t}})$-module $*$-algebra $\text{Func}(X)_q$ considered in the proceeding sections, all the conditions of Theorem 1.5 are satisfied. Thus, we can construct irreducible $*$-representations of $U_q\mathfrak{su}(\tilde{\mathfrak{t}})$ starting from one-dimensional $*$-representations of $\text{Func}(X)^{\text{inv}}_q$.

There is a map from the spectrum of $\text{Func}(X)^{\text{inv}}_q$ to $\mathfrak{h}^*$ by taking the restriction
of $J \circ \chi$ to $U_q \mathfrak{h}$ for any. Denote the image of $\chi$ by $\hat{\chi}$. Apparently, $\hat{\chi}$ belongs to $\mathfrak{t}^*$, the dual space to the Lie algebra of the maximal torus $T$ in $G$. We know that $\chi$ must belong to the spectrum of the restriction of an invariant integral $\int d\mu$ to $\text{Func}(X)^{inv}_q$. We see from (2.17)- (2.19) that the action of $U_q \mathfrak{g}$ induces shifts on the spectrum of $\text{Func}(X)^{inv}_q$ which correspond to shifts by the root vectors on $\mathfrak{h}^*$. Therefore, the image in $\mathfrak{h}^*$ of the spectrum of $\int d\mu$ is a subset of the shifted root lattice $\hat{\chi} + R$, where $R$ is the root lattice in $\mathfrak{h}^*$. It is easy to see that any character of $\text{Func}(X)^{inv}_q$ which corresponds to a point from the same subset gives rise to a unitary equivalent irreducible *-representation of $U_q \mathfrak{su}(\mathfrak{t})$. Therefore, we only need to investigate the conditions on a subset of $\mathfrak{h}^*$ to be the image of the spectrum of an invariant integral.

For any $p \in G^*$, let $O_p$ be the set of points of the form $S_w(p^\theta)$, where $p \mapsto p^\theta$ represents the left dressing action of $g \in G$ (whenever $g$ can act on $p$ – it is not always defined globally), and $S_w$ is the action of the Weyl group $W$ on $G^*$. Denote by $P_p$ the projection of $O_p$ onto $H^* \subset G^*$. Similarly, denote by $P_p$ the projection onto $\mathfrak{h}^* \subset \mathfrak{g}^*$ of the union of all $W$-translations of the coadjoint orbit passing through $p \in \mathfrak{g}^*$.

**Definition 1.9.** Let $R$ be the root lattice in $\mathfrak{h}^*$.

(1) We call a subset $Q \subset \mathfrak{h}^*$ of the form $\hat{\chi} + R$ $R$-connected if for any two points $a, b \in Q$, there exists a sequence $a_0 = a, a_1, ..., a_n = b$ such that $a_{k+1} - a_k$ is a root for any $k - 1, ..., n$.

(2) For any $p \in G^*$, an orbit lattice is a subset $Q$ of $\hat{\chi} + R$, where $\hat{\chi} \in \mathfrak{t}^*_+$, which satisfies the following condition: $Q \subset P_p$ and whenever $a \in Q$, $a \pm \alpha \in Q$ for any simple root $\alpha$, unless $a$ belongs to a part of the boundary of $P_p$ formed by a hyperplane orthogonal to $\alpha$.

(3) For any $p \in G^*$, a quantum orbit lattice is a subset $Q$ of $q^{\hat{\chi}+R}$, where $\hat{\chi} \in \mathfrak{t}^*_+$, which satisfies the following condition: $Q \subset P_p$ and whenever $q^a \in Q$, $q^{a \pm \alpha} \in Q$ for any simple root $\alpha$, unless $q^a$ belongs to a part of the boundary of $P_p$ formed by a hyperplane orthogonal to $\alpha$.

Consider the quotient of $U_q \mathfrak{g}_0$ over the ideal generated by the elements of the form
$z - \hat{\chi}(z)$, for any central element $z$. By Drinfeld's duality, we can consider $U_q \mathfrak{g}_0$ as the quantum algebra of functions on the dual Poisson Lie group $G_0^*$. As a special case of a more general construction described in [1], fix the new generators $E_i' = hE_i$, $F_i = hF_i$, and $H_i' = hH_i$, where $q = e^{-h}$ and $K_i = q^{-H_i}$. Then by taking the quasi-classical limit $h \to 0$, we get a subset in $G^*$. It is easy to show that it is of the form $\mathcal{P}_p$. Denote it by $Z_\chi$.

**Theorem 1.6**

1. A one-dimensional $*$-representation $\chi$ of $\text{Func}(X)_q$ gives rise to an irreducible $*$-representation of $U_q \mathfrak{g}_0$ (via the construction described in Theorem 1.5) if and only if $Q = q^{\delta + R} \cap Z_\chi$ is a quantum orbit lattice for $Z_\chi$.

2. There is a one-to-one correspondence between the unitary equivalence classes of irreducible $*$-representations of $U_q \mathfrak{su}(1, 1)$ and the pairs $(\mathcal{P}_p, Q)$, where $p \in SU(1, 1)^*$ and $Q$ is a quantum orbit lattice in $\mathcal{P}_p$.

3. There is a one-to-one correspondence between the unitary equivalence classes of irreducible $*$-representations of $U_\mathfrak{su}(1, 1)$ and the pairs $(\mathcal{P}_p, Q)$, where $p \in \mathfrak{su}(1, 1)^*$ and $Q$ is an orbit lattice in $\mathcal{P}_p$.

**Proof.** We know that the support $Q$ of the restriction to $\text{Func}(X)_q\text{inv}$ of any invariant integral on $\text{Func}(X)_q$ is a subset of $q^{\beta + R}$ for some $\beta$. If $q^\alpha \in Q$ and $q^{\alpha + \alpha_i} \not\in Q$, where $\alpha_i$ is a simple root, then $E_i \delta_\alpha = 0$, where $\delta_\alpha$ is the function on $Q$ taking value 1 at $q^\alpha$ and zero otherwise. But this precisely means that $q^\alpha$ lies on the part of the boundary of $Z_\chi$ formed by a hyperplane orthogonal to $\alpha_i$. Similarly, if $q^\alpha \in Q$ and $q^{\alpha - \alpha_i} \not\in Q$, then $F_i \delta_\alpha = 0$ and we get the same.

Thus we see that there exists an invariant integral on $\text{Func}(X)_q$ such that $\hat{\chi}$ belongs to the spectrum of its restriction to $\text{Func}(X)_q\text{inv}$ if and only if $Q$ is a quantum root lattice for $Z_\chi$. By Theorem 1.5, this is the sufficient and necessary condition for $W_\chi$ to give rise to an irreducible $*$-representation of $U_q \mathfrak{g}_0$.

The parts (2) and (3) are immediate consequences of the first part and the fact that the irreducible $*$-representations of $U_q \mathfrak{su}(1, 1)$ listed in Theorem 1.4 form a full list together with the trivial representation. But the latter arises from the trivial quantum orbit lattice (the unit element in $G^*$).
Example 1.10. For $\mathfrak{g}_0 = \mathfrak{su}(1,1)$ the orbit lattices are the sets of one of the following forms, where $\alpha$ is the simple root:

- $a + k\alpha$, where $a > 0$ and $k$ is a non-negative integer, for the coadjoint orbit passing through $a$,
- $a - k\alpha$, where $a < 0$ and $k$ is a non-positive integer, for the coadjoint orbit passing through $a$,
- $a + k\alpha$, where $-\frac{a}{2} < a < \frac{a}{2}$ and $k$ is any integer, for the coadjoint orbits passing through any of the points between $-a$ and $a$,
- $a + k\alpha$, for any integer $k$, for any coadjoint orbit which is a one-sheet hyperboloid,
- $0$.

According to Theorem 1.4, the orbit lattices of the first kind give rise to the anti-holomorphic discrete series representations, those of the second kind to the holomorphic discrete series representations, those of the third kind correspond to the tunnel effect giving rise to the complementary series representations, while those of the forth kind give rise to the principal continuous series representations. Of course, the trivial coadjoint orbit corresponds to the trivial representation.

In the quantum case, there are more sets of the form $\mathcal{P}_p \subset G^*$ than there are sets of the form $P_p \in \mathfrak{g}_0^*$. And, respectively, those quantum orbit lattices which come with the others correspond to the strange series representations.
Chapter 3

Super-Tensor Products

3.1 Algebras $\mathcal{R}_\pm$, $\mathfrak{g}$ and their $\ast$-Representations.

The Hopf $\ast$-algebra $\mathcal{R}_+ = C[SU(1,1)]_q$ is the pair $(\mathcal{R}, \ast)$, where $\mathcal{R} = C[SL_2(\mathbb{C})]_q$ (cf. [1]) is the Hopf algebra generated by all the matrix elements of finite-dimensional representations of the quantum enveloping algebra $U_{qs}^{SL_2(\mathbb{C})}$ (cf. [1, 5]), $\ast$ is the antilinear involutive algebra antiautomorphism and coalgebra automorphism of $\mathcal{R}$ given below.

Recall that $\mathcal{R}$ is generated by $t_{ij}$ ($i, j = 1, 2$) and relations

\[
\begin{align*}
t_{11}t_{12} &= qt_{12}t_{11}, & t_{11}t_{21} &= qt_{21}t_{11} \\
t_{12}t_{22} &= qt_{22}t_{12}, & t_{21}t_{22} &= qt_{22}t_{21} \\
t_{12}t_{21} &= t_{21}t_{12}, & t_{11}t_{22} - t_{22}t_{11} &= (q - q^{-1})t_{12}t_{21} \\
t_{11}t_{22} - qt_{12}t_{21} &= 1
\end{align*}
\]

(3.1)

\[
\Delta(t_{ij}) = \sum_{k=1,2} t_{ik} \otimes t_{kj}, \quad \varepsilon(t_{ij}) = \delta_{ij}
\]

\[S:\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \mapsto \begin{pmatrix} t_{22} & -q^{-1}t_{12} \\ -qt_{21} & t_{11} \end{pmatrix}.
\]
where $\Delta$ is the comultiplication, $S$ the antipode, and $\varepsilon$ the counit.

The antilinear involutive algebra anti-automorphism $\ast$ given by

$$
s_{11}^\ast = s_{22}, \quad s_{12}^\ast = qs_{21}
$$

(3.2)

equips $\mathfrak{g}_+$ with a Hopf $\ast$-algebra structure, i.e. the following conditions are satisfied (what is referred to as $\ast$'s being coalgebra automorphism):

$$(ab)^\ast = b^\ast a^\ast, \quad \Delta(a^\ast) = \Delta(a)^\ast$$

$$\bar{\omega}^2(a) = a, \quad \varepsilon(a^\ast) = \bar{\varepsilon}(a)$$

for each $a, b \in \mathfrak{g}$ where

$$\bar{\omega}(a) = (S(a))^\ast$$

(cf. [4, 13, 14, 21]).

The quasi-classical analog of $\mathfrak{g}_+$ is the algebra of regular functions on the real Poisson Lie group $SU(1, 1)$ considered as the pair $(SL_2(\mathbb{C}), \omega)$ where $\omega$ is an involutive antiholomorphic Poisson Lie group automorphism of $SL_2(\mathbb{C})$ such that $SU(1, 1) = \{g \in SL_2(\mathbb{C}) \mid \omega(g) = g\}$.

Consider the $\ast$-algebra $\mathfrak{g}_- = (\mathfrak{g}, \#)$ where $\#$ is the antilinear involutive algebra anti-automorphism of $\mathfrak{g}$ given by

$$t_{11}^\# = -t_{11}^\ast = -t_{22}, \quad t_{12}^\# = -t_{12}^\ast = -qs_{21}
$$

(3.3)

Note that $\#$ is not a coalgebra automorphism, so that $\mathfrak{g}_-$ is not a Hopf $\ast$-algebra.

The quasi-classical analog of $\mathfrak{g}_-$ is the algebra of regular functions on the left Poisson coset $SU(1, 1) \cdot w \in SL_2(\mathbb{C})$ where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

We will see soon that $\ast$-representations of $\mathfrak{g}_\pm$ are usually given by \textit{unbounded} operators. That is why one should carefully introduce the notion of $\ast$-representation of $\mathfrak{g}_\pm$. 

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Definition 2.1. Suppose that $\mathfrak{A}$ is a $\ast$-algebra, $H$ a Hilbert space. A left (right) $\mathfrak{A}$-module $V$ dense in $H$ is called unitarizable left (right) $\mathfrak{A}$-module if
\[ (a.v_1, v_2) = (v_1, a^*v_2) \] for each $a \in \mathfrak{A}, v_1, v_2 \in V$.

Throughout the chapter we use the notation $x = t_{12}t_{21}$. This element generates the subalgebra of spherical functions on the quantum group (see [13, 14, 28]).

Definition 2.2. A unitarizable left (resp. right) $\mathfrak{A}_\pm$-module $V$ is called self-adjoint if the operator defined on $V$ by the action of $a$ admits closure (denoted $\sigma_V(a)$) for any $a \in \mathfrak{A}_\pm$, and the operator $\sigma_V(x)$ is self-adjoint.

Definition 2.3. (1) Two self-adjoint unitarizable left (right) $\mathfrak{A}_\pm$-modules $V$ and $V'$ are said to be closure equivalent if $\sigma_V(a) = \sigma_{V'}(a)$ for any $a \in \mathfrak{A}_\pm$.

(2) A $\ast$-representation of $\mathfrak{A}_\pm$ is a closure equivalence class of self-adjoint unitarizable left $\mathfrak{A}_\pm$-modules (right ones give rise to $\ast$-antirepresentations).

(3) We say that a $\ast$-representation $\sigma$ is irreducible if any representative of $\sigma$ in the corresponding closure equivalence class is irreducible, and that $\sigma$ is unitarily equivalent to $\sigma'$ if there exist such representatives $V$ and $V'$ of $\sigma$ and $\sigma'$ respectively that $V$ is unitarily equivalent to $V'$.

The category of unitarizable left $\mathfrak{A}_\pm$-modules admits tensor product which equips it with the monoidal category structure. Given two unitarizable left $\mathfrak{A}_{i_1}$-modules $V_1$ and $V_2$, we define their tensor product as the unitarizable left $\mathfrak{A}_{i_1i_2}$-module $V_1 \otimes V_2$ such that
\[ a : v_1 \otimes v_2 \mapsto \Delta(a).(v_1 \otimes v_2) \]
for any $a \in \mathfrak{A}_{i_1i_2}$ and $v_i \in V_i$.

However, the category of unitarizable left $\mathfrak{A}_\pm$-modules does not prove to be helpful.

\[^1\text{The dot is the notation for the action on the module.}\]
in construct a quantum topological group because of an immense set of unitary equivalence classes of irreducible unitarizable left $\mathfrak{H}_\pm$-modules. Indeed, as we will see later, the tensor product of two simple unitarizable left $\mathfrak{H}_\pm$-modules usually is irreducible, which contradicts the geometrical intuition which tells us that the product of the two symplectic leaves corresponding to the tensor factors is not a single symplectic leaf anymore.

However, *-representations in their turn proved to be somewhat inconvenient. As is shown in Section 4, given two *-representations $\pi_1, \pi_2$, the unitarizable module $V_{\pi_1} \otimes V_{\pi_2}$ is not self-adjoint in general. Moreover, there does not exist a self-adjoint unitarizable left module $V$ which extends this tensor product within the same Hilbert space, i.e. such that $V_{\pi_1} \otimes V_{\pi_2} \subset V \subset \mathcal{H} = \overline{V_{\pi_1} \otimes V_{\pi_2}}$.

Essentially, that means that there is no way to define a natural tensor product in the category of *-representations of $\mathfrak{H}_+$. But, according to the observation, outlined in the introduction, there is another structure – that is so-called super-tensor products.

**Proposition 2.1.** (1) Each irreducible *-representation of $\mathfrak{H}_+$ is unitarily equivalent to one of the following:

- **one-dimensional *-representations** $\zeta_\varphi (\varphi \in [0, 2\pi])$, given by

$$\zeta_\varphi : \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \mapsto \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}$$

- **infinite-dimensional *-representations** $\pi_{\varphi, (\beta)} (\varphi \in [0, 2\pi], \beta \in (-\frac{1}{2}, \frac{1}{2}))$

  given by

$$\begin{align*}
\pi_{\varphi, (\beta)}(t_{11}) : e_k^+ & \mapsto (1 + q^{2k+2\beta-1})^{\frac{1}{2}}e_{k-1}^+ \\
\pi_{\varphi, (\beta)}(t_{12}) : e_k^+ & \mapsto e^{i\varphi}q^{k+\beta+\frac{1}{2}}e_k^+ \\
\pi_{\varphi, (\beta)}(t_{21}) : e_k^+ & \mapsto e^{-i\varphi}q^{k+\beta-\frac{1}{2}}e_k^+ \\
\pi_{\varphi, (\beta)}(t_{22}) : e_k^+ & \mapsto (1 + q^{2k+2\beta+1})^{\frac{1}{2}}e_{k+1}^+
\end{align*}$$

(3.5)
where \( \{ e_k^+ \}_{k \in \mathbb{Z}} \) is a certain orthonormal basis of the space of representation.

(2) Each irreducible \(*\)-representation of \( \mathcal{R}_- \) is unitarily equivalent to one of the infinite-dimensional \(*\)-representations \( \pi_\varphi^- \) \((\varphi \in [0, 2\pi)) \) given by

\[
\begin{align*}
\pi_\varphi^-(t_{11}) : e_k^- & \mapsto i(q^{-2(k+1)} - 1)^{1/2} e_{k+1}^- \\
\pi_\varphi^-(t_{12}) : e_k^- & \mapsto e^{i\varphi} q^{-k} e_k^- \\
\pi_\varphi^-(t_{21}) : e_k^- & \mapsto -e^{-i\varphi} q^{-k-1} e_k^- \\
\pi_\varphi^-(t_{22}) : e_k^- & \mapsto i(q^{-2k} - 1)^{1/2} e_{k-1}^- 
\end{align*}
\]  

(3.6)

where \( \{ e_k^- \}_{k \in \mathbb{Z}_+} \) is a certain orthonormal basis of the space of representation.

The proof is quite standard and is based on the idea that, given an irreducible \(*\)-representation \( \pi \), the spectrum of \( \pi(x) \) is the closure of a segment of a geometric progression with multiplier \( q^2 \) (because of \( t_{11} x = q^2 t_{11}, \ t_{22} x = q^{-2} t_{22} \)).

Thus, we see that

\[
\text{spec } \pi_\varphi^+ (x) = \mathcal{M}_+(\beta) \cup \{0\} , \quad \text{spec } \pi_\varphi^- (x) = (\mathcal{M}_-)^{-1} \\
\text{spec } \zeta_\varphi (x) = \{0\}
\]

where \( \mathcal{M}_+(\beta) = \{ q^{2(k+\beta)} \}_{k \in \mathbb{Z}}, \ \mathcal{M}_- = \{-q^{2k+1}\}_{k \in \mathbb{Z}_+} \).

**Remark 2.1.** Denote \( V_\varphi^+ (\beta) \) and \( V_\varphi^- \) the self-adjoint unitarizable left modules generated by \( \{ e_k^+ \}_{k \in \mathbb{Z}} \) and \( \{ e_k^- \}_{k \in \mathbb{Z}_+} \) respectively.

Consider the \(*\)-algebra \( \mathcal{S} = \mathcal{R}_+ \oplus \mathcal{R}_- \). It is easy to check that it can be equipped with a Hopf \(*\)-algebra structure as follows:

\[
\begin{align*}
\Delta(a, 0) &= \sum_k (a'_k, 0) \otimes (a''_k, 0) + \sum_k (0, a'_k) \otimes (0, a''_k) \\
\Delta(0, a) &= \sum_k (a'_k, 0) \otimes (0, a''_k) + \sum_k (0, a'_k) \otimes (a''_k, 0) \\
S(a, b) &= (S(a), S(b)) , \quad \varepsilon(a, b) = \varepsilon(a)
\end{align*}
\]
\((a, b \in \mathfrak{g})\) whenever \(\Delta(a) = \sum_k a'_k \otimes a''_k\).

The Hopf \(*\)-algebra \(\mathfrak{g}\) is a quantum analog of the algebra of regular functions on the Poisson Lie group \(SU(1, 1) \cup SU(1, 1) \cdot w \cong SU(1, 1) \rtimes \mathbb{Z}_2\), the action of \(\mathbb{Z}_2\) on \(SU(1, 1)\) given by conjugation by \(w\), which is nothing but the normalizer of \(SU(1, 1)\) in \(SL_2(\mathbb{C})\).

We call a unitarizable left \(\mathfrak{g}\)-module \(*\)-representation, if its restrictions to both \(\mathfrak{g}_+\) and \(\mathfrak{g}_-\) give rise to \(*\)-representations of \(\mathfrak{g}_\pm\). It is easy to see that each \(*\)-representation of \(\mathfrak{g}\) is the direct sum of its restrictions to \(\mathfrak{g}_+\) and \(\mathfrak{g}_-\), and the set of unitary equivalence classes of irreducible \(*\)-representations of \(\mathfrak{g}\) is just the union of those of \(\mathfrak{g}_+\) and \(\mathfrak{g}_-\).

Note that irreducible \(*\)-representations of \(\mathfrak{g}_+\) and \(\mathfrak{g}_-\) are pairs \((\Sigma, \beta_\Sigma)\) where \(\Sigma\) is a symplectic leaf of \(SU(1, 1)\) or \(SU(1, 1) \cdot w\) respectively, \(\beta_\Sigma\) is a character of the fundamental group \(\pi_1(\Sigma)\) of the leaf. Their quasi-classical analogues are realized in the sections of the corresponding linear bundles.

In particular, \(\zeta_\varphi\) corresponds to the one-dimensional leaf \(\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}\), while \(\pi_\varphi^{t, (\beta)}\) corresponds to the pair \((\Sigma_\varphi^+, \beta \in \mathbb{R}/\mathbb{Z})\) where \(\Sigma_\varphi^+\) is the two-dimensional symplectic leaf of \(SU(1, 1)\) given by \(t_{12} \neq 0, \arg t_{12} = \varphi\) (note that \(\Sigma_\varphi^+\) is equivalent to the outer part \(\mathcal{D}_+ = \{z \in \mathbb{C} \mid |z| > 1\}\) of the unit disc). At last, \(\pi_\varphi^-\) corresponds to the two-dimensional symplectic leaf \(\Sigma_\varphi^-\) of \(SU(1, 1)\) given by \(t_{12} \neq 0, \arg t_{12} = \varphi\) (note that \(\Sigma_\varphi^-\) is equivalent to the inner part \(\mathcal{D}_- = \{z \in \mathbb{C} \mid |z| < 1\}\) of the unit disc).

It is a little bit more convenient to parameterize irreducible \(*\)-representations of \(\mathfrak{g}\) by quadruples \((\varphi, C, \Sigma, \beta_\Sigma)\) where \(\varphi\) corresponds to a point of the maximal torus \(T_0 \subset SU(1, 1)\) of diagonal matrices, \(C\) is a Schubert cell of the flag manifold \(SL_2(\mathbb{C})R/B \cong CP^1\) (where \(B\) is the Borel subgroup of the upper-diagonal matrices), namely, \(\{\infty\}\) or \(C\), \(\Sigma\) is a non-degenerate symplectic leaf of \(C\) (that is, such that \(\dim \Sigma = \dim C\)), namely, \(\{\infty\}\) or \(\mathcal{D}_\pm\) respectively, \(\beta_\Sigma\) is a character of \(\pi_1(\Sigma)\). The non-degenerate symplectic leaves of the flag manifolds are the images of the corresponding symplectic leaves of \(SU(1, 1) \rtimes \mathbb{Z}_2 \subset SL_2(\mathbb{C})\) via the canonical projection.

Remark 2.2. From now on we fix \(\beta\) and consider the subcategory \(\mathcal{C}_\beta\) of such \(*\)-repre-
sentations of $\mathcal{O}$ that the spectrum of $\sigma(x)$ is a subset of $\mathfrak{m}_+^{(\beta)} \cup \mathfrak{m}_-^{-1} \cup \{0\}$ for each $*$-representation $\sigma$ from $C_\beta$. Sometimes $\beta$ will be omitted for convenience in such expressions as $\pi^{+,(\beta)}_\phi$, $V^{+,(\beta)}_\phi$ and the like.

3.2 The Normalizer of a Quantum Real Form.

Let $G_0$ be a real form of a simple complex Lie group $G$ such that there exists a compact Cartan subgroup of $G_0$, $\mathfrak{g}_0$ and $\mathfrak{g}$ the Lie algebras of $G_0$ and $G$ respectively. Recall that $\mathfrak{g}_0 = \{x \in \mathfrak{g} \mid \omega_0(x) = -x^* = x\}$ and $G_0 = \{g \in G \mid \omega(g) = g\}$ where $\omega_0$ is an antilinear involutive automorphism of $\mathfrak{g}$, and $\omega$ is the corresponding antiholomorphic involutive automorphism of $G$.

Choose a compact Cartan subalgebra $t \subset \mathfrak{g}_0$, and let $\mathfrak{h} = t \oplus t$ be the corresponding Cartan subalgebra in $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be the Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$ (which depends, of course, on the choice of positive roots).

Throughout the paper we consider the standard Poisson Lie group structure on $G_0$ given by the Manin triple $(\mathfrak{g}_R, \mathfrak{g}_0, \mathfrak{g}_0^*)$ where $\mathfrak{g}_0^* = \mathfrak{n}_+ \oplus \mathfrak{t}$. Both $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$ are isotopic with respect to the non-degenerate symmetric bilinear scalar product on $\mathfrak{g}_R$ given by imaginary part of the Killing form (the definition of Manin triple can be found in [1]).

The quantization of this Poisson Lie group structure is the quantum algebra $\mathfrak{a}_0 = \mathbb{C}[G_0)_q$ of regular functions. Recall that $\mathfrak{a}_0 = (\mathfrak{a}, \ast)$ where $\mathfrak{a} = \mathbb{C}[G]_q$ is the Hopf algebra generated by matrix elements of finite-dimensional representations of the quantum universal enveloping algebra $U_q \mathfrak{g}$ (cf. [1]), $\ast = S \circ \tilde{\omega}$ where $\tilde{\omega}$ is the quantization of $\omega_0$. Various examples of quantum real forms (and imaginary ones, that is when $|q| = 1$) can be found in [4].

Remark 2.3. Note that if we consider the standard Poisson Lie group structures corresponding to different choices of positive roots of $\mathfrak{h}$, we get, in general, non-isomorphic Poisson Lie groups parameterized by $W/W_1$ where $W$ is the Weyl group.

\footnote{This condition is necessary to consider a quantum real form (that is, when $q \in \mathbb{R} \setminus \{0\}$). For instance, in the case $G_0 = SL_n(\mathbb{R})$ (cf. [4]) we would have a quantum imaginary form instead (that is, when $|q| = 1$).}
of $g$ with respect to $\mathfrak{h}$, $W_1$ is a subgroup of $W$ described below in the section.

For instance, when $G = SL_{m+n}(\mathbb{C})$ and $G_0 = SU(m, n)$, the non-equivalent standard Poisson Lie group structures on $SU(m, n)$ give rise to quantum algebras of regular functions introduced in [4] and denoted there by $\mathbb{C}[SU(\pm 1, \pm 1, ..., \pm 1)]_q$ where the number of pluses and the number of minuses are $m$ and $n$ respectively.

Consider the double Poisson Lie group structure on $G_\mathbb{R}$ considered as a real group. Namely, we see from the above Manin triple that $\mathfrak{g}_\mathbb{R}$ is the double Lie algebra with respect to $\mathfrak{g}_0$. The induced real Poisson Lie group structure on $G_\mathbb{R}$ is given by the Manin triple $(\mathfrak{g}_\mathbb{R} \oplus \mathfrak{g}_\mathbb{R}, \mathfrak{g}_\mathbb{R}, \mathfrak{g}_0 \oplus \mathfrak{g}_0^*)$ where $\mathfrak{g}_\mathbb{R}$ is embedded into $\mathfrak{g}_\mathbb{R} \oplus \mathfrak{g}_\mathbb{R}$ as the diagonal.

Recall that the subgroup of zero-dimensional symplectic leaves of $G_0$ is the maximal compact torus $T_0 = \exp \mathfrak{t}_0$ and that of $G_\mathbb{R}$ is the maximal torus $T = \exp \mathfrak{t}$. The center $Z_0$ of $G_0$ (which coincides with the center of $G$ in the case when there exists a compact Cartan subgroup of $G_0$) is a finite subgroup of $T_0$.

As is well known, there exists a natural isomorphism $i : T \rightarrow T$ such that $i T_0 = T_0$ and $i Z_0 = Z_0$ where $Z_0$ is the intersection of $T_0$ with the center of $\mathfrak{t}^*$. Given a left (or right) pre-semishadow $(\mathbb{C}[G], \#_t)$, $(t \in T_0$, consider its quasi-classical analog $(\mathbb{C}[G], \#_{t'})$ where $t' = i(t)$, $f^*_{t'}(g) = f(\omega_{t'}(g)^{-1})$, $\omega_{t'}(g) = \omega(g t')$ (or $\omega_{t'}(g) = \omega(t' g)$ respectively). As follows from Theorem 3.1 (3), the set of equivalence classes of irreducible $*$-representations of this Poisson algebra, i.e. the set $X_{t'} = \{g \in G_\mathbb{R} | \omega_{t'}(g) = g\}$, must be either empty or coincide with a Poisson left coset $G_0 u \subset G_\mathbb{R}$ (or a Poisson right coset $u G_0$ respectively).

It is clear that the left (right) pre-semishadow $(\mathbb{C}[G], \#_t)$ is a left (right) semishadow if and only if $X_{t'}$ is not empty. Hence, shadows correspond to such Poisson cosets that $G_0 u = u G_0$ what means that $u$ belongs to the normalizer of $G_0$.

We see that a left coset $G_0 u$ is the set of fixed points with respect to the involution $g \mapsto \omega(g \cdot u^{-1}) \cdot u = \omega(g) \cdot \omega(u^{-1}) u$. Therefore, it is a Poisson submanifold of $G_\mathbb{R}$ if and only if $\omega(u^{-1}) u$ is a zero-dimensional symplectic leaf of $G$, that is, $\omega(u^{-1}) u \in T$ (in fact, it belongs to $T_0$), what is the direct analog of Theorem 3.2.

It is easy to show that it is equivalent to $u \in G_0 N(T)$ where $N(T)$ is the normalizer
of $T$. Therefore, equivalence classes of left semishadows of $C[G_0]_q$ are parameterized by $G_0 \backslash G_0 N(T)/T \simeq W_0 \backslash W$ where $W$ is the Weyl group of $g$, $W_0$ is the subgroup of $W$ (not normal in general) generated by simple reflections with respect to compact roots. Analogously one can show that right semishadows are naturally parameterized by $W/W_0$.

As to the shadows, they correspond to such $u \in G_R$ that $\omega(u^{-1})u$ belongs not simply to $T_0$ but to $Z_0$. This means that $ugu^{-1} = \omega(u)g\omega(u)^{-1}$ for each $g \in G$. It follows that the conjugation by $u$ commutes with $\omega$, therefore, $u \in N_G(G_0)$ where $N_G(G_0)$ is the normalizer of $G_0$ in $G$.

Thus, we see that shadows of $C[G_0]_q$ are parameterized by the finite group $\tilde{W} = N_G(G_0)/G_0$. Note that $\tilde{W}$ is in fact abelian, since the homomorphism $\bar{\tau} : \tilde{W} \to Z_0$ induced by $\tau : N_G(G_0) \to Z$, $g \mapsto \omega(g^{-1})g$ is easily seen to be injective.

Let us return to the quantum picture. One can show that the involution on $C[G]_q$ which defines the left semishadow corresponding to $W_0 w$ ($w \in W$) can be given by

$$\mathcal{J} = \mathcal{R}(\tilde{w}^{-1}) \circ \mathcal{L}(\tilde{w}) \quad (3.1)$$

where $\tilde{w}$ is the element of the quantum Weyl group corresponding to $w \in W$. Analogously, for the right semishadow corresponding to $wW_0$ one has

$$\mathcal{J} = \mathcal{L}(\tilde{w}^{-1}) \circ \mathcal{R}(\tilde{w}) \quad (3.2)$$

The quantum Weyl group was introduced in [26] and studied, for instance, in [8, 16, 24, 25, 26]. Recall that the quantum analog of $w \in W$ is a certain Gelfand-Naimark-Segal state $\hat{w} \in C[G]_q^*$ with respect to certain irreducible representations of $C[G]_q$ and certain vectors in the spaces of the representations.

As far as the shadows are concerned, note that $\tilde{W} \simeq W_1/W_0$ where $W_1$ is the subgroup of $W$ which consists of the elements whose action on $\mathfrak{h}$ commutes with the Cartan involution $\theta = \omega_0 \circ \omega_{0,\text{comp}}$ and preserves roots of $t$ in $\mathfrak{h}$. Therefore, the involution on $C[G]_q$ which defines the shadow corresponding to $\tilde{w} \in \tilde{W}$ can be given
by either (4.1) or (3.2) where \( w \in W_1 \) represents \( \bar{w} \).

Recall that the homomorphism \( \bar{\tau} : \bar{W} \to \bar{Z}_0 \) is injective. Therefore, the shadows corresponding to distinct elements of \( \bar{W} \) are not equivalent. Note also that, if \( G_0 \) is compact, \( \bar{W} \) is trivial.

Let us summarize the obtained results in the following statements.

**Theorem 2.1.** (1) The equivalence classes of left semishadows of \( C[G_0]_q \) are parameterized by \( W_0 \backslash W \), with the involution on \( C[G]_q \) given by (4.1) where \( w \) represents the corresponding coset from \( W_0 \backslash W \).

(2) The equivalence classes of right semishadows of \( C[G_0]_q \) are parameterized by \( W/W_0 \), with the involution on \( C[G]_q \) given by (3.2) where \( w \) represents the corresponding coset from \( W/W_0 \).

(3) The equivalence classes of shadows of \( C[G_0]_q \) are parameterized by the finite abelian group \( \bar{W} \cong W_1/W_0 \cong N_G(G_0)/G_0 \), with the involution on \( C[G]_q \) given by either (4.1) or (3.2) where \( w \in W_1 \) represents the corresponding element of \( W_1/W_0 \).

**Remark 2.4.** (1) The quasi-classical analogues of the left (or right) semishadows of \( C[G_0]_q \) are the algebras of regular functions on the Poisson left (or right) cosets of the form \( G_0w \) (or \( wG_0 \) respectively) where \( w \in N(T) \).

(2) The quasi-classical analogues of the shadows of \( C[G_0]_q \) are the algebras of regular functions on the connected components of \( N_G(G_0) \cong G_0 \times \bar{W} \), the normalizer of \( G_0 \) in \( G \).

Now we are going to construct the quantum algebra \( C[G_0 \rtimes \bar{W}]_q \) of regular functions on the quantum disconnected group \( G_0 \rtimes \bar{W} \). It is given by the following theorem.

**Theorem 2.2.** Define a \( \ast \)-algebra \( C[G_0 \rtimes \bar{W}]_q \) as the coproduct of all its shadows in the category of \( \ast \)-algebras:

\[
C[G_0 \rtimes \bar{W}]_q = \bigoplus_{\omega \in \bar{W}} (C[G]_q, \#_\omega) \tag{3.3}
\]
where $\tilde{\omega}$ defines the shadow corresponding to $\tilde{\omega} \in \tilde{W}$. This means that all the summands in (3.3) are $\ast$-subalgebras of the coproduct, and the product of any elements from distinct summands is zero.

The following formulae define a Hopf $\ast$-algebra structure on $\mathbb{C}[G_0 \times \tilde{W}]_q$:

$$
\Delta(j_{\tilde{\omega}}(a)) = \sum_{\tilde{\omega}_1 \tilde{\omega}_2 = \tilde{\omega}} (j_{\tilde{\omega}_1} \otimes j_{\tilde{\omega}_2}) \Delta(a) \quad (3.4)
$$

$$
S(j_{\tilde{\omega}}(a)) = j_{\tilde{\omega}^{-1}}(S(a)) \quad \varepsilon(j_{\tilde{\omega}}(a)) = \delta_{\tilde{\omega},1} \varepsilon(a)
$$

where $\tilde{\omega} \in \tilde{W}$, $a \in \mathbb{C}[G]_q$, $j_{\tilde{\omega}} : \mathbb{C}[G]_q \to \mathbb{C}[G_0 \times \tilde{W}]_q$ is the embedding of the $\tilde{\omega}$-th summand in (3.3).

Remark 2.5. When $G_0 = SU(1,1)$, the Hopf $\ast$-algebra $\mathbb{C}[G_0 \times \tilde{W}]_q$ is nothing but the Hopf $\ast$-algebra $\mathfrak{s} = \mathbb{C}[SU(1,1) \times \mathbb{Z}_2]_q$.

Remark 2.6. Note that the group $\tilde{W}$ is usually too small for the above construction to be sufficient to obtain a result in the general case similar to the result obtained in the following sections in the case $G_0 = SU(1,1)$. For instance, when $G_0 = SU(m,n)$, $\tilde{W} \cong \mathbb{Z}_2$ if $m = n$ and is trivial otherwise. So, while there are embeddings of $SU(2)$'s and $SU(1,1)$'s into $G_0$ corresponding to the embeddings of $\mathfrak{sl}(2)$-triples into $\mathfrak{g}$, they cannot be lifted to embeddings of the normalizers.

Recall, however, that we have several standard Poisson Lie group structures on $G_0$. They all induce the same Poisson Lie group structure on their double group $G_R$, so they can be considered as different Poisson Lie subgroups of $G_R$. It seems to be likely that a generalization of the result obtained in the present paper might involve in some sense all those Poisson Lie subgroups and their normalizers.

### 3.3 Geometric Realization of Tensor Products of Unitarizable Modules.

Throughout Sections 3 and 4 we consider the case of $\mathfrak{s} = \mathbb{C}[SU(1,1) \times \mathbb{Z}_2]_q$. In this section we make use of the left quantum adjoint action in order to obtain some
geometric realization of tensor products of irreducible self-adjoint unitarizable left \( \mathfrak{g} \)-modules. In the subsequent section this realization enables us to construct the \(*\)-representations (1.1),(1.2) of \( \mathfrak{g} \).

We call sometimes an \( \mathfrak{g} \)-module \( \mathfrak{g} \)-module if the action of \( \mathfrak{g}_\tau \subset \mathfrak{g} \) on it is trivial.

Let \( \text{Fun}_q^\pm \) be the \(*\)-algebra generated by \( t_{ij} \) \((i, j = 1, 2)\), functions of a real variable \( x \), relations (3.1) and

\[
\begin{align*}
    t_{11} f(x) &= f(xq) t_{11} ,
    t_{12} f(x) &= f(x) t_{12} ,
    t_{22} f(x) &= f(xq^{-2}) t_{22} ,
    t_{21} f(x) &= f(x) t_{21} ,
    t_{12} t_{21} &= x
\end{align*}
\]

with the involution given by (3.2) on \( \text{Fun}_q^+ \) and by (3.3) on \( \text{Fun}_q^- \).

The \( \mathfrak{g}_\tau \)-module \(*\)-algebra structure can be obviously extended from \( \mathfrak{g}_\tau \subset \text{Fun}_q^\pm \) to the whole \( \text{Fun}_q^\pm \).

Consider the \( \mathfrak{g}_\tau \)-module \(*\)-algebra \( \text{Fun}(\mathcal{H})_q^\pm \) of \( \text{Fun}_q^\pm \) generated by \( y = t_{11} t_{12} \), \( y^* = q^2 t_{22} t_{21} \), and functions of \( x \). Note that the involutions \(*\) and \( \dagger \) coincide on \( \text{Fun}(\mathcal{H})^\pm_q \), therefore, \( \text{Fun}(\mathcal{H})_q^+ = \text{Fun}(\mathcal{H})_q^- \) which we denote simply \( \text{Fun}(\mathcal{H})_q \) from now on.

\( \text{Fun}(\mathcal{H})_q \) first appeared in [14] as an algebra of functions on a quantum two-sheet hyperboloid \( \mathcal{H} \). One of its sheets can be realized as \( SU(1, 1)/T_0 \) and the other one as \( SU(1, 1) \cdot w / T_0 \) where \( w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

What we consider is a quantum analog of the transformation of the hyperboloid \( \mathcal{H} = \{ x \in \mathbb{R}, y \in \mathbb{C} \mid |y|^2 = x(x + 1) \} \) such that its symplectic leaf \( \{ x > 0 \} \) turns into \( \mathcal{D}_+ = \{ |z| > 1 \} \), the leaf \( \{ x \leq -1 \} \) into \( \mathcal{D}_- = \{ |z| < 1 \} \), and the zero-dimensional leaf \( \{ x = 0, y = 0 \} \) goes away into infinity \( \infty \in \mathbb{CP}^1 \approx SL_2(\mathbb{C})_\mathbb{R} / B \supset (SU(1, 1) \ltimes \mathbb{Z}_2)/T_0 \) where \( B \) is the Borel subgroup of upper-triangular matrices.

Namely, consider the \( \mathfrak{g}_\tau \)-module \(*\)-algebra \( \text{Fun}(\mathcal{C})_q \) generated by \( z = t_{11} t_{21}^{-1} \), \( z^* =
and functions of $r = x^{-1}$. In the new generators the relations look as follows:

$$zf(r) = f(rq^{-2})z, \quad zz^* = 1 + q^{-1}r$$
$$z^*f(r) = f(rq^2)z^*, \quad z^*z = 1 + qr$$

\begin{align*}
t_{11} : \quad z^k f(r) & \mapsto q^{-k-1}z^k \left(1 + q^{-1}r\right)f(rq^{-2}) - f(r) \\
t_{12} : \quad z^k f(r) & \mapsto -q^{-k-1}z^{k+1}f(r) - f(rq^2) \\
t_{21} : \quad (z^*)^k f(r) & \mapsto q^{k+1}(z^*)^{k+1}f(rq^{-2}) - f(r) \\
t_{22} : \quad (z^*)^k f(r) & \mapsto -q^{k-1}(z^*)^k f(r) - (1 + qr)f(rq^2) \tag{3.2}
\end{align*}

where $k \in \mathbb{Z}$ (as follows from (3.1), $z^{-1} = z^*(1 + q^{-1}r)^{-1}$).

There are the following unitary equivalence classes of infinite-dimensional irreducible \*$*$-representations of $Fun(C)_q$: $\pi^+$ (the restriction of $\pi^+_\varphi$) corresponding to $D_+$ and $\pi^-$ (the restriction of $\pi^-\varphi$) corresponding to $D_-$. There are also the one-dimensional \*$*$-representations $z \mapsto e^{i\varphi}$, $z^* \mapsto e^{-i\varphi}$, $f(r) \mapsto f(0)$ corresponding to the points of $S^1 = \{|z| = 1\}$.

Let $C_c^\infty(D_+)_q(\beta)$ and $C_c^\infty(D_-)_q$ be the ideals in $Fun(C)_q$ generated by those functions of $r$ whose supports are finite subsets of the geometric progressions $\mathcal{M}_\beta^+ = \{q^{2(k-\beta)}\}_{k \in \mathbb{Z}}$ and $\mathcal{M}_\beta^- = \{-q^{2k+1}\}_{k \in \mathbb{Z}^+}$ respectively (which are the sets of eigen-values of $r$ in the corresponding \*$*$-representations $\pi^+\beta$ and $\pi^-\beta$).

It is easy to see that $C_c^\infty(D_\pm)_q$ is an \*$*$-module \*$*$-algebra. By the construction, it can be thought of as a quantum analog of the algebra of smooth functions in $D_\pm$ with compact supports.

**Definition 2.4.** Suppose that $\mathfrak{a}_0$ is a Hopf \*$*$-algebra, $\mathcal{F}$ is an $\mathfrak{a}_0$-module \*$*$-algebra. A linear functional $\nu : \mathcal{F} \to \mathbb{C}$ is called quasi-invariant integral on $\mathcal{F}$ if there exists a positive group-like element $\chi \in \mathfrak{a}_0^*$ such that

$$\nu(a.f) = \langle \chi, a \rangle \nu(f) \tag{3.3}$$
for each $a \in \mathfrak{a}_0$, $f \in \mathcal{F}$.

If $\chi = 1$, $\nu$ is called invariant integral on $\mathcal{F}$.

**Proposition 2.2.** The linear functional $\nu_\pm : C^\infty_c(D_\pm)_q \to \mathbb{C}$ given by

$$
\nu_\pm (f) = (q^{-1} - q) \cdot tr \pi^\pm(f) r
$$

is a quasi-invariant integral on $C^\infty_c(D_\pm)_q$, the associated element $\chi \in \mathfrak{r}_+$ given by $\chi = \chi_0^2 = \chi_0^* \chi_0$ where the group-like element $\chi_0$ is given by

$$
\chi_0 = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}
$$

**Proof.** It is easy to see that

$$
\nu_\pm (e_k f(r)) = \delta_{k,0} \cdot (q^{-1} - q) \cdot \sum_{r \in \mathcal{M}_\pm} f(r) r
$$

(3.4)

The remainder of the proof is just a straightforward computation with use of (3.2) and (3.4). One can show that the quasi-invariant integral is unique up to multiplying by a positive constant.

It is easy to see that the irreducible *-representation $\pi^\pm$ of $C^\infty_c(D_\pm)_q$ is faithful and unique up to unitary equivalence. Therefore, $C^\infty_c(D_\pm)_q$ can be identified with the image

$$
\mathcal{F}(V^\pm_\varphi) = C^\infty_c(D_\pm)_q |_{V^\pm_\varphi}
$$

which is the algebra of such linear operators in $V^\pm_\varphi$ that their matrices with respect to the canonical basis $\{ e^{\pm}_k \}$ contain finitely many non-zero elements.

Let us equip $\mathcal{F}(V^\pm_\varphi)$ with an $\mathfrak{r}_+$-module structure by twisting that on $C^\infty_c(D_\pm)_q$:

$$
a : f |_{V^\pm_\varphi} \mapsto (\gamma(a).f) |_{V^\pm_\varphi}
$$

(3.5)
where \( a \in \mathfrak{R}_+, \ f \in C_c^\infty(D_\pm)_q \) and \( \gamma = R(\chi^{-1}_0) \) is an automorphism of \( \mathfrak{R} \).

Consider the scalar product on \( F(V^\pm_\varphi) \) given by

\[
(f_1, f_2) = \nu_\pm(f_2^* f_1)
\]

**Theorem 2.3.** The \( \mathfrak{R}_+ \)-module \( F(V^\pm_\varphi) \) is unitarizable and unitary equivalent to \( V^\pm_\varphi \otimes V^\pm_{\pi+\varphi} \), a unitary intertwiner given by

\[
\begin{align*}
   e_m^\pm \otimes e_n^\pm & \mapsto (q^{-1} - q)^{-\frac{1}{2}} e_{\pm i(m-n)\varphi} r_n^\pm \left| r_n^\pm \right|^{-\frac{1}{2}} \\
   & \times (-q r_n^\pm)^{-\frac{1}{2}} z^{\mp(m-n)} \delta_n^\pm(r)
\end{align*}
\]

where \( r_n^{(+)} = q^{-2(n+\beta)} \in \mathfrak{m}_+^{-\beta}, \ r_n^{(-)} = -q^{2n+1} \in \mathfrak{m}_-, \) and \( \delta_n^\pm(r) \) is the “\( \delta \)-function” on \( \mathfrak{m}_\pm \) which takes the value 1 at the point \( r_n^{(\pm)} \) and 0 at all other points, \((a)_\alpha = (a)_\infty/(q^{2a})_\infty, (a)_\infty = \prod_{k=0}^{\infty}(1 - aq^{2k})\).

**Proof.** That \( F(V^\pm_\varphi) \) is unitarizable is provided by (3.3), (3.5) and (3.6).

Let us prove that \( F(V^\pm_\varphi) \) is unitarily equivalent to \( V^\pm_\varphi \otimes V^\pm_{\pi+\varphi} \). Consider the conjugate \( \mathfrak{R}_+ \)-module \( (V^\pm_\varphi)^* \) defined as follows:

\[
\langle a.\psi, v \rangle = \langle \psi, S(a).v \rangle
\]

for each \( a \in \mathfrak{R}_+, \ \psi \in (V^\pm_\varphi)^*, \ v \in V^\pm_\varphi \). Note that \((V^\pm_\varphi)^*\) is not unitarizable.

Since the action of \( \mathfrak{R}_+ \) on \( C_c^\infty(D_\pm)_q \) comes from the quantum adjoint action, we see that the \( \mathfrak{R}_+ \)-modules \( C_c^\infty(D_\pm)_q \) and \( V^\pm_\varphi \otimes (V^\pm_\varphi)^* \) are equivalent, the intertwiner being given by the obvious identifications of these both to a space of finite-dimensional linear operators in \( V^\pm_\varphi \). As soon as we twist the action on \( C_c^\infty(D_\pm)_q \) so as to get the action on \( F(V^\pm_\varphi) \), this intertwiner becomes a unitary operator.

Recall now that \( \Delta(\gamma(a)) = (id \otimes \gamma)\Delta(a) \) for each \( a \in \mathfrak{R}_+ \). Therefore, \( F(V^\pm_\varphi) \) is unitarily equivalent to \( V^\pm_\varphi \otimes (V^\pm_\varphi)^* \) where \((V^\pm_\varphi)^*\) is the unitarizable module obtained from \((V^\pm_\varphi)^*\) by twisting the action of \( \mathfrak{R}_+ \) by \( \gamma \). It is an easy computation to see that
$(V_\varphi^\pm)_\gamma^*$ is unitary equivalent to $V_{n+\varphi}^\pm$, the unitary intertwiner given by

$$J : e_n^\pm \mapsto (q^{-1} - q)^{-\frac{1}{2}} \cdot |r_n^{(\pm)}|^{-\frac{1}{2}} \psi_n^\pm$$

where $\{\psi_n^\pm\}$ is the basis of $(V_\varphi^\pm)_\gamma^*$ dual to the basis $\{e_n^\pm\}$ of $V_\varphi^\pm$ (that is, $\langle \psi_m^\pm, e_n^\pm \rangle = \delta_{mn}$). This proves our assertion.

The explicit expression (3.7) is obtained as follows. The right hand side is equal to $(q^{-1} - q)^{-\frac{1}{2}} \cdot |r_n^{(\pm)}|^{-\frac{1}{2}} E_{mn}$ where $E_{mn}$ is the operator in $V_\varphi^\pm$ given by $E_{mn} : e_k^\pm \mapsto \delta_{kn} e_m^\pm$.

Now we use the formulae (3.5),(3.6).

Let $\mathcal{F}(V_{\varphi_2}^\pm, V_{\varphi_1}^\pm)$ and $\mathcal{F}(V_{\varphi_2}^\pm, V_{\varphi_1}^\mp)$ be the vector spaces of such linear operators from $V_{\varphi_2}^\pm$ to $V_{\varphi_1}^\pm$ and to $V_{\varphi_1}^\mp$ respectively that their matrices with respect to the canonical bases $\{e_n^\pm\}$ contain finitely many non-zero elements. Consider the $\mathfrak{g}$-module structure on $\mathcal{F}(V_{\varphi_2}^\pm, V_{\varphi_1}^\pm)$ and $\mathcal{F}(V_{\varphi_2}^\pm, V_{\varphi_1}^\mp)$ given by

$$a : f \mapsto \sum_k \pi_{\varphi_1}^\pm (a_k') f \pi_{\varphi_2}^\pm (S(\gamma(a_k')))$$

and

$$a : f \mapsto \sum_k \pi_{\varphi_1}^\mp (a_k') f \pi_{\varphi_2}^\mp (S(\gamma(a_k')))$$

respectively whenever $\Delta(a) = \sum_k a_k' \otimes a_k''$ (and, therefore, $\Delta(\gamma(a)) = \sum_k a_k' \otimes \gamma(a_k'')$).

Consider the scalar products on $\mathcal{F}(V_{\varphi_2}^\pm, V_{\varphi_1}^\pm)$ and $\mathcal{F}(V_{\varphi_2}^\pm, V_{\varphi_1}^\mp)$ given by

$$(f_1, f_2) = \nu_{\pm}((f_1, f_2)_F) \text{ where } (f_1, f_2)_F = f_2^* f_1$$

is an $\mathcal{F}(V_{\varphi_2}^\pm)$-valued scalar product which is $\mathcal{F}(V_{\varphi_2}^\pm)$-linear with respect to the right action of $\mathcal{F}(V_{\varphi_2}^\pm)$.

The idea of the proof of the following theorem does not differ significantly from that of the proof of Theorem 2.2.

**Theorem 2.4.** (1) The $\mathfrak{g}_+\text{-module } \mathcal{F}(V_{\varphi_2}^\pm, V_{\varphi_1}^\pm)$ is unitarizable and unitarily equivalent to $V_{\varphi_1}^\pm \otimes V_{\varphi_2}^\pm$.

(2) The $\mathfrak{g}_-\text{-module } \mathcal{F}(V_{\varphi_2}^\pm, V_{\varphi_1}^\mp)$ is unitarizable and unitarily equivalent to $V_{\varphi_1}^\mp \otimes V_{\varphi_2}^\pm$.

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3.4 Super-Tensor Products of Irreducible $\ast$-Representations.

Now we use the geometric realization of tensor products of irreducible self-adjoint unitarizable $\mathfrak{g}$-modules obtained in the previous section in order to construct the super-tensor products (1.1) and (1.2). But first we explain why we cannot correctly define tensor products otherwise.

The following result (actually, its $\mathfrak{g}_{+}$-part) was first obtained in [30]. We give another proof based on the geometric realization what provides us also with a nice quasi-classical analogues of the results.

**Theorem 2.5.** (1) There does not exist a self-adjoint unitarizable $\mathfrak{g}_{+}$-module $V$ dense in the Hilbert space $\mathfrak{h} = V_{\varphi}^{\pm} \otimes V_{\pi+\varphi}^{\pm}$ and such that

$$V_{\varphi}^{\pm} \otimes V_{\pi+\varphi}^{\pm} \subset V$$

(2) There does not exist self-adjoint unitarizable $\mathfrak{g}_{\pm}$-modules $V_{\pm}$ dense in the Hilbert spaces $\mathfrak{h}_{\pm} = V_{\varphi_{1}}^{\pm} \otimes V_{\pi+\varphi_{2}}^{\pm}$ and $\mathfrak{h}_{-} = V_{\varphi_{1}}^{\mp} \otimes V_{\pi+\varphi_{2}}^{\pm}$ respectively and such that

$$V_{\varphi_{1}}^{\pm} \otimes V_{\pi+\varphi_{2}}^{\pm} \subset V_{+}, \quad V_{\varphi_{1}}^{\mp} \otimes V_{\pi+\varphi_{2}}^{\pm} \subset V_{-}$$

**Remark 2.7.** In other words, there is no correctly defined tensor products $V_{\varphi_{1}}^{\pm} \otimes V_{\pi+\varphi_{2}}^{\pm}$ and $V_{\varphi_{1}}^{\mp} \otimes V_{\pi+\varphi_{2}}^{\pm}$.

**Proof of Theorem 2.5.** (2) Consider the action of $x = t_{12}t_{21}$ in the subspace of $\mathfrak{h}$ generated by the vectors of the form $e_{k}^{\pm} \otimes e_{k}^{\pm}$. By (3.7), the closure of this operator is unitary equivalent to the closure of the operator given by the action of $\gamma(x) = x$ in the subspace of $C_{c}^{\infty}(\mathcal{D}_{\pm})_{q}$ generated by functions $f(r)$.

The $L^{2}$-closure of this subspace is $L^{2}(\mathfrak{m}_{\pm}, d\nu_{\pm})$ where the measure $d\nu_{\pm}$ is given
by $\int f(r) d\nu_\pm = \nu_\pm(f(r))$ and $\mathcal{M}_+$ is supposed to be $\mathcal{M}_+^{(-3)}$ (recall that $r = x^{-1}$). By (3.2), our operator is the minimal closed operator in this space given by the second order $q$-difference expression

$$x : f(r) \mapsto -(q^{-1} - q)^2 \cdot D(1 + r)Df(r) \quad (3.1)$$

where $(Df)(r) = \frac{f(qr^{-1}) - f(1 - rq)}{rq - 1 - rq}$.

This operator is symmetric but not self-adjoint. Therefore, $V^\pm_\varphi \otimes V^\pm_{\pi^+ \varphi}$ does not give rise to a $\ast$-representation of $\mathfrak{A}_+$. Assume that there exists a self-adjoint extension $V$ of $V^\pm_\varphi \otimes V^\pm_{\pi^+ \varphi}$, and come to a contradiction.

By the assumption, the closure $\sigma(x)$ of the operator given by the action of $x$ in $V$ restricted to $L^2(\mathfrak{M}_\pm, d\nu_\pm)$ is a self-adjoint extension of the minimal closed operator given by (3.8).

As is well known, any its self-adjoint extension is given by a boundary condition of the form

$$\cos \alpha \cdot f(\pm 0) + \sin \alpha \cdot (Df)(\pm 0) = 0 \quad (3.2)$$

$L^2(\mathfrak{M}_\pm, d\nu_\pm)$ is invariant also with respect to $\sigma(t_{11})$, as follows from (3.2). However, as (3.2) shows, $\sigma(t_{11})$ does not respect the initial domain of $\sigma(x)$ as it is given by (3.2). Thus, we have come to a contradiction.

(2) This statement follows from the previous one in the following way. For instance, assume that there exists a self-adjoint extension $\tilde{\mathcal{F}}(V^\pm_\varphi, V^\pm_{\varphi_1})$ of $\mathcal{F}(V^\pm_\varphi, V^\pm_{\varphi_1})$.

Let $\tilde{\mathcal{F}}(V^\pm_\varphi)$ be the maximal algebra of operators in $V^\pm_\varphi$ such that $\forall f \in \tilde{\mathcal{F}}(V^\pm_\varphi, V^\pm_{\varphi_1})$ for each $v \in \tilde{\mathcal{F}}(V^\pm_\varphi, V^\pm_{\varphi_1}), f \in \tilde{\mathcal{F}}(V^\pm_\varphi)$. One can show that $\tilde{\mathcal{F}}(V^\pm_\varphi)$ can be equipped with an $\mathfrak{A}_+$-module structure which extends that of $\mathcal{F}(V^\pm_\varphi)$ so that it gives rise to a $\ast$-representation of $\mathfrak{A}_+$, what contradicts with the previous statement.

Remark 2.8. We will see in Section 5 that the quasi-classical analog of the $\mathfrak{A}_+$-module $\ast$-algebra structure on $C^\infty_c(\mathcal{D}_\pm)_q$ is the local action of the dual Poisson Lie group $SU(1, 1)^*$ (isomorphic to the group of the matrices of the form $\begin{pmatrix} t & z \\ 0 & t^{-1} \end{pmatrix}$) in $\mathcal{D}_\pm$ by translations and dilations.
Thus, the quasi-classical analog of the above negative result is such an obvious fact that this local action cannot be extended to a global one. The obstacle at \( r = \pm 0 \) which prevents to construct a self-adjoint extension of the tensor product in the proof of Theorem 2.5 corresponds to the obstacle at \( S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \} \).

However, although the local action of \( SU(1,1)^* \) cannot be extended to a global one on \( D_+ \) and \( D_- \) separately, it can be extended to a global action on \( \mathbb{C} = \overline{D_+} \cup \overline{D_-} \).

This observation prompts us what to do in the quantum case. Namely, one should consider the problem of self-adjoint extensions of \( V_{\varphi_1}^{+(\beta)} \otimes V_{\pi+\varphi_1}^{+(\beta)} \otimes V_{\varphi_2}^{-} \otimes V_{\pi+\varphi_1}^{-} \) using its realization as

\[
\mathcal{F}(V_{\varphi_1}^{+(\beta)}) \oplus \mathcal{F}(V_{\varphi_2}^{-}) = (C_c^\infty(D_+)_q \oplus C_c^\infty(D_-)_q)|_{\pi_1^{+(\beta)} \oplus \pi_2^-}.
\]

Let \( C_c^\infty(\mathbb{C})_q \) be the extension of \( C_c^\infty(D_+)_q \oplus C_c^\infty(D_-)_q \) which is the ideal in \( Fun(\mathbb{C})_q \) generated by those functions of \( r \) whose supports are compact subsets of \( \mathfrak{d}(-\beta) = \mathfrak{d}_+(-\beta) \cup \mathfrak{d}_- \cup \{0\} \) and which are smooth at zero. Consider the extension

\[
\tilde{\mathcal{F}}^{(\beta)}_{\varphi_1,\varphi_2} = C_c^\infty(\mathbb{C})_q|_{\pi_1^{+(\beta)} \oplus \pi_2^-}
\]

of \( \mathcal{F}(V_{\varphi_1}^{+(\beta)}) \oplus \mathcal{F}(V_{\varphi_2}^{-}) \), and equip it with the \( \mathcal{A}_+ \)-module structure given by (3.5) and with the scalar product given by (3.6).

**Theorem 2.6.** The unitarizable \( \mathcal{A}_+ \)-module \( \tilde{\mathcal{F}}^{(\beta)}_{\varphi_1,\varphi_2} \) is self-adjoint. We denote the corresponding \( \ast \)-representation of \( \mathcal{A}_+ \) by \( \pi_{\varphi_1}^{+(\beta)} \otimes \pi_{\pi+\varphi_1}^{+(\beta)} \otimes \pi_{\varphi_2}^{-} \otimes \pi_{\pi+\varphi_1}^{-} \).

**Proof.** Indeed, the minimal closed operator in \( L^2(\mathfrak{d}^{(-\beta)}, dv) \), where \( dv = dv_+ + dv_- \), given by the second order \( q \)-difference expression (3.8) and the boundary condition

\[
f(+0) = f(-0) \quad (Df)(+0) = (Df)(-0) \quad (3.3)
\]

is easily seen to be self-adjoint. The operators given by the action of \( x \) in other \( x \)-invariant subspaces generated by elements of the form \( z^k f(r) \) for each fixed \( k \in \mathbb{Z} \).
can be considered analogously (another way is to notice that all these parts of the operator given by the action of $x$ are intertwined by the action of powers of $t_{12}$ or $t_{21}$).

Finally, it is easy to see that all operators given by the actions of $t_{ij}$ ($i, j = 1, 2$) respect the smoothness condition. This proves our assertion.

**Remark 2.9.** $C_c^\infty(\mathbb{C})_q$ can be thought of as a quantum analog of the algebra of smooth functions on $\mathbb{C}$ with compact supports.

Consider now the problem of self-adjoint extensions of the unitarizable $\mathfrak{g}$-module $V^+_{\varphi_1} \otimes V^\pm_{\varphi_2} \otimes V^-_{\varphi_3} \otimes V^\mp_{\varphi_4}$ denoted for convenience by $V_\pm$.

Note that $V_\pm$ is endowed with an $\mathfrak{g}$-equivariant Hilbert $(\tilde{\mathcal{F}}_{\varphi_1, \varphi_3}, \tilde{\mathcal{F}}_{\varphi_2, \varphi_4})$-bimodule structure. This means that there are the left $\tilde{\mathcal{F}}_{\varphi_1, \varphi_3}$-module and the right $\tilde{\mathcal{F}}_{\varphi_2, \varphi_4}$-module structures $m_l : \tilde{\mathcal{F}}_{\varphi_1, \varphi_3} \otimes V_\pm \to V_\pm$ and $m_r : V_\pm \otimes \tilde{\mathcal{F}}_{\varphi_2, \varphi_4} \to V_\pm$ respectively given by

\[
\begin{align*}
(f_1, f_3) \otimes (u', u'') &\mapsto (f_1 u', f_3 u'') \\
(u', u'') \otimes (f_2, f_4) &\mapsto (u' f_2, u'' f_4)
\end{align*}
\]

where $(u', u'') \in V_\pm$, $(f_1, f_3) \in \tilde{\mathcal{F}}_{\varphi_1, \varphi_3}$ as well as the $\tilde{\mathcal{F}}_{\varphi_1, \varphi_3}$-linear scalar product $(\cdot, \cdot)_l : V_\pm \otimes V_\pm \to \tilde{\mathcal{F}}_{\varphi_1, \varphi_3}$ and the $\tilde{\mathcal{F}}_{\varphi_2, \varphi_4}$-linear scalar product $(\cdot, \cdot)_r : V_\pm \otimes V_\pm \to \tilde{\mathcal{F}}_{\varphi_2, \varphi_4}$ given by

\[
\begin{align*}
((v_1, v_2), (v_3, v_4))_l &= (v_1 v_3^*, v_2 v_4^*) \\
((v_1, v_2), (v_3, v_4))_r &= (v_3 v_1^*, v_4 v_2^*)
\end{align*}
\]

such that $m_l, m_r, (\cdot, \cdot)_l$ and $(\cdot, \cdot)_r$ are $\mathfrak{g}$-module morphisms.

Let $\tilde{\mathcal{F}}_\pm$ be the maximal $\mathfrak{g}$-equivariant Hilbert $(\tilde{\mathcal{F}}_{\varphi_1, \varphi_3}, \tilde{\mathcal{F}}_{\varphi_2, \varphi_4})$-bimodule which extends $V_\pm$. The following theorem can be deduced from Theorem 2.6.

**Theorem 2.7.** The unitarizable $\mathfrak{h}_\pm$-module $\tilde{\mathcal{F}}_\pm$ is self-adjoint. We denote the corresponding $*$-representation of $\mathfrak{h}_\pm$ by $\pi_\pm^+$ $\otimes \pi_\pm^0$ $\otimes \pi_3^-$ $\otimes \pi_4^\mp$. 

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Another way to do it is to construct explicitly those super-tensor products in a way similar to the construction shown in Theorem 2.6. The key step here is to find an appropriate self-adjoint extension of the operator given by the action of $x$ restricted to a certain $x$-invariant subspace.

For instance, the self-adjoint operator $(\pi^{+,(\beta)} \circ \pi^{+,(\beta)} \circ \pi^{-,(\beta)} \circ \pi^{-,(\beta)})(x)$ restricted to the subspace generated by vectors of the form $e_k^+ \otimes e_k^+$ is unitarily equivalent to the operator in $L^2(\mathcal{M}(-\beta), d\nu)$ given by (3.8) and (3.3), the intertwiner given by (3.7) where $m = n = k$, the right hand side multiplied by $(-1)^k e^{ik(\varphi_2^{(s)} - \varphi_1^{(s)\pm})}$.

Also, the self-adjoint operator $(\pi^{+,\beta}(\beta) \circ \pi^{-,\beta} \circ \pi^{-,\beta} \circ \pi^{+,\beta})(x)$ restricted to the subspace generated by vectors of the form $e_k^+ \otimes e_k^+$ ($k \in \mathbb{Z}_+$) is unitarily equivalent to the operator in $L^2(\mathcal{M}, d\nu)$, where $\mathcal{M} = \mathcal{M}_+ \cup \mathcal{M}_- \cup \{0\}$, $\mathcal{M}_+ = \{\pm q^{2k-\beta}\}_k \in \mathbb{Z}_+$, and $\int f(t) d\nu = (q^{-1} - q) \cdot \sum_{t \in \mathcal{M}} t f(t)$, given by the second order $q$-difference expression

$$x : f(t) \mapsto (q^{-1} - q)^2 \cdot [p(t) + D\sqrt{(1 - q^{-\beta}t)(1 + q^{\beta + 1}t)} D\kappa_+(t) + D\sqrt{(1 + q^{-\beta}t)(1 - q^{\beta + 1}t)} D\kappa_-(t)] f(t)$$

(3.4)

where $p(t)$ is a certain function and the boundary condition (3.3), the intertwiner given by

$$e_{\pm k}^+ \otimes e_{\mp k}^+ \mapsto (q^{-1} - q)^{-1/2} \cdot i^k e^{ik(\varphi_2^{(s)} - \varphi_1^{(s)^\pm})} t_k^+(\pm) \frac{1}{2} \delta_k^{(\pm)}(t)$$

where $\varphi_i^{(s)} = \varphi_i^{(s)}$, $\varphi_i^{(s)} = \varphi_{i+2}$, $t_k^+(\pm) = \pm q^{2k-\beta}$, $\delta_k^{(\pm)}$ is the "$\delta$-function" on $\mathcal{M}$ at the point $t^+_k$, and $\kappa_+(t) = \sum_{k \in \mathbb{Z}_+} \delta_k^{(\pm)}(t)$ is the characteristic function of $\mathcal{M}_\pm$.

The following theorem is proven in Appendix B. It confirms our right to fix $\beta$ and develop an independent theory for each fixed value of $\beta$.

Namely, consider the category $C_\beta$ of *-representations of $\mathfrak{S}$ such that the spectrum of the operator which represents $x = (x, x)$ is contained in $\mathcal{M}_+^{(\beta)} \cup \mathcal{M}_-^{(\beta)} \cup \{0\}$. Theorem 2.8 shows that $C_\beta$ is closed with respect to super-tensor products.
Theorem 2.8.

\[ \zeta_{\varphi_1} \otimes \pi_{\varphi_2}^{+, (\beta)} \simeq \pi_{\varphi_1 + \varphi_2}^{+, (\beta)}, \quad \zeta_{\varphi_1} \otimes \pi_{\varphi_2}^{-} \simeq \pi_{\varphi_1 + \varphi_2}^{-} \]
\[ \pi_{\varphi_1}^{+, (\beta)} \otimes \zeta_{\varphi_2} \simeq \pi_{\varphi_1 - \varphi_2}^{+, (\beta)}, \quad \pi_{\varphi_1}^{-} \otimes \zeta_{\varphi_2} \simeq \pi_{\varphi_1 - \varphi_2}^{-} \]
\[ \zeta_{\varphi_1} \otimes \zeta_{\varphi_2} \simeq \zeta_{\varphi_1 + \varphi_2} \]

(3.5)

Remark 2.10. Note that, although the "super-tensor" products (3.6) and (3.7) are unitarily equivalent for different values of \( \beta \), there is no canonical unitary equivalence, since the corresponding quantum Clebsch-Gordan coefficients depend on \( \beta \) (see Appendix B).

3.5 The Super-Tensor Products and the Dressing Action on the Flag Manifold.

In this section we consider the quasi-classical analogues of the results obtained above. First of all, we must consider the quasi-classical analog of the left (right) quantum adjoint action. This can be done on a more general level.

Let \( G_0 \) be a real form of a simple complex Lie group \( G \) such that there exists a compact Cartan subgroup of \( G_0 \). Consider the standard Poisson Lie group structure on \( G_0 \). This structure as well as the induced real Poisson Lie group structure on \( G_0 \) are described in Section 2.

Note that one can consider two different quasi-classical analogues of the quantum algebra \( C[G_0]_q \) of regular functions. The first one is the Poisson Hopf *-algebra \( C[G_0] \) of regular functions on the Poisson algebraic group \( G_0 \), the Poisson brackets given by

\[ \{ f_1 \mod h \hat{\alpha}_0, f_2 \mod h \hat{\alpha}_0 \} = h^{-1} [f_1, f_2] \mod h \hat{\alpha}_0 \]

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where $\hat{\mathfrak{a}}_0 = C[G_0]_q \otimes C[[h]]$ is the QFSH-algebra obtained as the quantization of $C[G_0] \simeq \hat{\mathfrak{a}}_0 / h\hat{\mathfrak{a}}_0$, according to [1] (we suppose $q = e^{-\frac{\hbar}{2}} \in C[[h]]$).

The second analog is the universal enveloping algebra $U_{g_0^*}$ of the dual Lie bialgebra. More precisely, these two analogues correspond to some QFSH- and QUE-algebras respectively which are not isomorphic over $C[[h]]$, but become isomorphic over $C$ once $\hbar$ is fixed.

As is well known from [1], there is a covariant functor which gives an equivalence of the categories of QFSH- and QUE-algebras. For instance, the QUE-algebra corresponding to the QFSH-algebra $C[SU(1, 1)]_q \otimes C[[h]]$ is generated over $C[[h]]$ by $t'_{ij} = h^{-\delta_{ij}} t_{ij}$ ($i, j = 1, 2$). Its quasi-classical analog is $Usu(1, 1)^*$. The precise description of the functor can be found in [1].

To consider the quasi-classical analog of the left (right) quantum adjoint action of $C[G_0]_q$ on itself, one should combine the above two quasi-classical analogues so that we get a left (right) action of $U_{g_0^*}$ on $C[G_0]$. Let us show that it is the local right (left) dressing action of $g_0^*$ on $G_0$ made into a left (right) action in the usual way. It is just what is called for convenience in this paper left (right) dressing action.

Recall first the definition of the right (left) dressing action (cf. [23, 18]). Suppose that $g_0^*$ is realized as the Lie algebra of right (left) invariant differential 1-forms on $G_0$, the Lie brackets given by

$$[\alpha_1, \alpha_2] = -d(\pi\alpha_1, \alpha_2) + L_{\pi\alpha_2}\alpha_1 - L_{\pi\alpha_1}\alpha_2$$

where $L$ stands for the Lie derivative, $\langle \cdot, \cdot \rangle : TG_0 \times T^*G_0 \to C$ is the natural pairing between the tangent and cotangent bundles, $\pi : T^*G_0 \to TG_0$ is the bundle map associated with the Poisson manifold structure on $G_0$.

For each $\alpha \in g_0^*$, let $\alpha_r$ ($\alpha_l$ respectively) be the right (left) invariant differential 1-form on $G_0$ such that $\alpha_r(1) = \alpha$ ($\alpha_l(1) = \alpha$ respectively). The map $\alpha \mapsto -\pi\alpha_r$ ($-\pi\alpha_l$ respectively) from $g_0^*$ into the Lie algebra of smooth vector fields on $G_0$ is a Lie algebra homomorphism (antihomomorphism).

The vector fields $-\pi\alpha_r$ ($-\pi\alpha_l$ respectively) are called right (left) dressing fields.
They give rise to a *local* left (right) action of $G_0^*$ on $G_0$ called *right (left) dressing action*.

Consider the quasi-classical analog of $ad_q$. When $h$ tends to zero, the action of $C[G_0]_q$ on itself given by

$$ a : b \mapsto h^{-1}(ad_q - \varepsilon(a))b $$

tends to the action of $C[G_0]$ on itself given by

$$ a : b(g) \mapsto \{a(gg'), b(g)\}|_{g' = g^{-1}} $$

It is easy to see that it is nothing but the differentiation along the right dressing field $-\pi((da)(1, \cdot))$. The quasi-classical limit of $ad'_q$ can be considered analogously.

At last, the quasi-classical analog of the action of $C[G_0]_q$ on its left (right) semishadows is easily seen to be the local right (left) dressing action of $G_0^*$ on $G_R$ restricted to the corresponding Poisson left (right) coset, since $G_0^*$ is canonically embedded into $G_R^* \simeq G_0 \times G_0^*$.

As follows from (3.4), the left (right) quantum adjoint action of $C[G_0 \rtimes \hat{W}]_q$ is, in fact, a $C[G_0]_q$-action. Its quasi-classical analog can be considered also as the local right (left) dressing action of $(G_0 \rtimes \hat{W})^* \simeq G_0^*$ on $G_0 \rtimes \hat{W}$.

Now we consider the quasi-classical analogues of the results obtained in Sections 3 and 4. Suppose $G_0 = SU(1, 1)$, $G = SL_2(C)$. Recall that in this case $\hat{W} \simeq Z_2$ and $SU(1, 1) \rtimes Z_2$ is embedded into $SL_2(C)$ as $SU(1, 1) \cup SU(1, 1) \cdot w$ where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Recall also that $SU(1, 1)^*$ is isomorphic to the group of translations and dilations of the plane and is embedded into $SL_2(C)$ as the group of matrices of the form $\begin{pmatrix} t & z \\ 0 & t^{-1} \end{pmatrix}$ where $t > 0, z \in \mathbb{C}$.

It underlines the negative result of Theorem 2.4 that the *local* right (left) dressing action of $SU(1, 1)^*$ on either $SU(1, 1)$ or $SU(1, 1) \rtimes Z_2$ cannot be extended to a *global* one.

Let us compare our situation with the case of the compact real form $G_0$ of $G$. 

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In this case the Iwasawa's decomposition holds and the global right (for instance) dressing action \( g_- : g_+ \mapsto g_+^g \) where \( g_+ \in G_0 \), \( g_- \in G_0^* \) can be given by

\[
g_- g_+ = g_+^{g_-} g'_-
\]

where \( g'_- \in G_0^* \) provided by that the multiplication map \( G_0 \times G_0^* \to G_\mathbb{R} \) is bijective.

In general, \( G_0G_0^* \) is not even dense in \( G_\mathbb{R} \). However, there is still the following fact.

**Proposition 2.3.** The multiplication map \( (SU(1,1) \times \mathbb{Z}_2) \times SU(1,1)^* \to SL_2(\mathbb{C}) \) is injective and its image is dense in \( SL_2(\mathbb{C}) \).

In what follows we denote \( G_0 = SU(1,1), \ G = SL_2(\mathbb{C}), \ B \subset SL_2(\mathbb{C}) \) is the Borel subgroup of upper-triangular matrices. We consider below only the case of the left quantum adjoint action and the right dressing action. Another case can be considered analogously and does not contain anything new.

Since \( B \) is a Poisson Lie subgroup of \( G_\mathbb{R} \), the flag manifold \( G_\mathbb{R}/B \) is endowed with a Poisson manifold structure, and the local right dressing action of \( G_0^* \subset G_\mathbb{R}^* \) on \( G_\mathbb{R} \) induces a local action on \( G_\mathbb{R}/B \) which we call also right dressing action.

Note that, since \( G_\mathbb{R}/B \) is compact, this local action can be extended to a global one. The \( G_0^* \)-orbits of this action are the Schubert cells \( C_1 = \{ \infty \} \) and \( C_w = CP^1 \setminus \{ \infty \} \) parameterized, as is well known, by the Weyl group \( W \), \( G_0^* \) acting on \( C_w \approx \mathbb{C} \) by translations and dilations.

If \( G_0 \) were \( SU(2) \), the corresponding \( G_0^* \)-orbits would be the same. However, while in this case they are symplectic leaves, if \( G_0 \) is \( SU(1,1) \), this is not the case.

Indeed, \( G_0G_0^* / B \approx \mathcal{D}_+ \cup \{ \infty \} \) and \( G_0wG_0^* / B \approx \mathcal{D}_- \) are Poisson submanifolds of \( G_\mathbb{R}/B \). It is easy to see that the symplectic leaves of \( G_\mathbb{R}/B \approx CP^1 \) are \( \{ \infty \}, \mathcal{D}_+, \mathcal{D}_- \) and each point of \( S^1 = \partial \mathcal{D}_\pm \).

Let us call a symplectic leaf of a Schubert cell non-degenerate if its dimension is the same as the dimension of the cell. Recall the parameterization of irreducible \( * \)-representations of \( \mathfrak{g} \) by quadruples \( (t,C,\Sigma,\beta) \) where \( t \in T_0, C \) is a Schubert cell.
of the flag manifold $G_{\mathbb{R}}/B$, $\Sigma$ is a non-degenerate symplectic leaf of $C$, $\beta$ is a unitary character of the fundamental group $\pi_1(\Sigma)$ of $\Sigma$. Thus, the one-dimensional $\ast$-representations $\zeta_\varphi$ correspond to the leaf $\{\infty\}$, the infinite-dimensional ones $\pi^\pm_\varphi$ to the leaf $\mathcal{D}_\pm$.

As follows from the geometric realization considered in Sections 4 and 5, the quasi-classical analogues of the tensor products $V_{\zeta_\varphi} \otimes V_{\zeta_\varphi}$, $V^{+, (\beta)}_\varphi \otimes V^{+, (\beta)}_{\varphi^+}$ and $V^\varphi_\varphi \otimes V^\varphi_{\varphi^+}$ of unitarizable $\mathfrak{g}_+$-modules is the local right dressing action of $SU(1,1)^*$ on the symplectic leaves $\{\infty\}$, $\mathcal{D}_+$ and $\mathcal{D}_-$ respectively, while the quasi-classical analogues of the super-tensor products $\zeta_\varphi \otimes \zeta_\varphi$ and $\pi^{+, (\beta)}_{\varphi_1} \otimes \pi^{+, (\beta)}_{\varphi^+} \oplus \pi^-_{\varphi_2} \otimes \pi^-_{\varphi^+}$ is the global right dressing action of $SU(1,1)^*$ on the Schubert cells $\{\infty\}$ and $\mathcal{C}$ respectively. The negative result of Theorem 2.4 corresponds to the obvious fact that the local action of $SU(1,1)^*$ on $\mathcal{D}_\pm$ by translations and dilations cannot be extended to a global one.
Appendix A

Module *-Algebras

Throughout the thesis we deal constantly with the $\mathfrak{A}$-module *-algebras, where $\mathfrak{A}$ is a Hopf *-algebra. Here we give the necessary definitions and a brief account of their properties. As well as remind the important definition of the adjoint action of a Hopf algebra on itself.

**Definition 3.1.** (1) Suppose that $\mathcal{F}$ is both a *-algebra and a left (resp. right) $\mathfrak{A}_0$-module. Then, $\mathcal{F}$ is called left (resp. right) $\mathfrak{A}_0$-module *-algebra if

- the multiplication map $m : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$ is an $\mathfrak{A}_0$-module morphism,
- The *-structures in $\mathfrak{A}_0$ and $\mathcal{F}$ are compatible in the following sense:

$$ (a.f)^* = \bar{\omega}(a).f^* \quad (A.1) $$

for each $a \in \mathfrak{A}_0$, $f \in \mathcal{F}$, where $\bar{\omega}(a) = (S(a))^*$.

(2) Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are left (resp. right) $\mathfrak{A}_0$-module algebras. A map $A : \mathcal{F}_1 \to \mathcal{F}_2$ is called left (resp. right) $\mathfrak{A}_0$-module *-algebra morphism if $A$ is both an algebra *-homomorphism and a left (right) $\mathfrak{A}_0$-module morphism.

**Remark 3.1.** Suppose that $\mathfrak{A}$ is a Hopf algebra, $\mathcal{F}$ is both an algebra and a left (right) $\mathfrak{A}$-module. The definition of left (right) $\mathfrak{A}$-module algebra can be obtained by forgetting about the *-structure and the condition (A.1).
The chapter about the super-tensor products uses also the auxiliary concept of a 'shadow' of an \( \mathfrak{A} \)-module \( * \)-algebra. The definition follows.

**Definition 3.2.** (1) Let \( \mathcal{F} \) be a left (resp. right) \( \mathfrak{A} \)-module algebra, \( (\mathcal{F}, \ast) \) a left (resp. right) \( \mathfrak{A}_0 \)-module \( * \)-algebra. If a \( * \)-algebra \( (\mathcal{F}, \sharp) \) equipped with the action of \( \mathfrak{A} \) on \( \mathcal{F} \) becomes a left (right) \( \mathfrak{A}_0 \)-module \( * \)-algebra, we will call it left (right) pre-semishadow of \( (\mathcal{F}, \ast) \).

Two left (right) pre-semishadows are called *equivalent* if they are equivalent as left (right) \( \mathfrak{A}_0 \)-module \( * \)-algebras.

(2) A left (right) pre-semishadow \( (\mathcal{F}, \sharp) \) of \( (\mathcal{F}, \ast) \) is called left (right) semishadow if there exists at least one unitarizable left (right) \( (\mathcal{F}, \sharp) \)-module.

**Definition 3.3.** Let \( (\mathcal{F}, \ast) \) be a \( \mathfrak{A}_0 \)-bimodule \( * \)-algebra. If a \( * \)-algebra \( (\mathcal{F}, \sharp) \) equipped with the left and right actions of \( \mathfrak{A}_0 \) is both a left and a right semishadow of \( (\mathcal{F}, \ast) \), we will call it *shadow* of \( (\mathcal{F}, \ast) \).

One of the examples of \( \mathfrak{A}_0 \)-module \( * \)-algebra is the algebra of functions on (quantum) real \( G_0 \)-spaces where \( G_0 \) is a real Lie group.

For any Hopf algebra \( \mathfrak{A} \), consider the so-called *restricted dual* Hopf algebra \( \mathfrak{A}^* \) generated by the linear functionals on \( \mathfrak{A} \) such that for any \( \xi \in \mathfrak{A}^* \), there exists a two-sided ideal \( J_\xi \subseteq \mathfrak{A} \) annihilated by \( \xi \). The following lemma is well known.

**Lemma 3.1.** For any Hopf \( * \)-algebra \( \mathfrak{A}_0 \), \( \mathfrak{A}_0^* \) is also a Hopf \( * \)-algebra, with the canonical structure given by

\[
\langle \xi \eta, a \rangle = \langle \xi \otimes \eta, \Delta(a) \rangle, \quad \langle \Delta(\xi), a \otimes b \rangle = \langle \xi, ab \rangle \\
\langle S(\xi), a \rangle = \langle \xi, S(a) \rangle, \quad \varepsilon(\xi) = \langle \xi, 1 \rangle \\
\langle \xi^*, a \rangle = \langle \xi, \overline{\omega(a)} \rangle, \quad \overline{\omega}(\xi), a) = \overline{\langle \xi, a^* \rangle},
\]

(A.2)

where \( a, b \in \mathfrak{A}_0, \xi, \eta \in \mathfrak{A}_0^*, \langle , \rangle : \mathfrak{A}_0^* \otimes \mathfrak{A}_0 \to \mathbb{C} \) is the natural pairing.

Under the assumptions of Lemma 3.1, there exist canonical left and right \( \mathfrak{A}_0^* \)-module
*-algebra structures on $\mathfrak{a}_0$ given by the well known right regular representation and the left regular antirepresentations respectively:

$$\mathcal{R}(\xi)a = \langle \text{id} \otimes \xi, \Delta(a) \rangle, \quad \mathcal{L}(\xi)a = \langle \xi \otimes \text{id}, \Delta(a) \rangle$$

where $\xi \in \mathfrak{a}_0^*, a \in \mathfrak{a}_0$. When $\mathfrak{a}_0 = C[G_0]_q$ is the quantum algebra of regular functions, $\mathcal{R}$ and $\mathcal{L}$ are the quantum analogues of the right and left regular representations of $G_0$ in functions on $G_0$ respectively.

Remark 3.2. Note that the left or right semishadows of $C[G_0]_q$ with respect to $\mathcal{R}$ or $\mathcal{L}$ are easily seen to be the quantum analogues of the algebras of regular functions on Poisson right cosets $gG_0 \subset G_\mathbb{R}$ or left ones $G_0g$ (where $g \in G_\mathbb{R}$) respectively.

Therefore, if we consider $C[G_0]_q$ as the corresponding $U_q G_0$-bimodule $*$-algebra, its shadows correspond to the connected components of the Poisson Lie group $N_G(G_0)$, the normalizer of $G_0$ in $G_\mathbb{R}$.

The following example of $\mathfrak{a}_0$-module $*$-algebra is of a particular interest for us. The proposition given below is a direct consequence of the definition of Hopf $*$-algebra.

**Proposition 3.1.** Let $\mathfrak{a}_0$ be a Hopf $*$-algebra. The left and right quantum adjoint actions $ad_q$ and $ad'_q$ of $\mathfrak{a}_0$ on itself given by

$$ad_q(a)b = \sum_k a'_k b S(a''_k), \quad ad'_q(a)b = \sum_k S(a'_k) ba''_k$$

whenever $\Delta(a) = \sum_k a'_k \otimes a''_k$ ($a, b \in \mathfrak{a}_0$) equip $\mathfrak{a}_0$ with an $\mathfrak{a}_0$-bimodule $*$-algebra structure.

This case is considered in the present paper for quantum algebras of regular functions on real quantum groups. Some simple examples follow.

**Example 3.1.** Recall that we denoted $C[SL_2(\mathbb{C})]_q$ by $\mathfrak{a}$ for short, and the involution which makes it into $C[SU(1,1)]_q$ by $\ast$. It is easy to show that $\mathfrak{a}_+ = (\mathfrak{a}, \ast)$ and $\mathfrak{a}_- = (\mathfrak{a}, \#)$ are the only pre-shadows (and shadows) of $\mathfrak{a}_+$, and that they are not
Example 3.2. Consider $\mathbb{C}[SU(2)]_q = (\mathfrak{g}, \#)$ where $\mathfrak{g}$ is given by

$$t_{11}^\mathfrak{g} = t_{22}, \quad t_{12}^\mathfrak{g} = -qt_{21}$$

It is easy to see that the only pre-shadows of $(\mathfrak{g}, \#)$ are $(\mathfrak{g}, \#)$ and $(\mathfrak{g}, b)$ where $b$ is given by

$$t_{11}^b = -t_{11}^\mathfrak{g} = -t_{22}, \quad t_{12}^b = -t_{12}^\mathfrak{g} = qt_{21}$$

However, $(\mathfrak{g}, b)$ is not a shadow because of

$$t_{11}^b t_{11} + t_{21}^b t_{21} = -1 \quad \text{(A.3)}$$

We will see that when $G_0$ is compact, $\mathbb{C}[G_0]_q$ has only one shadow, namely, itself.

Denote by $\mathcal{T}$ the subgroup of non-zero group-like elements of $\mathfrak{a}^*$, that is, $\mathcal{T} = \{ t \in \mathfrak{a}^* \mid t \neq 0, \Delta(t) = t \otimes t \}$. Denote $\mathcal{T}_0$ the subgroup of non-zero Hermitian group-like elements of $\mathfrak{a}_0^*$, that is, $\mathcal{T}_0 = \{ t \in \mathcal{T} \mid t^* = t \}$.

**Theorem 3.1.** (1) $(\mathfrak{g}, \#)$ is a left or right pre-semishadow of $\mathfrak{a}_0 = (\mathfrak{a}, \ast)$ if and only if there exists $t_0 \in \mathcal{T}_0$ such that

$$\# = \ast \circ R(t_0) \quad \text{or} \quad \# = \ast \circ L(t_0)$$

respectively.

(2) Left or right pre-semishadows $(\mathfrak{g}, \#)$ and $(\mathfrak{g}, \#')$ are equivalent if and only if there exists $t \in \mathcal{T}$ such that

$$\# = R(t) \circ \#' \circ R(t^{-1}) \quad \text{or} \quad \# = L(t) \circ \#' \circ L(t^{-1})$$

respectively.

(3) $(\mathfrak{g}, \#)$ is a left (or right) pre-semishadow of $\mathfrak{a}_0$ with respect to $ad_q$ (or $ad'_q$) if and only if it is a right pre-semishadow with respect to $L$ (or a left one with respect
to \( R \).

The theorem immediately follows from the following easy lemma.

**Lemma 3.2.** Let \( A \) be a Hopf algebra, and suppose that \( A \) is equipped with the left or right \( A \)-module algebra structure given by \( \text{ad}_q \) or \( \text{ad}'_q \) respectively. A linear map \( \gamma : A \to A \) is a left or right \( A \)-module algebra automorphism if and only if there exists \( t \in T \) such that \( \gamma = \mathcal{R}(t) \) or \( \gamma = \mathcal{L}(t) \) respectively.

**Theorem 3.2.** The left pre-semishadow \((A, \bowtie)\) is a left semishadow if and only if there exists \( u \in A_0^* \) such that \( t = \omega(u^{-1})u \), \( c(u) = 1 \).

**Corollary 3.1.** \((A, \bowtie)\) is a shadow if and only if there exists \( u \in A_0^* \) such that \( \omega(u^{-1})u \in Z_0 \) where \( Z_0 \) is the intersection of \( T_0 \) and the center of \( A_0^* \), and

\[
\bowtie = \mathcal{R}(u^{-1}) \circ \circ \mathcal{R}(u) = \\
= \mathcal{L}(u^{-1}) \circ \circ \mathcal{L}(u)
\]

\[\text{For right semishadows the condition looks as follows: } t = u\omega(u^{-1}).\]
Appendix B

Decomposition of the Super-Tensor Products

This appendix is devoted to the proof of Theorem 2.8. Of course, (3.5) does not require a special consideration. We prove (3.6) below, (3.7) can be proven analogously and even much simpler.

Denote in short
\[\pi \overset{\text{def}}{=} \pi_{\psi_1}^+ \otimes \pi_{\psi_2}^+ \oplus \pi_{\psi_3}^- \otimes \pi_{\psi_4}^-\]  \hspace{1cm} (B.1)

and consider the self-adjoint operator \(\pi(x)\). It is clear that what we really need to obtain the decomposition of \(\pi\) is to know the spectrum of \(\pi(x)\). The subspace \(L_m\) (\(m \in \mathbb{Z}\)) generated by \(e_{k-m}^\pm \otimes e_k^\pm\) \((k \in \mathbb{Z})\) is easily seen to be \(\pi(x)\)-invariant. Let \(\pi(x)_m\) be the restriction of \(\pi(x)\) to \(L_m\).

It is easy to show that \(\pi(x) \geq 0, \text{Ker } \pi(x) = \{0\}\). The unitary operator \(u = q^{\frac{1}{2}} \pi(t_{12})\pi(x)^{-\frac{1}{2}}\) intertwines \(\pi(x)_m\) and \(\pi(x)_{m+1}\), therefore, all the self-adjoint operators \(\pi(x)_m\) \((m \in \mathbb{Z})\) are unitarily equivalent.

Consider, for instance, \(\pi(x)_0\). It is unitarily equivalent to the operator \(A\) in \(L^2(\mathfrak{M}^{(-\beta)} , d\nu)\) given by the second order \(q\)-difference expression (6.1) and the boundary condition (3.3).

First of all, it is clear that the spectrum of \(A\) is \textit{simple}. It is clear also that it is a union of some geometric progressions with ratio \(q^2\) because of \(t_{11}x = q^2x_{11}, t_{22}x = \)
It is easy to show that $\pi$ can be decomposed into a direct integral of the irreducible representations of the form $\pi^{+,(\beta')}_{\psi}$ (no one-dimensional ones, since $\text{Ker}A = \{0\}$). That which geometric progressions comprise the spectrum of $A$ indicates the possible values of $\beta'$ for the irreducible components (note that if $\pi^{+,(\beta')}_{\psi}$ occurs in the decomposition for some $\beta'$ and $\psi$, this is the case for the same $\beta'$ and all values of $\psi$).

The following proposition immediately implies (3.6).

**Proposition 4.1.** The spectrum of $A$ is simple and equal to $\mathfrak{m}_+^{(\beta)} \cup \{0\}$ (where $\beta$ is the same as fixed in (B.1)), $\text{Ker}A = \{0\}$.

**Proof.** The standard way to prove it is to study the asymptotic behavior of eigenfunctions of $A$ at infinity. Consider the function

$$f_{\lambda}(t) = \Phi_1\left(\frac{-qt^{-1}}{q^2}; q^2, q^2\lambda t\right) \quad (B.2)$$

where $(a)_k = (1 - a)(1 - aq^2)...(1 - aq^{2(k-1)})$ and

$$\Phi_1\left(\frac{a}{b}; q^2, t\right) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)}(a)_k x^k}{(b)_k (q^2)_k}$$

is a basic hypergeometric function.

The function $f_{\lambda}(t)$ generates the one-dimensional space of solutions of

$$-(q^{-1} - q)^2 \cdot (D(1 + t)Df)(t) = \lambda f(t)$$

$$f(+0) = f(-0), \quad (Df)(+0) = (Df)(-0)$$

Since $A \geq 0$ and $\text{Ker}A = \{0\}$, we can suppose $\lambda > 0$.

We compare $A$ with some operator $A_0$ with known spectrum. The operator $A_0$ is given in $L^2(\mathfrak{m}_+^{(-\beta)}, d\nu_+^{(-\beta)})$ by the second order $q$-difference expression

$$A_0 = -(q^{-1} - q)^2 \cdot DtD$$

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and the boundary condition

\[ \lim_{k \to +\infty} (f(q^{2(k-\beta)}) - f(q^{2(k-\beta+1)})) = 0 \]  \hspace{1cm} \text{(B.3)}

The eigen-functions of \( A_0 \) are the so-called zero index Hahn-Exton q-Bessel functions (introduced in [3])

\[ f_\lambda^{(0)}(t) = \Phi_1 \left( \frac{0}{q^2}; q^2, q^2\lambda t \right) \]

and the spectrum of \( A_0 \) is simple and equal to \( m_+^{(\beta)} \cup \{0\}, \text{Ker}A_0 = \{0\} \).

This was announced in [12] where the operator \( A_0 \) appeared in harmonic analysis on the quantum group \( M(2) \) of the motions of the plane (note that the same operator appears also in the problem of decomposition of tensor products of irreducible \(*\)-representations of the quantum algebra of regular functions).

The proof which was not included in [12] is based on the fact that, when \( \beta = \frac{1}{2}, \) \( A_0 \) can be approximated by some simpler operators \( A_0^{(m)} = q^{2m}T^{-m}A_0^{(0)}T^m \) in the sense that the operators \( A_0 - A_0^{(m)} \) are bounded and converge to zero as \( m \to +\infty \). The operator \( T \) is the shift \((Tf)(t) = f(q^2t)\) and the operator \( A_0^{(0)} \) given in \( L^2(\{q^{2k+1}\}_{k \geq -m}, d\nu_+) \) by the second order q-difference expression

\[ A_0^{(0)} = -(q^{-1} - q)^2 \cdot Dt(1-t)D \]

and the boundary condition (B.3) appeared in [27] in harmonic analysis on the quantum group \( SU(2) \).

The spectrum of \( A_0^{(0)} \) is well-known. It is simple and consists of the values

\[ \lambda_l = \frac{(q^{-l} - q^{-l+1})(q^{-l+1} - q^{-l-1})}{(q^{-l} - q^{-l-1})(q^2 - 1)} \]  \hspace{1cm} (\( l \in \frac{1}{2} \mathbb{Z} \)) of the quadratic Casimir element in the finite-dimensional irreducible \(*\)-representations of \( U_q\mathfrak{su}(2) \). Therefore, we know the spectra of the operators \( A_0^{(m)} \), hence the spectrum of \( A_0 \). Note that, if we know it for just one value of \( \beta \), it is easy to obtain it for all values of \( \beta \).

The following lemma can be proved by some straightforward calculations. The
convergence is always provided by terms of the form $q^{k^2}$.

**Lemma 4.1.** The series in the right hand side of the following identity converges to the left hand sides absolutely, uniformly in compacta:

$$\Phi_1^1\left(\begin{array}{c} a \\ b \end{array} ; q^2, x \right) = \sum_{k=0}^{\infty} \frac{q^{2(k-1)}(ax)^k}{(b)(q^2)_k} \Phi_1^1\left(\begin{array}{c} 0 \\ q^{2k}b \end{array} ; q^2, q^{2k}x \right)$$

(B.4)

**Corollary 4.1.** Put $a = -qt^{-1}, b = q^2, x = q^2 \lambda t$ into (B.3). We get

$$\Phi_1^1\left(\begin{array}{c} -qt^{-1} \\ q^2 \end{array} ; q^2, q^2 \lambda t \right) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(2k+1)} \lambda^k}{(q^2)_k^2} \times$$

$$\times \Phi_1^1\left(\begin{array}{c} 0 \\ q^{2(k+1)} \end{array} ; q^2, q^{2(k+1)} \lambda t \right)$$

Consider the operator $B : f(t) \mapsto \frac{\int(t) - \int(q^2t)}{t}$.

**Lemma 4.2.** The right hand side of the following identity converges to the left hand side absolutely, uniformly in compacta:

$$\Phi_1^1\left(\begin{array}{c} -\sqrt{qt^{-1}} \\ q^2 \end{array} ; q^2, q^2 \lambda t \right) = (-qB)_{\Phi_1^1}\left(\begin{array}{c} 0 \\ q^2 \end{array} ; q^2, q^2 \lambda t \right)$$

**Proof.** By the recurrence formula (cf. [3])

$$B : \Phi_1^1\left(\begin{array}{c} 0 \\ b \end{array} ; q^2, cx \right) \mapsto -\frac{c}{1-b} \Phi_1^1\left(\begin{array}{c} 0 \\ q^2b \end{array} ; q^2, q^2cx \right)$$

we obtain from (B.4) that

$$\Phi_1^1\left(\begin{array}{c} -qt^{-1} \\ q^2 \end{array} ; q^2, q^2 \lambda t \right) = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q^2)_k} B^k \Phi_1^1\left(\begin{array}{c} 0 \\ q^2 \end{array} ; q^2, q^2 \lambda t \right) =$$
Proof of Prop.A.1 (continued). Suppose $\lambda \in \mathfrak{m}_+^{(\beta)}$. By [12, Prop.11], $f^{(0)}_\lambda(t)$ vanishes faster than $t^{-n}$ for any $n \in \mathbb{N}$ as $t \to +\infty$, $t \in \mathfrak{m}_+^{(-\beta)}$. Note that Prop.11 in [12] is a simple consequence of the orthogonality relation for the matrix elements of irreducible $*$-representations of $U_q \mathfrak{sl}(2)$ which are expressed in terms of the Hahn-Exton $q$-Bessel functions and the recurrence formulae for these functions (cf. [3]).

It follows that the functions

$$f^{(k)}_\lambda(t) \overset{def}{=} (-qB)^k f^{(0)}_\lambda(t) \quad (k \in \mathbb{Z}_+)$$

also vanish faster than $t^{-n}$ for any $n \in \mathbb{N}$ as $t \to +\infty$, $t \in \mathfrak{m}_+^{(-\beta)}$.

Note that all the above lemmas remain valid when both sides of the identities are multiplied by $t^n$ for any $n \in \mathbb{N}$. Therefore, $t^n f^{(k)}_\lambda(t)$ converge to $t^n f_\lambda(t)$ for any $n \in \mathbb{N}$ absolutely, uniformly in compacta as $k \to +\infty$. Hence, $f_\lambda(t)$ also vanishes faster than $t^{-n}$ for any $n \in \mathbb{N}$ as $t \to +\infty$, $t \in \mathfrak{m}_+^{(-\beta)}$.

This implies that each point of the geometric progression $\mathfrak{m}_+^{(\beta)}$ is an eigen-value of $A$. To show that $\mathfrak{m}_+^{(\beta)} \cup \{0\}$ is the whole spectrum of $A$ note that, since for each eigen-value $\lambda$ of $A_0$ the corresponding eigen-function $f^{(0)}_\lambda$ vanishes faster than $t^{-n}$ for any $n \in \mathbb{N}$ as $t \to +\infty$, $t \in \mathfrak{m}_+^{(-\beta)}$, for any $\lambda'$ which does not belong to the spectrum of $A_0$ the corresponding eigen-function $f^{(0)}_\lambda$ grows faster than $t^n$ for any $n \in \mathbb{N}$ as $t \to +\infty$, $t \in \mathfrak{m}_+^{(-\beta)}$.

It follows in a similar way that, for each $\lambda' \notin \mathfrak{m}_+^{(\beta)} \cup \{0\}$, the corresponding eigen-function $f_{\lambda'}(t)$ of $A$ grows faster than any polynomial as $t \to +\infty$, $t \in \mathfrak{m}_+^{(-\beta)}$. This proves Prop.A.1.

In the conclusion, I would like to note that, according to (3.6) and (3.7), one can define the *Clebsch-Gordan coefficients for quantum algebra of functions* ("even" and
"odd") as follows:

\[
e_{m}^{\pm}(\varphi_{1}) \otimes e_{n}^{\pm}(\varphi_{2}) = \sum_{k \in \mathbb{Z}} \int_{0}^{2\pi} \left[ \begin{array}{ccc} \varphi_{1} & \varphi_{2} & \varphi \\ m & n & k \end{array} \right]_{q,\beta}^{(\pm)} e_{k}^{\pm}(\varphi) d\varphi
\]

As follows from (3.7), (B.2) and (3.2), the "even" Clebsch-Gordan coefficients are expressed in terms of the functions

\[
_{1} \Phi_{1} \left( \begin{array}{c} -qt^{-1} \\ q^{2} q^{2(k+1)} \lambda t \end{array} ; q^{2}, q^{2(k+1)} \lambda t \right) = \text{const} \cdot (B^{k} \phi)(t)
\]

where \( e_{k}^{\pm}(\varphi) \) stands for the canonical basis of \( V_{\varphi}^{\pm} \) given by (3.5), (3.6).

As follows from (3.7), (B.2) and (3.2), the "even" Clebsch-Gordan coefficients are expressed in terms of the functions

\[
_{1} \Phi_{1} \left( \begin{array}{c} -qt^{-1} \\ q^{2} q^{2(k+1)} \lambda t \end{array} ; q^{2}, q^{2(k+1)} \lambda t \right) = \text{const} \cdot (B^{k} \phi)(t)
\]

\( (k \in \mathbb{Z}) \) where \( \lambda \in \mathfrak{m}_{\beta}^{(k+1)} \), \( t \in \mathfrak{m}_{\beta}^{(-k)} \). This follows, by the way, that the Clebsch-Gordan coefficients do depend on \( \beta \) (see the remark at the end of Sect.6).

As was shown in [13, 14, 21], the matrix elements \( t_{ij}^{e} \in \mathbb{C}[SU(1, 1)]_{q}^{*} \) of irreducible \(*\)-representations of \( U_{q}su(1, 1) \) are expressed in terms of the \( q \)-Jacobi functions. As far as we know precise expression of the Clebsch-Gordan coefficients for quantum algebra of functions, we can apply the technique of [7] to obtain an addition formula for the \( q \)-Jacobi functions. Namely, one should apply the operator given by the action of

\[
\Delta(t_{ij}^{e}) = \sum_{k} t_{ik}^{e} \otimes t_{kj}^{e}
\]

to the right hand sides and the left hand sides of (B.5). Therefore, the functions (B.6) appear in that addition formula which involves the "even" Clebsch-Gordan coefficients.
Bibliography


