Price of Anarchy in a Bertrand Oligopoly Market

by

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Abstract

The price of anarchy quantifies the inefficiency that occurs in the total system objective in the user optimization as compared to the system optimization setting. It is well known that this inefficiency occurs due to lack of coordination among the competitors in the system. In this thesis, we study the price of anarchy in a Bertrand oligopoly market by comparing the total profits in the two settings. The main contribution of this thesis is a lower and an upper bound for the price of anarchy that only depends on the price sensitivity matrix characterizing the demand sellers face. We first derive these bounds for a symmetric affine demand model. Using the same approach, we also provide a lower bound for asymmetric affine demand as well as a lower and an upper bound for nonlinear demand. These bounds are easy to compute. In addition, we illustrate that the worst-case price of anarchy value occurs for a uniform demand model when quality differences do not exist among sellers. This implies that in many real-world instances where quality differences exist, the performance under the user optimization may in fact be close to what is achieved under system optimization. We illustrate several insights on the bounds we present through simulations.

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Chapter 1

Introduction

1.1 Motivation

The use of central coordination to achieve a system-wide optimal objective is seldom feasible. It is usually unacceptable as users may not have an incentive to comply with the central directives. Although classic problems in operations research assume that there is a central authority that has the power to control the system, recently there has been a trend to acknowledge this difficulty, understand its consequences, and design systems that achieve coordination by other means. These models invariably include economic and game-theoretic aspects as a way to model user behavior. Some examples that have been or can be modeled from that perspective include the Internet, wireless networks, road traffic networks, transit networks, evacuation systems, distribution systems, auctions, and facility location problems, just to mention a few.

The price of anarchy has been defined in the seminal work by Koutsoupias and Papadimitriou [18] as a measure of the extent to which competition approximates cooperation. In general, it is the worst-case ratio between the value of the user optimum and that of the social optimum objectives. Usually, the equilibrium state has been taken to be that of a Nash equilibrium [22] - a state in which no user wishes to deviate unilaterally from its own strategy in order to improve the value of its private objective.
In this research project, we study the price of anarchy in a Bertrand oligopoly market [3] where sellers compete through prices. It is different from Cournot competition [8] as in the latter, sellers compete through quantities. The comparison between the two types of competitions has been discussed in Farahat and Perakis [11], [12]. In our model, there is a demand function (representing the customers in an aggregate format) that depends only on the prices set by the sellers. We show the existence and uniqueness of pricing policies, and evaluate the price of anarchy of such a market for affine symmetric and asymmetric demand functions. In particular, we discuss several special classes of demand functions and propose a lower and an upper bound under some assumptions. We extend the same approach to a nonlinear demand model to derive a lower bound and an upper bound in order to quantify the inefficiency between user and system optimum. Finally, we discuss some insights on the bounds we present using simulation.

1.2 Literature Review

There is rich literature in economics on price and quantity competition, including the seminal models by Bertrand and Cournot ([3], [8]) mentioned in the previous section. Kirman and Sobel [17] develop a multi-period model of oligopoly where a set of competing firms decide in each period the price and the production level in the face of random demand. They show the existence of equilibrium price and quantity strategies for the firm. Rosen [26] proves the existence and uniqueness results for general oligopolistic games. He shows existence of the payoff function of a seller under concavity with respect to his own strategy space and convexity of the joint strategy space. He also establishes uniqueness under strict diagonal dominance of the Hessian matrix of the payoff functions. Murphy et al. [21] analyze equilibrium strategies in a single-period quantity competition model using mathematical programming results.

It has been well-known that the presence of competition in various settings intro-
duces inefficiencies in the system (that is, it decreases the overall system profit). The concept was first proposed by Koutsoupias and Papadimitriou [18] for a particular telecommunication network model, and has received considerable attention ever since. Papadimitriou [23] coined the term "price of anarchy" to measure the ratio of the performance of the decentralized system over the worst performance of the decentralized system. Dafermos and Sparrow [10] used the terms "user-optimization" and "system-optimization" in order to distinguish between Nash equilibrium when users act unilaterally in their own self interest and when users are forced to select the routes that optimize the total network efficiency. The recent book by Roughgarden [29] provides an extensive coverage of results on the price of anarchy.

Many studies have shown that the loss of efficiency from decentralization depends on the nature of the game (i.e., atomic or non-atomic). In atomic games, a common resource is shared among a finite number of players, each using a non-infinitesimal amount of it (e.g., see Koutsoupias and Papadimitriou [18], Cominetti, Correa and Stier-Moses [6]). In non-atomic games, the common resource is shared among an infinite number of players, each using only an infinitesimal amount of it. One of the most famous results in non-atomic games is the price of anarchy bound $\frac{4}{3}$ in transportation networks with linear travel costs, derived by Roughgarden and Tardos [28], [29], and extended to separable travel time functions for capacitated networks as described in Correa, Schulz and Stier-Moses [7]. Perakis [24] generalizes these results to asymmetric and non-separable cost functions. Other price of anarchy results are obtained in network resource allocation games (see Johari and Tsitsiklis [16]) and network pricing games (see Acemoglu and Ozdaglar [1]).

Although the concept of price of anarchy has been used in the context of networks primarily in the literature, those problems are non-atomic in nature (Cominetti, Correa, and Stier-Moses [6] is an exception). The problem we consider in this thesis differs from this body of literature in that it is an atomic game where players maximize payoffs and every seller has market power. Thus, the direct application of results obtained in the non-atomic game literature to the pricing problem is not appropriate.
Besides the pricing problem we discussed above, another example of atomic games arises in the area of supply chains. The literature on competitive pricing for supply chains is surveyed in Chan et al [5], where they quantify the inefficiency by comparing the variance of orders with the variance of demand. Perakis and Roels [25] analyze different supply chain configurations and compute the price of anarchy between the integrated supply chain and the decentralized supply chain. Focusing on the effect of competition, Martinez-de-Albeniz and Simchi-Levi [19] compute a price of anarchy of $\frac{4}{3}$ in a procurement game with option contracts. It is crucial to note that the cited literature mostly compares efficiencies for cost minimizing games. In this thesis, we have a profit maximizing game instead. In addition, we consider non-separable functions as the demand functions also depend on the pricing strategies of the competitors.

1.3 Thesis Outline and Main Contribution

This section presents an outline of this work and an overview of the main contributions. The thesis is divided into 7 chapters, including the current one and the conclusion at the end. It is advised to read them in order because all of them draw on concepts from earlier chapters. We explicitly point out such relations in the chapter-by-chapter discussion that follows.

Chapter 2 offers a review of the central concepts needed for the subsequent chapters. We present the notation, fundamental concepts such as the system optimum and the user equilibrium, and well-known results related to those concepts. Next, we list the assumptions imposed on our analysis and explain the reasons behind them. In the following sections, we review the central concept of the price of anarchy, which is used to measure the ratio of the total revenue generated under user optimization and under system optimization. For completeness, in the final section, we also show the existence and uniqueness of solutions for the user and system optimization problems.

In Chapter 3, we evaluate the closed-form solution for the price of anarchy using the affine demand model. We apply the first order optimality conditions to obtain the
user and system optimal solutions respectively. Notice that the closed-form solutions can be applied to both symmetric and asymmetric affine demand models. In the last section of this chapter, we compare the optimal prices of the user and system optimization and prove that higher prices are charged in the system optimization system.

In Chapter 4, we evaluate the closed-form solution for the price of anarchy using the general uniform demand model. We first obtain the price of anarchy when there are no quality differences among the sellers, that is, the values of \( d_i \) are the same, and denote this quantity by \( POA_u \). Next, we obtain the closed-form solution for the price of anarchy when quality differences exist, i.e., \( d = (d_1, d_2, \ldots, d_n) \), where values of \( d_i \) differ from each other. We show that \( POA_u \) is the worst-case bound. In addition, we show that the price of anarchy increases when the coefficient of variation and/or the number of sellers in the market increases.

Chapter 5 presents the most important findings in this thesis, a lower and an upper bound for the price of anarchy. We also include some simulation results to show the tightness of the bounds. It is important to note that the bounds are valid irrespective of the quality differences among sellers. In the last section of the chapter, we show that the argument can also be applied to the asymmetric demand model by introducing some constants that measure the degree of asymmetry of the price sensitivity matrix.

The price of anarchy for the nonlinear model is presented in Chapter 6. We derive a lower and an upper bound for the price of anarchy using a similar argument as in the affine demand model in Chapter 5.

Finally, Chapter 7 concludes the thesis and discusses the future work that can be extended from the current model.

The results presented in this thesis can also be found in Farahat, Perakis and Sun [13].
Chapter 2

Model Description

In this chapter we lay the groundwork for this thesis. Section 2.1 presents the specifics of the model together with the notation. Next, we describe the assumptions imposed on our analysis. We formally introduce the user and system optimization problems in Section 2.3. In the final section, we also show the existence and uniqueness of the user and the system optimization problems.

2.1 Notation and Terminology

This section introduces the concepts and notation that will be used throughout this thesis. We use the same notations, assumptions and setup as in Farahat and Perakis [11], [12].

Consider an oligopoly market consisting of $n$ sellers offering substitutable differentiated products. We denote $I$ the set of sellers. A single seller is denoted by $i \in I$. We denote the set of all competitors of $i$ by $-i$. The price set by seller $i$ is denoted by $p_i$. The pricing policy variables for the entire set of sellers are represented by an $n$-dimension vector $p = (p_1, p_2, \ldots, p_n)$.

For the affine demand model, seller $i$'s share of demand is denoted by $q_i = d_i -$
\[ a_i p_i + \beta_{-i} \bar{p}_{-i}, \text{ where } \alpha_i \text{ and } \beta_{-i} \text{ are positive price sensitivity indicators with respect to seller } i's \text{ own prices and that of his competitors } -i \text{ respectively. } \]

We define the price sensitivity matrix \( B \) as follows.

\[
B = \begin{bmatrix}
\alpha_1 & -\beta_{1,2} & \ldots & \ldots & -\beta_{1,n} \\
-\beta_{2,1} & \alpha_2 & \ldots & \ldots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
-\beta_{n,1} & \ddots & \ddots & \alpha_n & -\beta_{n-1,n}
\end{bmatrix}
\]

The price sensitivity ratio for seller \( i \) is defined as \( r_i = \sum_{j \neq i} \frac{\beta_{ij}}{\alpha_i} \).

\( Z_{\text{UO}} \) and \( Z_{\text{SO}} \) are used to denote total revenue generated under the user and system optimization respectively.

In this thesis, we consider several special cases of the affine demand function: (i) the general uniform model, (ii) the uniform model with respect to own sensitivities and (iii) uniform model with respect to competitors' sensitivities.

The general uniform model

The price sensitivity is the same for all sellers, \( \alpha_i = \alpha, \beta_i = \beta, q_i = d_i - \alpha p_i + \beta \bar{p}_{-i}, \forall i = 1, 2, \ldots, n. \) For simplicity, \( \beta_{ij} \) is set to 1.

\[
B_u = \begin{bmatrix}
\alpha & -1 & \ldots & -1 \\
-1 & \alpha & \ldots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
-1 & \ldots & -1 & \alpha
\end{bmatrix}, \quad \Gamma_u = \text{diag}(B_u) = \begin{bmatrix}
\alpha & 0 \\
0 & \alpha
\end{bmatrix}, \quad = \alpha I \quad (2.1)
\]

where \( I \) is an \( n \times n \) identity matrix.

22
The uniform model with respect to own sensitivities

The price sensitivities with respect to competitors $\beta_{ij}$ vary while $\alpha$ is the same for all sellers.

$$B = \begin{bmatrix} \alpha & -\beta_{1,2} & \ldots & -\beta_{1,n} \\
-\beta_{2,1} & \alpha & \ldots & \vdots \\
\vdots & \ddots & \ddots & -\beta_{n-1,n} \\
-\beta_{n,1} & \ldots & -\beta_{n,n-1} & \alpha \end{bmatrix} \quad \Gamma = \begin{bmatrix} \alpha \\
\alpha \\
\vdots \\
0 \end{bmatrix} = \alpha \mathbf{I} \quad (2.2)$$

The uniform model with respect to competitors' sensitivities

While keeping $\beta_{ij} = \beta$ for all sellers, $\alpha_i$ differs across all sellers.

$$B = \begin{bmatrix} \alpha_1 & -\beta & \ldots & -\beta \\
-\beta & \alpha_2 & \ldots & \vdots \\
\vdots & \ddots & \ddots & -\beta \\
-\beta & \ldots & -\beta & \alpha_n \end{bmatrix} \quad \Gamma = \begin{bmatrix} \alpha_1 \\
\alpha_2 \\
\vdots \\
0 \end{bmatrix} \quad (2.3)$$

Notice that vector $\mathbf{d} = (d_1, \ldots, d_n)$ denotes the quantity differences among sellers, that is, the quantity sold when prices are zero. We will consider two cases:

1. When there are no quality differences, i.e., $\mathbf{d} = (d_1, \ldots, d)$.

2. When there are quality differences, i.e., $\mathbf{d} = (d_1, \ldots, d_n)$.

### 2.2 Assumptions and Discussions

In this section, we describe the assumptions imposed on the models that we study in this thesis. Consider an oligopoly consisting of $n$ sellers offering differentiated substitutable products. Our analysis is restricted to models that satisfy the following assumptions:
Assumption 1. Demand is a function of the prices, that is, \( q = d - Bp \).

\( B \) is an \( n \times n \) matrix of demand sensitivities to price changes. Specifically, if \( b_{ij} \) is the \((i, j)\)th element of the matrix \( B \), then \(-b_{ij}\) is the change in quantity allocated to each seller \( i \) as a result of a unit change in the price charged by seller \( j \), holding all other prices constant.

Assumption 2. \( d \geq 0 \).

\( d \) is an \( n \) dimensional vector representing demand when all sellers set their prices to zero. It indicates the quality differences among all sellers. Naturally, we assume that \( d \) is nonnegative.

Assumption 3. \( B \) is a symmetric matrix.

Unless stated otherwise, we require \( B \) to be symmetric in our analysis. However, Section 5.5.1 is an exception when we extend our analysis to an asymmetric matrix \( B \).

Assumption 4. \( \text{diag}(B) > 0 \) and \( \text{offdiag}(B) \leq 0 \).

For any square matrix \( M \), let \( \text{diag}(M) \) denote the diagonal part of the matrix \( M \). Similarly, let \( \text{offdiag}(M) \) denote the off-diagonal of \( M \), \( \text{offdiag}(M) = M - \text{diag}(M) \).

We assume that the demand sensitivity matrix \( B \) exhibits the following sign pattern which is equivalent to a demand system with non-positive own sensitivities and non-negative cross-sensitivities (that is, gross substitutes).

Assumption 5. \( B \) is column diagonally dominant; that is, \( |b_{ii}| \geq \sum_{j \neq i} |b_{ij}| \) for all \( i, j \in \mathcal{I} \).

If \( |b_{ii}| > \sum_{j \neq i} |b_{ij}| \) for all \( i \), we refer to \( B \) as strictly diagonally dominant. Assumption 5 is applicable to markets where the total demand is decreasing with prices. More explicitly, consider the change in total market demand as a result of a unit increase
in the price charged by seller \( i \), holding all other prices constant. This change is equal to \(-|b_{ii}| + \sum_{j \neq i} |b_{ij}|\) for all \( i, j \in I \). Column diagonal dominance indicates a negative change and follows from the law of demand.

It should be noted that Assumptions 4 and 5 imply that \( B \) is an M-matrix.

**Assumption 6.** For the nonlinear demand model, the demand function \( q_i \) is a continuous and concave function of the prices, and is at least twice differentiable with respect to \( p \). Furthermore, we assume that:

\[
\frac{\partial q_i}{\partial p_i} < 0, \\
\frac{\partial q_i}{\partial p_j} \geq 0, \quad j \in -i
\]

Assumption 6 states that the demand decreases strictly if the seller increases his price and increases if his competitors increase their prices. This assumption is equivalent to saying that the Jacobian matrix of the demand function, \( J_q(p) = \left[ \begin{array}{cccc} \frac{\partial q_1}{\partial p_1} & \cdots & \frac{\partial q_1}{\partial p_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial q_n}{\partial p_1} & \cdots & \frac{\partial q_n}{\partial p_n} \end{array} \right] \),

has negative diagonals and nonnegative off-diagonal elements.

**Assumption 7.** \(-J_q(p)\) is symmetric and strictly diagonally dominant.

Diagonal dominance of the matrix implies that a seller’s demand is more sensitive to his own price changes than to those of his competitors. Assumptions 6 and 7 have a similar interpretation for \(-J_q(p)\) - when the price sensitivity parameters would be replaced with partial derivatives of the demand with respect to the corresponding prices.

Notice that under Assumptions 6 and 7, \(-J_q(p)\) is an M-matrix, which is also positive definite.

**Remark 2.2.1.** For the affine case, \( B = -J_q(p) \).
2.3 Problem Description

In this section, we formally introduce the user and system optimization problems. We also define the concept of the price of anarchy, one of the key notions that we study in this thesis.

2.3.1 User Optimization

User optimization is decentralized in the sense that every seller optimizes for himself. Each seller sets his price as the best response price to his competitors' optimal prices. Given all competitors' optimal pricing, $\bar{p}_{-i}$, the best pricing policy of seller $i$ is the optimal solution of the best response optimization problem. This is a Nash equilibrium problem, which is defined below.

**Definition 2.3.1. Nash equilibrium policies:** The pricing policies for each seller are Nash equilibrium pricing policies if no single seller can increase his payoff by unilaterally changing his policy.

The definition implies that each seller sets his equilibrium pricing policy as the best response to the equilibrium pricing policies of his competitors. Each seller maximizes his revenue given the competitors' optimal prices by solving the optimization problem UO.

\[
\begin{align*}
\text{UO - Affine} & : \max_{p_i} \quad p_i \cdot (d_i - \alpha_i p_i + \beta_{-i} \bar{p}_{-i}) \\
\text{UO - Nonlinear} & : \max_{p_{UO,i}} \quad p_{UO,i} \cdot q_i(p_{UO,i}, \bar{p}_{UO,i}) \\
\end{align*}
\]
We use \( p_{\text{uo}} \) to denote the optimal pricing policies of all sellers under the user optimization and the total profit obtained is represented by \( Z_{\text{uo}} \).

### 2.3.2 System Optimization

Under system optimization, a central authority is optimizing the total profit of the system and forcing all sellers to comply. To achieve that, the central authority could solve the optimization problem whose optimal solution maximizes the total profit.

\[
\text{SO}_\text{- AFFINE}: \quad \max_{\mathbf{p}} \quad \sum_{i=1}^{n} p_i \cdot (d_i - \alpha_i p_i + \beta_{-i} p_{-i}) \\
\text{s.t.} \quad p_i \geq 0 \quad \forall i = 1, 2, \ldots, n
\]

\[
\text{SO}_\text{- NONLINEAR}: \quad \max_{\mathbf{p}_{\text{so}}} \quad \sum_{i=1}^{n} p_{\text{so},i} \cdot q_{\text{so},i}(\mathbf{p}_{\text{so}}) \\
\text{s.t.} \quad p_{\text{so},i} \geq 0 \quad \forall i = 1, 2, \ldots, n
\]

We denote the system optimal price vector and total profit by \( \mathbf{p}_{\text{so}} \) and \( Z_{\text{so}} \) respectively. Beckmann, McGuire and Winsten [2] proved that the first-order optimality conditions can be used to easily characterize the optimal solution to this problem.

### 2.3.3 The Price of Anarchy

Although it is well-known that system optimization yields higher total profit, in many real-world applications there are independent users who will only accept directions given by the central authority if these are in their own interest or if there are incentives that encourage them to do so. Assuming that they will follow directions given by an authority may not be realistic. As a result, it is important to determine the price of
anarchy, that is, how much is lost due to lack of coordination in terms of profit. In this thesis, we quantify the price of anarchy as the ratio of the total profit generated under user optimization over that of system optimization.

\[ POA = \frac{Z_{uo}}{Z_{so}} = \frac{\text{puo}^T \text{quo}}{\text{pso}^T \text{qso}} \]

### 2.4 Existence and Uniqueness

In this section, we show the existence and uniqueness of the optimal solution for both the user optimization and system optimization problems.

#### 2.4.1 User Optimization Optimum

We change the maximization problem into a minimization problem (2.4), where \( q_i \) is a concave and continuous function of the prices as stated in Assumption 6.

\[
\begin{align*}
\text{UO} &: \min_{p_i} -p_i \cdot q_i(p_i, \bar{p}-i) \\
\text{s.t.} & \quad p_i \geq 0
\end{align*}
\]

**Proposition 2.4.1.** Under Assumption 6, there exists a solution \( \text{puo} \) to the user optimization problem 2.4.

**Proof.** Using the result of Debreu, Glicksberg, Fan 1952 [14], there exists a Nash equilibrium solution since the objective function is continuous and concave with respect to \( p_i \) and the feasible region is a convex set.

**Proposition 2.4.2.** Under Assumptions 6 and 7, there exists a unique solution \( \text{puo} \) to the user optimization problem.

In order to prove Proposition 2.4.2, we first need to establish the following properties.
Using the first order optimality conditions and ignoring the constraint \( p \geq 0 \). For an optimal solution to exist, 
\[
\frac{\partial z_i}{\partial p_i} = -q_i - \frac{\partial q_i}{\partial p_i} p_i = 0, \text{ for some } p_i.
\]

Let \( \Gamma(p) = \text{diag}(-Jq(p)) \), we rewrite the previous optimality condition as 
\[
F(p) := -q(p) + \Gamma(p)p = 0 \quad (2.5)
\]
\[
p = \Gamma(p)^{-1}q(p) \quad (2.6)
\]

**Lemma 2.4.3.** *Under Assumption 6, \( p \geq 0 \).*

*Proof.* The concavity of the demand function \( q_i \) implies that, 
\[
-q_i(p) + q_i(0) \leq -\nabla q_i(p)(p - 0),
\]
where \( \nabla q_i(p) \) is the gradient of \( q_i \) with respect to vector \( p \).

In matrix form, 
\[
q(p) \geq q(0) + Jq(p)p \quad (2.7)
\]

Substitute Equation (2.6) into (2.7),
\[
q(p) \geq q(0) + Jq(p)\Gamma(p)^{-1}q(p)
\]
\[
\Gamma(p)\Gamma(p)^{-1}q(p) \geq q(0) + Jq(p)\Gamma(p)^{-1}q(p)
\]
\[
(\Gamma(p) - Jq(p))\Gamma(p)^{-1}q(p) \geq q(0) \quad \text{(Using (2.6), and the M-matrix property)}
\]
\[
p \geq (\Gamma(p) - Jq(p))^{-1}\Gamma(p)q(0)
\]

Notice that \( \Gamma(p) - Jq(p) \) is an M-matrix, that is, \( (\Gamma(p) - Jq(p))^{-1} \geq 0 \). \( \Gamma(p) \) is a nonnegative diagonal matrix and \( q(0) \) is also nonnegative, thus, \( p \geq 0 \) \( \square \)
Proof for Proposition 2.4.2. We have shown in Lemma 2.4.3, that if $p$ in (2.6) satisfies the first order conditions, it also satisfies the nonnegativity constraint, and thus, it is a feasible solution to the user optimization problem (2.4).

Using $J\Gamma(p)$ to denote the Jacobian matrix of $\Gamma(p)$, the Jacobian matrix of vector function $F$ can be written as Equation (2.8),

$$JF = -Jq(p) + J\Gamma(p) + \Gamma(p)$$

Assumption 6 and 7 imply that $-Jq(p)$ is positive definite, and both $J\Gamma(p)$ and matrix $\Gamma(p)$ are diagonal matrices with nonnegative entries. Thus, $JF$ is also positive definite. Hence the system of nonlinear equations has a unique solution. \qed

2.4.2 System Optimization Optimum

The system optimization objective function can be written in vector form as $p^Tq$. Under Assumptions 6 and 7, the Hessian matrix of the objective function is positive definite, which implies that there exists a unique solution to the optimization problem. For more details, we refer readers to Beckmann, McGuire and Winsten [2].
Chapter 3

General Affine Demand Model and Price Comparison

In this chapter, we evaluate the closed-form solution for the price of anarchy using the affine demand model. By applying the first order optimality conditions, we obtain the user and system optimal solutions respectively. Notice that the closed-form solution can be applied to both symmetric and asymmetric affine demand models. In the last section of the chapter, we compare the optimal prices of the user and system optimization problems and prove that higher prices are charged in the system optimization setting.
3.1 Closed-Form Solutions for the General Affine Demand Model

3.1.1 User Optimization Optimal Solution

We denote $z_i$ as the total profit generated by seller $i$.

$$z_i = p_i \cdot q_i(p) = p_i \cdot (d_i - \alpha_ip_i + \beta_i \bar{p}_i).$$

Using the first order optimality conditions, we differentiate the objective function with respect to $p_i$ and assemble all the components $\frac{\partial z_i}{\partial p_i}$ in the following vector, $d - Bp - \Gamma p$.

To compute the optimal price, $p_{UO}$, set this vector to zero,

$$d - Bp_{UO} - \Gamma p_{UO} = 0$$
$$\Rightarrow B + \Gamma p_{UO} = d$$
$$\Rightarrow p_{UO} = (B + \Gamma)^{-1}d.$$

Notice that since $B + \Gamma$ is an M-matrix, then $p_{UO} \geq 0$.

The optimal total profit generated in the market, $Z_{UO}$ is the sum of the profits earned by all sellers in the market.

$$Z_{UO} = z_1 + z_2 + \ldots + z_n$$
$$= (p_{UO})^T(d - Bp_{UO})$$
$$= ((B + \Gamma)^{-1}d)^T(d - B(B + \Gamma)^{-1}d)$$
$$= ((B + \Gamma)^{-1}d)^T(d - (-\Gamma + B + \Gamma)(B + \Gamma)^{-1}d)$$
$$= ((B + \Gamma)^{-1}d)^T(d + \Gamma(B + \Gamma)^{-1}D - d)$$
$$= ((B + \Gamma)^{-1}d)^T\Gamma((B + \Gamma)^{-1}d).$$
3.1.2 System Optimization Optimal Solution

The objective function of system optimization can be written in matrix form as,

$$Z_{SO} = p_{SO}^T (d - Bp_{SO}).$$

Since it is an unconstrained optimization problem with a concave objective, we can differentiate it with respect to the price vector $p_{SO}$ and solve for the optimal $p_{SO}$ by setting the gradient to zero.

$$\nabla Z_{SO} = d - (B + B^T)p_{SO} = 0.$$

Let $S = \frac{B + B^T}{2}$,

$$\Rightarrow p_{SO} = \frac{1}{2} S^{-1} d.$$

Notice that since $S$ is an M-matrix, $p_{SO} \geq 0$. The optimal total profit generated by the system optimization is given as

$$Z_{SO} = p_{SO}^T (d - Bp_{SO})$$

$$= (\frac{1}{2} S^{-1} d)^T (d - \frac{1}{2} BS^{-1} d)$$

$$= \frac{1}{2} d^T S^{-1} d - \frac{1}{4} S^{-1} d$$

$$= (\frac{1}{2} S^{-1} d)^T (\frac{1}{2} d)$$

$$= \frac{1}{4} d^T B^{-1} d.$$ (3.3)
3.1.3 The Price of Anarchy

The price of anarchy is computed as the ratio of the total profit generated under the user optimization and the system optimization.

\[
POA = \frac{Z_{UO}}{Z_{SO}}
\]

\[
= \frac{((B + \Gamma)^{-1}d)^T \Gamma ((B + \Gamma)^{-1}d)}{\frac{1}{2}d^TS^{-1}d}
\]

\[
= \frac{4((B + \Gamma)^{-1}d)^T \Gamma ((B + \Gamma)^{-1}d)}{d^TS^{-1}d}.
\]  

Remark 3.1.1. Although the model used in the thesis uses zero cost per unit, the model can be easily extended to include a fixed cost per unit. If we define \( \tilde{d} = d - Bc \), one could easily show that \( Z_{UO} \) and \( Z_{SO} \) have the same form as we have shown earlier except \( d \) would be replaced by \( \tilde{d} \) instead.

3.2 Price Comparison

In this section, we compare the optimal prices between the user and system optimization.

**Proposition 3.2.1.** Under Assumptions 2, 6 and 7, \( p_{so} \geq p_{uo} \).

**Proof.** Recall \( p_{so} = \frac{1}{2}S^{-1}d \), and \( p_{uo} = (B + \Gamma)^{-1}d \).

Since \( B^T \) is an M-matrix,

\[
B^T \leq \Gamma.
\]

Hence,

\[
(B + B^T) \leq (B + \Gamma)
\]

\[
\Rightarrow \ 2S \leq (B + \Gamma)
\]
Since both $S$ and $B + \Gamma$ are M-matrices,

$$\frac{1}{2}S^{-1} \geq (B + \Gamma)^{-1}$$

From Assumption 2, $d \geq 0$, it follows that

$$p_{so} = \frac{1}{2}S^{-1}d \geq p_{uo} = (B + \Gamma)^{-1}d.$$
Chapter 4

Price of Anarchy for the General Uniform Demand Model

In this chapter, we evaluate the closed-form solution for the price of anarchy using the general uniform demand model. We first obtain the price of anarchy when there are no quality differences among the sellers, that is, the values of \( d_i \) are the same, and denote this quantity by \( POA_u \). Next, we obtain the closed-form solution for the price of anarchy when quality differences exist, i.e., \( d = (d_1, d_2, \ldots, d_n) \) where values of \( d_i \) differ from each other. We show that \( POA_u \) is the worst-case bound when there are quality differences among the sellers. In addition, we show that the price of anarchy increases when the coefficient of variation and/or the number of sellers in the market increases.

4.1 The General Uniform Model without Quality Differences

To gain additional intuition on the definition of the price of anarchy, in this section, we look at the general uniform model (2.1). In this section, the model excludes quality...
differences between sellers, that is, \( d_i \) is the same for all sellers, i.e., \( d = (d, d, \ldots, d) = de \), where \( e \) is an \( n \) dimensional unit vector.

### 4.1.1 The Closed-Form Solutions

**Theorem 4.1.1.** Under Assumptions 1 to 5, the price of anarchy for the general uniform demand model is

\[
POA_u = \frac{4(1 - r)}{(2 - r)^2},
\]

where \( r \) is the sensitivity ratio, \( r = \frac{n-1}{a} \).

To prove this theorem, we first establish the following propositions.

**Proposition 4.1.2.** The optimal total profit generated under the user optimization is given as,

\[
Z_{uo} = \frac{(\alpha)(n)(d^2)}{(2\alpha + 1 - n)^2}.
\]

**Proof.** Since the general uniform demand model is used, \( \Gamma = \alpha I \), where \( I \) is an identity matrix of dimension \( n \times n \). Equation (3.2) becomes

\[
Z_{uo} = ((B + \alpha I)^{-1}d)^T(\alpha I)((B + \alpha I)^{-1}d)
= \alpha \| (B + \alpha I)^{-1}d \|^2.
\]
Expand $(B + \alpha I)^{-1}$ as follows,

$$(B + \alpha I)^{-1} = \begin{bmatrix} 2\alpha & -1 \\ \vdots & \ddots \\ -1 & 2\alpha \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2\alpha + 1 - 1 & -1 \\ \vdots & \ddots \\ -1 & 2\alpha + 1 - 1 \end{bmatrix}^{-1}$$

$$= [(2\alpha + 1)I - H]^{-1}, \quad \text{where } H = \begin{bmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{bmatrix}$$

$$= \frac{1}{2\alpha + 1}[I - \frac{1}{2\alpha + 1}H]^{-1}$$

$$= \frac{1}{2\alpha + 1}[I + \frac{1}{2\alpha + 1}H + \frac{1}{(2\alpha + 1)^2}H + \frac{1}{(2\alpha + 1)^3}H + \ldots].$$

Since $H^2 = nH, H^3 = n^2H, \ldots, H^k = n^{k-1}H$, it follows that

$$(B + \alpha I)^{-1}$$

$$= \frac{1}{2\alpha + 1}[I + \frac{1}{2\alpha + 1}H + \frac{n}{(2\alpha + 1)^2}H + \frac{n^2}{(2\alpha + 1)^3}H + \ldots]$$

$$= \frac{1}{2\alpha + 1}[I + \frac{1}{(2\alpha + 1)^2} + \frac{n}{(2\alpha + 1)^3} + \frac{n^2}{(2\alpha + 1)^4} + \ldots]H$$

$$= \frac{1}{2\alpha + 1}[I + \frac{1}{(2\alpha + 1)(2\alpha + 1 - n)}H$$

$$= \begin{bmatrix} \frac{1}{2\alpha + 1} + \frac{1}{(2\alpha + 1)(2\alpha + 1 - n)} & \frac{1}{(2\alpha + 1)(2\alpha + 1 - n)} \\ \frac{1}{(2\alpha + 1)(2\alpha + 1 - n)} & \frac{1}{2\alpha + 1} + \frac{1}{(2\alpha + 1)(2\alpha + 1 - n)} \end{bmatrix}.$$
Therefore,

\[(B + \alpha I)^{-1}d = (B + \alpha I)^{-1}(de)\]
\[= d \left( \frac{1}{2\alpha + 1} + \frac{1}{(2\alpha + 1)(2\alpha + 1 - n)} + \frac{n - 1}{(2\alpha + 1)(2\alpha + 1 - n)} \right) e\]
\[= d \left( \frac{1}{2\alpha + 1} + \frac{n}{(2\alpha + 1)(2\alpha + 1 - n)} \right) e\]
\[= \frac{d}{2\alpha + 1 - n} e.\]

This in turn gives rise to

\[Z_{UO} = \alpha \|(B + \alpha I)^{-1}d\|^2\]
\[= \alpha \left[ \frac{d}{2\alpha + 1 - n} e \right]^T \left[ \frac{d}{2\alpha + 1 - n} e \right]\]
\[= \alpha(n) \left( \frac{d}{2\alpha + 1 - n} \right)^2\]
\[= \frac{\alpha(n)(d^2)}{(2\alpha + 1 - n)^2}.\]

\[\Box\]

**Proposition 4.1.3.** The optimal total profit generated under system optimization is given as,

\[Z_{SO} = 1 \frac{(n)d^2}{4\alpha + 1 - n}.\]

**Proof.** As given in Equation (3.3), the closed-form for the total optimal profit generated under the system optimization is given as

\[Z_{SO} = \frac{1}{4} d^T S^{-1} d = \frac{1}{4} d^T B^{-1} d.\]

Notice in this case, B is symmetric, thus, S = B.
We expand out $B^{-1}$:

$$
B^{-1} = \begin{bmatrix}
\alpha & -1 \\
\vdots & \\
-1 & \alpha \\
\end{bmatrix}^{-1} = \begin{bmatrix}
\alpha + 1 - 1 & -1 \\
\vdots & \\
-1 & \alpha + 1 - 1 \\
\end{bmatrix}^{-1} = [(\alpha + 1)I - H]^{-1}.
$$

Using an approach similar to that used earlier for expressing $(B + \alpha I)^{-1}$, we obtain

$$
B^{-1} = \frac{1}{\alpha + 1}I + \frac{1}{(\alpha + 1)(\alpha + 1 - n)}H.
$$

Substituting the results we obtained earlier, the closed form solution for $Z_{SO}$ is as follows,

$$
Z_{SO} = \frac{1}{4}d^T d\left[\frac{1}{\alpha + 1} + \frac{n}{(\alpha + 1)(\alpha + 1 - n)}\right]e
= \frac{1}{4}d^T (\frac{d(\alpha + 1)}{(\alpha + 1)(\alpha + 1 - n)})e
= \frac{1}{4}d^T (\frac{d}{(\alpha + 1 - n)})e
= \frac{1}{4}\frac{d^2}{\alpha + 1 - n} e^T e
= \frac{1}{4}\frac{(n)d^2}{\alpha + 1 - n}.
$$

\[ \square \]

*Proof for Theorem 4.1.1.* Substituting the results of Propositions 4.1.2 and 4.1.3, the
price of anarchy equals

\[
POA_u = \frac{\alpha(n)(d^2)}{(2\alpha + 1 - n)^2} \frac{1}{\frac{(n)d^2}{4\alpha + 1 - n}} \frac{4\alpha(\alpha + 1 - n)}{(2\alpha + 1 - n)^2}.
\]

Dividing both the denominator and numerator by \(\alpha^2\),

\[
POA_u = \frac{4(1 - \frac{n-1}{\alpha})}{(2 - \frac{n-1}{\alpha})^2} \frac{4(1 - r)}{(2 - r)^2}.
\]

(4.1)

\[
\Box
\]

4.1.2 Discussions

From Equation (4.1), we see that for the general uniform demand case with no quality differences among the sellers, i.e., the same \(d\) for all sellers, the price of anarchy \(POA_u\) only involves the price sensitivity ratio, \(r\), and is independent of the number of sellers.

The result is expected since every seller charges the same price and sells the same quantity, and therefore, comparing the total profit generated by all \(n\) sellers in the market is equivalent to comparing that of one seller under either user optimization or system optimization.

Figure 4-1 shows the price of anarchy achieved under this model. \(POA_u\) decreases with increasing \(r\). In addition, it decreases slowly for small \(r\), and the negative slope is steeper for large \(r\). For instance, when \(r\) is smaller than 0.66, \(POA_u\) stays above 0.75 = \(\frac{3}{4}\), which implies that when the buyers are not extremely sensitive to the price changes, the total profit generated by user optimization is close to that of the system.
optimization despite lack of central coordination. On the other hand, when $r$ is large (e.g., $r = 0.95$), $POA_u$ plunges to 0.18, indicating a higher inefficiency of the user equilibrium.

4.2 The General Uniform Model with Quality Differences

In this section, we analyze the role of quality differences in the general uniform demand model (2.1), where $\alpha$ is the same for all sellers and $\beta_{i,j} = 1$ for all $i, j$. We have defined the realized affine demand as $q = d - Bp$. $d_i$ represents quality characteristics of seller $i$ because it represents the quantity sold when seller sets his price to zero.

In Section 4.1, we have shown in Theorem 4.1.1 that when $d = (d, d, \ldots, d)$, i.e., $d_i$ is same for all sellers, the price of anarchy equals $POA_u = \frac{4(1-r)}{(2-r)^2}$. In the following
sections, we are going to prove that when quality differences exist among sellers, that is, \( \mathbf{d} = (d_1, d_2, \ldots, d_n) \) and the values of \( d_i \) differ from each other, \( POA_u \) is the worst-case bound. In fact, as more variations exist in terms of quality difference, the price of anarchy improves.

### 4.2.1 Coefficient of Variation

**Derivation**

**Definition 4.2.1.** The coefficient of variation (CV) is a measure of dispersion of a probability distribution. It is defined as the ratio of the standard deviation to the mean of the variable demand.

Define \( d_i = \bar{d} + \varepsilon_i \), where \( \bar{d} \) is the mean of the variable demand, i.e., \( \bar{d} = \frac{\sum_{i=1}^{n} d_i}{n} \).

\[
CV^2 = \left( \frac{\text{standard deviation}}{\text{mean}} \right)^2
= \frac{\text{variance}}{\text{mean}^2}
= \frac{1}{n} \sum_{i=1}^{n} (d_i - \bar{d})^2
= \frac{\sum_{i=1}^{n} (d_i - \bar{d})^2}{\bar{d}^2}.
\]

**Lemma 4.2.1.**

\[
CV^2 = \frac{n \sum_{i=1}^{n} d_i^2 - (\sum_{i=1}^{n} d_i)^2}{(\sum_{i=1}^{n} d_i)^2} \quad (4.2)
= \frac{n \sum_{i=1}^{n} \varepsilon_i^2 - (\sum_{i=1}^{n} \varepsilon_i)^2}{(\sum_{i=1}^{n} d_i)^2}.
\]
Proof. From Definition 4.2.1,

\[ CV^2 = \frac{1}{n} \sum_{i=1}^{n} d_i^2 - \frac{2}{n} \bar{d} \sum_{i=1}^{n} d_i + \bar{d}^2. \]

We expand \( \sum_{i=1}^{n} d_i^2 \) and \((\sum_{i=1}^{n} d_i)^2\) as follows,

\[
\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (\bar{d} + \varepsilon_i)^2
= \sum_{i=1}^{n} (\bar{d}^2 + 2\bar{d}\varepsilon_i + \varepsilon_i^2)
= nd^2 + 2\bar{d}\sum_{i=1}^{n} \varepsilon_i + \sum_{i=1}^{n} \varepsilon_i^2. \tag{4.4}
\]

\[
\left( \sum_{i=1}^{n} d_i \right)^2 = \left( \sum_{i=1}^{n} (\bar{d} + \varepsilon_i) \right)^2
= \left( nd + \sum_{i=1}^{n} \varepsilon_i \right)^2
= n^2\bar{d}^2 + 2n\bar{d}\sum_{i=1}^{n} \varepsilon_i + \left( \sum_{i=1}^{n} \varepsilon_i \right)^2. \tag{4.5}
\]

Substitute Equations (4.4) and (4.5) to obtain the results in Lemma 4.2.1. \qed

Properties of \( CV^2 \)

Proposition 4.2.2. \( CV^2 \) can only take values between 0 and \( n - 1 \), i.e.,

\[ 0 \leq CV^2 \leq n - 1. \]
Proof. From Equation (4.2),

$$CV^2 = \frac{n \sum_{i=1}^{n} d_i^2}{(\sum_{i=1}^{n} d_i)^2} - 1.$$

Using the Cauchy-Schwarz inequality, it follows that

$$n \sum_{i=1}^{n} d_i^2 \geq \left( \sum_{i=1}^{n} d_i \right)^2$$

$$\Rightarrow CV^2 \geq 0.$$

Because the sum of the squares is always less than or equal to the square of the sum when $d_i$ is nonnegative,

$$\sum_{i=1}^{n} d_i^2 \leq \left( \sum_{i=1}^{n} d_i \right)^2$$

$$\Rightarrow CV^2 \leq n - 1.$$

We now consider the two extreme cases of $CV^2$:

**Lemma 4.2.3.** $CV^2 = 0$ iff there are no quality differences among all sellers.

Proof. When $CV^2 = 0$, it is equivalent to $n \sum_{i=1}^{n} d_i^2 = (\sum_{i=1}^{n} d_i)^2$. This only occurs when all the sellers have the same demand function, i.e., $d = (d, d, \ldots, d)$. That is,

$$CV^2 = \frac{n \sum_{i=1}^{n} d_i^2}{(\sum_{i=1}^{n} d_i)^2} - 1 = 0,$$

$$\Leftrightarrow n \sum_{i=1}^{n} d_i^2 = \left( \sum_{i=1}^{n} d_i \right)^2.$$
Lemma 4.2.4. \( CV^2 = n - 1 \) iff only one seller has non-zero demand when all sellers set their prices to zero.

Proof. When \( d \) has only one non-zero term, i.e., \( d = (0, \ldots, d, 0, \ldots) \),

\[
\begin{align*}
\therefore n \sum_{i=1}^{n} d_i^2 &= nd^2 \\
\left( \sum_{i=1}^{n} d_i \right)^2 &= d^2 \\
\therefore CV^2 &= \frac{nd^2}{d^2} - 1 = n - 1.
\end{align*}
\]

\[ \square \]

4.2.2 The Closed-Form Solutions

User Optimization Optimal Solution

For the general uniform model, the optimal revenue generated by user optimization is given by \( Z_{UO} = \alpha \|(B + \alpha I)^{-1}d\|^2 \).

Proposition 4.2.5.

\[
Z_{UO} = \alpha \left( \frac{(2\alpha + 1)^2 nd^2 + (2\alpha + 1)^2 \sum_{i=1}^{n} \epsilon_i}{(2\alpha + 1)^2(2\alpha + 1 - n)^2} \right) + \frac{(2\alpha + 1 - n)^2 \sum_{i=1}^{n} \epsilon_i^2 + (4\alpha + 2 - n)(\sum_{i=1}^{n} \epsilon_i)^2}{(2\alpha + 1)^2(2\alpha + 1 - n)^2} \right).
\]
Proof. We substitute \( \mathbf{d} = (d_1, d_2, \ldots, d_n) \),

\[
(B + \alpha I)^{-1} \mathbf{d} = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
\frac{1}{(2\alpha + 1)(2\alpha + 1 - n)} & \frac{1}{(2\alpha + 1)(2\alpha + 1 - n)} & \cdots & \frac{1}{(2\alpha + 1)(2\alpha + 1 - n)} \\
\frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
\frac{1}{(2\alpha + 1)(2\alpha + 1 - n)} & \frac{1}{(2\alpha + 1)(2\alpha + 1 - n)} & \cdots & \frac{1}{(2\alpha + 1)(2\alpha + 1 - n)} \\
\frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
\frac{1}{(2\alpha + 1)(2\alpha + 1 - n)} & \frac{1}{(2\alpha + 1)(2\alpha + 1 - n)} & \cdots & \frac{1}{(2\alpha + 1)(2\alpha + 1 - n)}
\end{bmatrix}
\begin{bmatrix}
d_1 \\
\vdots \\
d_n
\end{bmatrix}
\]

Next, \( \| (B + \alpha I)^{-1} \mathbf{d} \|^2 \) is evaluated as follows,

\[
\| (B + \alpha I)^{-1} \mathbf{d} \|^2 = ((B + \alpha I)^{-1} \mathbf{d})^T (B + \alpha I)^{-1} \mathbf{d}
\]

\[
= \sum_{i=1}^{n} \left( \frac{d_i(2\alpha + 1 - n) + \sum_{i=1}^{n} d_i}{(2\alpha + 1)(2\alpha + 1 - n)} \right)^2
\]

\[
= \frac{\sum_{i=1}^{n} [d_i^2(2\alpha + 1 - n)^2 + 2(2\alpha + 1 - n)d_i \sum_{i=1}^{n} d_i + (\sum_{i=1}^{n} d_i)^2]}{(2\alpha + 1)^2(2\alpha + 1 - n)^2}
\]

\[
= \frac{(2\alpha + 1 - n)^2 \sum_{i=1}^{n} d_i^2 + 2(2\alpha + 1 - n)(\sum_{i=1}^{n} d_i)^2 + n(\sum_{i=1}^{n} d_i)^2}{(2\alpha + 1)^2(2\alpha + 1 - n)^2}
\]

\[
= \frac{(2\alpha + 1 - n)^2 \sum_{i=1}^{n} d_i^2 + [2(2\alpha + 1 - n) + n](\sum_{i=1}^{n} d_i)^2}{(2\alpha + 1)^2(2\alpha + 1 - n)^2}.
\]
Substituting Equation (4.4) and (4.5) into the equation above, we obtain

\[
\| (B + \alpha I)^{-1} d \|^2 \\
= \frac{(2\alpha + 1 - n)^2 (nd^2 + 2d \sum_{i=1}^n \varepsilon_i + \sum_{i=1}^n \varepsilon_i^2)}{(2\alpha + 1)^2(2\alpha + 1 - n)^2} + \\
\frac{[2(2\alpha + 1 - n) + n](n^2 d^2 + 2n d \sum_{i=1}^n \varepsilon_i + (\sum_{i=1}^n \varepsilon_i)^2)}{(2\alpha + 1)^2(2\alpha + 1 - n)^2}
\]

\[
= \frac{(nd^2 + 2d \sum_{i=1}^n \varepsilon_i)[(2\alpha + 1 - n)^2 + 2(2\alpha + 1 - n)n + n^2]}{(2\alpha + 1)^2(2\alpha + 1 - n)^2} + \\
\frac{(2\alpha + 1 - n)^2 \sum_{i=1}^n \varepsilon_i^2 + [2(2\alpha + 1 - n) + n](\sum_{i=1}^n \varepsilon_i)^2}{(2\alpha + 1)^2(2\alpha + 1 - n)^2}
\]

\[
= \frac{(2\alpha + 1)^2 nd^2 + (2\alpha + 1)^2 2d \sum_{i=1}^n \varepsilon_i}{(2\alpha + 1)^2(2\alpha + 1 - n)^2} + \\
\frac{(2\alpha + 1 - n)^2 \sum_{i=1}^n \varepsilon_i^2 + (4\alpha + 2 - n)(\sum_{i=1}^n \varepsilon_i)^2}{(2\alpha + 1)^2(2\alpha + 1 - n)^2}.
\]

Thus, the optimal total profit under the user optimization is obtained as

\[
Z_{UO} = \alpha \| (B + \alpha I)^{-1} d \|^2 \\
= \alpha \left( \frac{(2\alpha + 1)^2 nd^2 + (2\alpha + 1)^2 2d \sum_{i=1}^n \varepsilon_i}{(2\alpha + 1)^2(2\alpha + 1 - n)^2} \right) + \\
\alpha \left( \frac{(2\alpha + 1 - n)^2 \sum_{i=1}^n \varepsilon_i^2 + (4\alpha + 2 - n)(\sum_{i=1}^n \varepsilon_i)^2}{(2\alpha + 1)^2(2\alpha + 1 - n)^2} \right).
\]

System Optimization Optimal Solution

The optimal total profit under system optimization is given as

\[
Z_{SO} = \frac{1}{2} d^T B^{-1} d.
\]
Proposition 4.2.6.

\[ Z_{SO} = \frac{1}{4} \frac{(\alpha + 1)nd^2 + (\alpha + 1)2d\sum_{i=1}^{n} \varepsilon_i + \sum_{i=1}^{n} \varepsilon_i^2 + (\sum_{i=1}^{n} \varepsilon_i)^2}{(\alpha + 1)(\alpha + 1 - n)} \]

Proof. Substituting \( \mathbf{d} = (d_1, d_2, \ldots, d_n) \),

\[ Z_{SO} = \frac{1}{4} \mathbf{d}^T \mathbf{B}^{-1} \mathbf{d} \]

\[ = \frac{1}{4} \mathbf{d}^T \left( \frac{1}{\alpha + 1} I + \frac{1}{(\alpha + 1)(\alpha + 1 - n)} \mathbf{H} \right) \mathbf{d} \]

\[ = \frac{1}{4} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}^T \begin{bmatrix} \frac{1}{\alpha + 1} + \frac{1}{(\alpha + 1)(\alpha + 1 - n)} & 1 \\ 1 & \frac{1}{(\alpha + 1)(\alpha + 1 - n)} \\ \vdots & \vdots \\ \frac{1}{\alpha + 1} + \frac{1}{(\alpha + 1)(\alpha + 1 - n)} & \frac{1}{\alpha + 1} \end{bmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \]

\[ = \frac{1}{4} \frac{(\alpha + 1 - n) \sum_{i=1}^{n} d_i^2 + (\sum_{i=1}^{n} d_i)^2}{(\alpha + 1)(\alpha + 1 - n)} \]

When we substitute Equation (4.4) and (4.5), we obtain the equality,

\[ Z_{SO} = \frac{(\alpha + 1 - n) (nd^2 + 2d\sum_{i=1}^{n} \varepsilon_i + \sum_{i=1}^{n} \varepsilon_i^2) + (n^2 d^2 + 2nd \sum_{i=1}^{n} \varepsilon_i + (\sum_{i=1}^{n} \varepsilon_i)^2)}{4(\alpha + 1)(\alpha + 1 - n)} + \]

\[ = \frac{1}{4} \frac{(\alpha + 1)nd^2 + (\alpha + 1)2d\sum_{i=1}^{n} \varepsilon_i + (\alpha + 1 - n) \sum_{i=1}^{n} \varepsilon_i^2 + (\sum_{i=1}^{n} \varepsilon_i)^2}{(\alpha + 1)(\alpha + 1 - n)} \]

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4.2.3 Price of Anarchy in terms of $J(\varepsilon)$

**Proposition 4.2.7.** When quality differences exist among the sellers, the price of anarchy is obtained as

$$ POA = POA_u \cdot J(\varepsilon), $$

where $POA_u$ is the price of anarchy for the general uniform demand model without quality differences and

$$ J(\varepsilon) = \frac{nd^2 + 2d \sum_{i=1}^{n} \varepsilon_i + \frac{1}{(2a+1)^2} \left[ (2a + 1 - n)^2 \sum_{i=1}^{n} \varepsilon_i^2 + (4a + 2 - n) \left( \sum_{i=1}^{n} \varepsilon_i \right)^2 \right]}{nd^2 + 2d \sum_{i=1}^{n} \varepsilon_i + \frac{1}{(\alpha+1)} \left[ (\alpha + 1 - n) \sum_{i=1}^{n} \varepsilon_i^2 + (\sum_{i=1}^{n} \varepsilon_i)^2 \right]}. $$

**Proof.** Substituting $Z_{UO}$ and $Z_{SO}$ obtained in the previous sections,

$$ POA = \frac{4a(\alpha + 1)(\alpha + 1 - n)}{(2a + 1)^2(2a + 1 - n)^2} \cdot \frac{(2a + 1)^2 nd^2 + (2a + 1)^2 2d \sum_{i=1}^{n} \varepsilon_i}{(\alpha + 1)nd^2 + (\alpha + 1)2d \sum_{i=1}^{n} \varepsilon_i + (\alpha + 1 - n) \sum_{i=1}^{n} \varepsilon_i^2 + (\sum_{i=1}^{n} \varepsilon_i)^2} + \frac{(2a + 1 - n)^2 \sum_{i=1}^{n} \varepsilon_i^2 + (4a + 2 - n) (\sum_{i=1}^{n} \varepsilon_i)^2}{(\alpha + 1)nd^2 + (\alpha + 1)2d \sum_{i=1}^{n} \varepsilon_i + (\alpha + 1 - n) \sum_{i=1}^{n} \varepsilon_i^2 + (\sum_{i=1}^{n} \varepsilon_i)^2}. $$

Dividing both the denominator and numerator by $(\alpha + 1)(2a + 1)^2$,

$$ POA = \frac{4a(\alpha + 1 - n)}{(2a + 1 - n)^2} \cdot \frac{nd^2 + 2d \sum_{i=1}^{n} \varepsilon_i + \frac{1}{(2a+1)^2} \left[ (2a + 1 - n)^2 \sum_{i=1}^{n} \varepsilon_i^2 + (4a + 2 - n) \left( \sum_{i=1}^{n} \varepsilon_i \right)^2 \right]}{nd^2 + 2d \sum_{i=1}^{n} \varepsilon_i + \frac{1}{(\alpha+1)} \left[ (\alpha + 1 - n) \sum_{i=1}^{n} \varepsilon_i^2 + (\sum_{i=1}^{n} \varepsilon_i)^2 \right]}. $$

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Dividing $\alpha^2$ and substitute the price sensitivity ratio, $r = \frac{n-1}{a}$, we obtain

$$POA = \frac{4(1-r)}{(2-r)^2} \cdot J(\varepsilon) = POA_u \cdot J(\varepsilon),$$

where

$$J(\varepsilon) = \frac{nd^2 + 2d \sum_{i=1}^{n} \varepsilon_i + \frac{1}{(2\alpha+1)^2} \left[ (2\alpha + 1 - n)^2 \sum_{i=1}^{n} \varepsilon_i^2 + (4\alpha + 2 - n) (\sum_{i=1}^{n} \varepsilon_i)^2 \right]}{nd^2 + 2d \sum_{i=1}^{n} \varepsilon_i + \frac{1}{(\alpha+1)^2} \left[ (\alpha + 1 - n) \sum_{i=1}^{n} \varepsilon_i^2 + (\sum_{i=1}^{n} \varepsilon_i)^2 \right]}.$$

(4.6)

4.2.4 Price of Anarchy in terms of $CV^2$

In this section, we examine $J(\varepsilon)$ and express it only in terms of $n$, $r$ and $CV^2$.

**Proposition 4.2.8.** $J(\varepsilon) = \frac{1 + \left( \frac{(n-1)(2-r)}{2(n-1)+r} \right)^2 CV^2}{1 + \frac{(n-1)(1-r)}{n-1+r} CV^2}$

**Proof.** From Equation (4.6),

$$J(\varepsilon) = \frac{nd^2 + 2d \sum_{i=1}^{n} \varepsilon_i + \frac{1}{(2\alpha+1)^2} \left[ (2\alpha + 1 - n)^2 \sum_{i=1}^{n} \varepsilon_i^2 + (4\alpha + 2 - n) (\sum_{i=1}^{n} \varepsilon_i)^2 \right]}{nd^2 + 2d \sum_{i=1}^{n} \varepsilon_i + \frac{1}{(\alpha+1)^2} \left[ (\alpha + 1 - n) \sum_{i=1}^{n} \varepsilon_i^2 + (\sum_{i=1}^{n} \varepsilon_i)^2 \right]}$$

$$= \frac{nd^2 + 2d \sum_{i=1}^{n} \varepsilon_i + \frac{1}{(2\alpha+1)^2} \left[ (2\alpha + 1 - n)^2 n \sum_{i=1}^{n} \varepsilon_i^2 + (4\alpha + 2 - n)n (\sum_{i=1}^{n} \varepsilon_i)^2 \right]}{nd^2 + 2d \sum_{i=1}^{n} \varepsilon_i + \frac{1}{(\alpha+1)^2} \left[ (\alpha + 1 - n) n \sum_{i=1}^{n} \varepsilon_i^2 + n (\sum_{i=1}^{n} \varepsilon_i)^2 \right]}$$

$$= \frac{n^2d^2 + 2n\sum_{i=1}^{n} \varepsilon_i + \frac{1}{(2\alpha+1)^2} \left[ (2\alpha + 1 - n)^2 n \sum_{i=1}^{n} \varepsilon_i^2 + (4\alpha + 2 - n)n (\sum_{i=1}^{n} \varepsilon_i)^2 \right]}{n^2d^2 + 2n\sum_{i=1}^{n} \varepsilon_i + \frac{1}{(\alpha+1)^2} \left[ (\alpha + 1 - n) n \sum_{i=1}^{n} \varepsilon_i^2 + n (\sum_{i=1}^{n} \varepsilon_i)^2 \right]}$$

$$= \frac{\left( \sum_{i=1}^{n} d_i \right)^2 + \frac{2\alpha+1-n}{2\alpha+1} \left( n \sum_{i=1}^{n} \varepsilon_i^2 - (\sum_{i=1}^{n} \varepsilon_i)^2 \right)}{\left( \sum_{i=1}^{n} d_i \right)^2 + \frac{\alpha+1-n}{\alpha+1} \left( n \sum_{i=1}^{n} \varepsilon_i^2 - (\sum_{i=1}^{n} \varepsilon_i)^2 \right)}.$$
Divide both the numerator and denominator by \((\sum_{i=1}^{n} d_i)^2\), and use Equation (4.3) to reach an equation that is only in terms of \(\alpha, n\) and \(CV^2\), that is,\

\[
J(\alpha, n, CV^2) = \frac{1 + (\frac{2\alpha+1-n}{2\alpha+1})^2 CV^2}{1 + \frac{\alpha+1-n}{\alpha+1} CV^2}.
\]

Equation (4.7) can be expressed in terms of the price sensitivity ratio \(r\), where \(r = \frac{n-1}{\alpha}\),\

\[
J(r, n, CV^2) = \frac{1 + \left(\frac{(n-1)(2-r)}{2(n-1)+r}\right)^2 CV^2}{1 + \frac{(n-1)(1-r)}{n-1+r} CV^2}.
\]

4.2.5 Properties of \(J(n, r, CV^2)\)

**Proposition 4.2.9.** For the general uniform model with quality differences, \(POA \geq POA_u\), where \(POA_u\) is the price of anarchy when quality differences do not exist among sellers.

**Proof.** Given Equation (4.8), if we subtract the denominator from the numerator, we obtain \(\frac{n(n-1)r^2CV^2}{(2(n-1)+r)^2(n-1+r)}\), which implies that when \(n \geq 2\), \(J(r, n, CV^2) \geq 1\). Since \(POA = POA_u \cdot J(r, n, CV^2)\), it follows that \(POA\) is always equal to or greater than \(POA_u\), i.e., \(POA_u\) is the lower bound for \(POA\). □

**Corollary 4.2.10.** A lower bound to \(POA\) in the general uniform demand case is, \(POA_u = \frac{4(1-r)}{(2-r)^2}\), when \(CV^2_{min} = 0\) (that is, when quality differences are absent).

**Corollary 4.2.11.** An upper bound to \(POA\) in the general uniform demand case is \(POA_u \cdot \frac{1 + \left(\frac{(n-1)(2-r)}{2(n-1)+r}\right)^2 (n-1)}{1 + \frac{(n-1)^2(1-r)^2}{n-1+r}}\), when \(CV^2\) takes its maximum value, i.e., \(CV^2_{max} = n - 1\).
Proposition 4.2.12. POA has the following properties:

1. If \( r \) and \( n \) are fixed, POA increases with increasing \( CV^2 \).

2. If \( r \) and \( CV^2 \) are fixed, POA increases with increasing \( n \).

Proof: 1. Rewrite \( J(r, n, CV^2) \) as follows:

\[
J(r, n, CV^2) = 1 + \frac{(n-1)(2-r)}{2(n-1)+r} CV^2
\]

\[
= 1 + \frac{(n-1)(2-r)}{2(n-1)+r} \left( \frac{1}{1 + \frac{(n-1)(1-r)}{n-1+r} CV^2} \right) CV^2
\]

\[
= 1 + \frac{nr^2(n-1)(n-2+r)}{(2(n-1)+r)^2(n-1+r)} \cdot \frac{1}{CV^2} + \frac{(n-1)(1-r)}{n-1+r}
\]

Therefore, for any particular \( r \) and \( n \) (\( n \geq 2 \)), \( J(r, n, CV^2) \) increases when \( CV^2 \) increases.

2. We show in Appendix B that for any particular \( r \) and \( CV^2 \),

\[
\frac{\partial J(r, n, CV^2)}{\partial n} \geq 0.
\]

Thus, POA increases when \( n \) increases. \( \square \)

4.2.6 Numerical Results

In the following section, we illustrate some of the results obtained earlier through numerical examples. The purpose of this exercise is to show the general trends for the price of anarchy when factors such as \( n \), \( r \) and \( CV^2 \) are varied.

Fix \( CV^2 \), vary \( n \) and \( r \)

We fix \( CV^2 \) to its two extreme values, i.e., \( CV^2 = 0 \) and \( n - 1 \).
When \( CV^2 = 0 \), as we showed in Corollary 4.2.11, \( POA = POA_u \). Therefore, if we vary the number of sellers, it does not affect the price of anarchy.

When \( CV^2 = n - 1 \), only one seller in the market has the nonzero demand. We use \( r = [0.3, 0.666, 0.75, 0.85, 0.95] \) and plot the results in Figure 4-2. The results have also been tabulated in Tables 4.1 and 4.2, which summarize the improvement in \( POA \) in a more concise way.

**Conclusions**

1. \( POA_u \) is the lower bound for \( POA \). From Figure 4-2, given a particular price sensitivity ratio \( r \), \( POA \) is consistently higher than \( POA_u \).
2. When \( n = 2 \), there is a gap between \( POA \) and \( POA_u \), which indicates the improvement when quality differences exist in the market. In addition, the difference between the two values is larger when \( POA_u \) is low.

3. As the number of sellers increases, \( POA \) increases. However, the slope of \( POA \) is relatively steeper initially, then gradually becomes gentler. This implies that when there are few sellers in the market, a small increase in the number of sellers boosts the price of anarchy considerably. Nevertheless, as more and more sellers enter the market, the price of anarchy increases at a slower rate.

4. From Tables 4.1 and 4.2, we observe that the improvement in \( POA \) is far more prominent for small \( POA_u \). Thus, it implies that in a given market, despite a high price sensitivity ratio (e.g. \( r = 0.95 \)) and consequently a low \( POA_u \) (e.g., \( POA_u = 0.1814 \)), if there are variations in the quality differences among the sellers, it is possible to achieve a much higher price of anarchy (e.g., when \( n = 15 \), \( POA = 0.5051 \) and the percentage increase is 178.4%).

**Fix \( r \), vary \( n \) and \( CV^2 \)**

In the simulation, \( r \) is fixed to be 0.666, and \( CV^2 \) is varied from 0 to \( n - 1 \) and \( n = [2, 5, 10, 15] \). The result is shown as Figure 4-3. If demand is general uniform without quality differences, i.e., \( CV^2 = 0 \), \( POA_u = 0.75 = \frac{3}{4} \).
Figure 4-3: Fix $r = 0.666$, Vary $n$ and $CV^2$
Discussions

1. When $CV^2 = 0$, the price of anarchy equals $POA_u$, which is the lowest possible value for a given $r$. This matches our intuition as $CV^2 = 0$ means every seller has the same uniform demand function. Increasing the number of sellers does not affect the optimal revenue earned by each seller, and consequently the price anarchy remains the same.

2. For a particular coefficient of variation value, when the quality differences are allowed to vary among the sellers, the price of anarchy increases as the number of sellers increases as shown in Figure 4-3.

3. The increase in $POA$ is larger for a small increase of sellers in the market. For instance, the gap between the $POA$ curves when $n = 5$ and $n = 10$ is larger than that between $n = 10$ and $n = 15$. 
Chapter 5

Bounds for the Price of Anarchy

In this chapter, we obtain a lower bound and an upper bound for the price of anarchy in terms of the eigenvalues of matrix $G$ for matrix $B$, where $G = \Gamma^{-\frac{1}{2}}B\Gamma^{-\frac{1}{2}}$. Furthermore, the bounds can be expressed in terms of the price sensitivity ratio, which is easier to compute. It is crucial to note that the bounds are valid irrespective of the vector $d$ variations. We also include some simulation results to show the tightness of the bounds. We first consider a symmetric matrix $B$, and in the last section of the chapter, we extend the analysis to an asymmetric matrix $B$ and present a lower bound that is in terms of the minimum eigenvalue of $G$ and some constants that measure the degree of asymmetry. The results covered in this chapter can also be found in Farahat, Perakis and Sun [13].

5.1 Preliminary Work

We rewrite the equation for the price of anarchy (3.5) so that we can express the bounds in terms of eigenvalues of $G$, where $G = \Gamma^{-\frac{1}{2}}B\Gamma^{-\frac{1}{2}}$. 

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Lemma 5.1.1. Under Assumption 3, given a price sensitivity matrix \( B \),

\[
POA = \frac{4w^T w}{w^T (G + G^{-1} + 2I) w}.
\] (5.1)

Proof. In Chapter 4, we showed that for affine demand,

\[
POA = \frac{4((B + \Gamma)^{-1} d)^T \Gamma ((B + \Gamma)^{-1} d)}{d^T B^{-1} d}.
\] (5.2)

Substitute \( w = \Gamma^{\frac{1}{2}} (B + \Gamma)^{-1} d \), Equation (5.2) becomes

\[
POA = \frac{4w^T w}{w^T \Gamma^{-\frac{1}{2}} (B + \Gamma) \Gamma^{-\frac{1}{2}} (B + \Gamma) \Gamma^{-\frac{1}{2}} w}
\]

\[
= \frac{4w^T w}{w^T \Gamma^{-\frac{1}{2}} (I + \Gamma B^{-1}) (B + \Gamma) \Gamma^{-\frac{1}{2}} w}
\]

\[
= \frac{4w^T w}{w^T \Gamma^{-\frac{1}{2}} (B + \Gamma + \Gamma B^{-1} \Gamma) \Gamma^{-\frac{1}{2}} w}
\]

\[
= \frac{4w^T w}{w^T \Gamma^{-\frac{1}{2}} (B + 2\Gamma + \Gamma B^{-1} \Gamma) \Gamma^{-\frac{1}{2}} w}
\]

\[
= \frac{4w^T w}{w^T (\Gamma^{-\frac{1}{2}} B + 2\Gamma^{\frac{1}{2}} + \Gamma^{\frac{1}{2}} B^{-1} \Gamma) \Gamma^{-\frac{1}{2}} w}
\]

\[
= \frac{4w^T w}{w^T (\Gamma^{-\frac{1}{2}} B \Gamma^{-\frac{1}{2}} + \Gamma^{\frac{1}{2}} B \Gamma^{\frac{1}{2}} + 2I) w}.
\]

Let matrix \( G = \Gamma^{-\frac{1}{2}} B \Gamma^{-\frac{1}{2}} \), which is also a symmetric diagonally dominant M-matrix (refer to definition of M-matrix in Appendix A) with diagonals equal to 1 and off-diagonal elements given by

\[
g_{i,j} = \frac{B_{i,j}}{\sqrt{B_{i,i} B_{j,j}}} = \frac{-\beta_{i,j}}{\sqrt{\alpha_{i} \alpha_{j}}}.
\] (5.3)
Therefore, the price of anarchy can be expressed as

\[ POA = \frac{4w^T w}{w^T (G + G^{-1} + 2I) w}. \] (5.4)

\[ \Box \]

5.2 Lower Bound for Symmetric Matrix B

In the following section, we first derive a lower bound for the price of anarchy in terms of the minimum eigenvalue of matrix G, which we denote as \( POA(\lambda_{\text{min}}(G)) \). Next, applying Gersgorin's Theorem, we obtain another lower bound in terms of the maximum price sensitivity ratio of G, namely, \( POA(r_{\text{max}}(G)) \). Finally in the end of this section, we show that for the uniform demand model with respect to own sensitivities, the lower bound can be further simplified to only consist of \( r_{\text{max}}(B) \).

5.2.1 POA in terms of \( \lambda_{\text{min}}(G) \)

**Theorem 5.2.1.** Under Assumptions 4 and 5,

\[ POA \geq \frac{4\lambda_{\text{min}}(G)}{(1 + \lambda_{\text{min}}(G))^2}. \]

In order to prove Theorem 5.2.1, we first establish the following proposition.

**Proposition 5.2.2.** Under Assumptions 4 and 5,

\[ \max_{\lambda_i(G)} \left[ \lambda_i(G) + \frac{1}{\lambda_i(G)} \right] = \lambda_{\text{min}}(G) + \frac{1}{\lambda_{\text{min}}(G)}. \]

**Proof.** Let \( x = \lambda(G) \) where \( \lambda(G) \) is an eigenvalue of the matrix \( G \), and \( f(x) = x + \frac{1}{x} \).

For \( x > 0 \), \( \frac{df}{dx} = \frac{2}{x^2} > 0 \). Therefore, \( f(x) \) is a convex function as shown in Figure 5-1.
Convexity ensures that the maximum of function $f(x)$ occurs at an extreme point of the feasible region. Therefore, we only need to analyze the behavior of the function at $\lambda_{\text{min}}(G)$ and $\lambda_{\text{max}}(G)$.

Recall that $G$ is a symmetric diagonally dominant M-matrix with diagonals equal to 1 and negative off-diagonal elements. Let $G = I - R$ where $I$ is an identity matrix and $R$ is a nonnegative matrix with zero diagonal and each off-diagonal element equals to $i_{B_{ij}} O_i$.

In addition,

$$r_{i,j} = \frac{B_{i,j}}{\sqrt{B_{i,i} B_{j,j}}} = \frac{\beta_{i,j}}{\sqrt{\alpha_i \alpha_j}}.$$ 

In addition,

$$\lambda(G) = 1 - \lambda(R) \geq 0$$

$$\lambda_{\text{min}}(G) = 1 - \lambda_{\text{max}}(R).$$ \hspace{1cm} (5.5)

Since $R$ is a nonnegative matrix, the Perron-Frobenius Theorem states that $\rho(R)$ is
an eigenvalue of $\mathbf{R}$ of largest modulus, that is,

$$
\lambda_{\text{max}}(\mathbf{R}) \geq |\lambda(\mathbf{R})|
$$

$$
\Rightarrow \lambda_{\text{max}}(\mathbf{R}) \geq |\lambda_{\text{min}}(\mathbf{R})|
$$

$$
\therefore \lambda_{\text{max}}(\mathbf{R}) > 0
$$

$$
\Rightarrow \lambda_{\text{min}}(\mathbf{G}) = 1 - \lambda_{\text{max}}(\mathbf{R}) \leq 1
$$

$$
\Rightarrow 1 - \lambda_{\text{max}}(\mathbf{R}) \leq 1 - |\lambda_{\text{min}}(\mathbf{R})| \leq |1 - \lambda_{\text{min}}(\mathbf{R})|.
$$

(5.6)

From Equation (5.6), we conclude that $\lambda_{\text{min}}$ is further away from 1 as compared to other eigenvalues of $\mathbf{G}$.

Since $\mathbf{G}$ is positive definite, i.e., $\lambda_{\text{max}}(\mathbf{G}) \geq \lambda_{\text{min}}(\mathbf{G}) > 0$, from Figure 5-1, there are only 2 possibilities to locate $\lambda_{\text{min}}(\mathbf{G})$:

1. Both $\lambda_{\text{max}}(\mathbf{G})$ and $\lambda_{\text{min}}(\mathbf{G}) \leq 1$

   Since $f(x)$ is monotonically decreasing from 0 to 1 and $\lambda_{\text{min}}(\mathbf{G})$ is further away from 1 than $\lambda_{\text{max}}(\mathbf{G})$

   $$
   \Rightarrow f_{\text{max}}(x) \text{ occurs at } x = \lambda_{\text{min}}(\mathbf{G}).
   $$

2. $\lambda_{\text{min}}(\mathbf{G}) \leq 1$ and $\lambda_{\text{max}}(\mathbf{G}) \geq 1$

   If $\lambda_{\text{min}}(\mathbf{G})$ differs from 1 by a positive amount, $\varepsilon$, i.e., $\lambda_{\text{min}}(\mathbf{G}) = 1 - \varepsilon$,

   then $\lambda_{\text{max}}(\mathbf{G}) \leq 1 + \varepsilon$.

   $$
   f(x = \lambda_{\text{min}}(\mathbf{G})) \quad = \quad 1 - \varepsilon + \frac{1}{1 - \varepsilon} = 2 + \frac{\varepsilon^2}{1 - \varepsilon}.
   $$

(5.7)

Because $f(x)$ is monotonically increasing for $x \geq 1$, and $\lambda_{\text{max}}(\mathbf{G}) \geq 1$

$$
\Rightarrow f(x = \lambda_{\text{max}}(\mathbf{G})) \quad \leq \quad 1 + \varepsilon + \frac{1}{1 + \varepsilon} = 2 + \frac{\varepsilon^2}{1 + \varepsilon}.
$$

(5.8)

Comparing Equation (5.7) and (5.8), we conclude that $f(\lambda_{\text{min}}(\mathbf{G})) > f(\lambda_{\text{max}}(\mathbf{G}))$.

This implies that $f(x)$ reaches its maximum when $x = \lambda_{\text{min}}(\mathbf{G})$.  

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\[ f_{\text{max}}(x) = \lambda_{\text{max}}(G + G^{-1}) = \lambda_{\text{min}}(G) + \frac{1}{\lambda_{\text{min}}(G)}. \]

With the result from Proposition 5.2.2, we are ready to prove Theorem 5.2.1.

**Proof for Theorem 5.2.1.** Since \( G \) is a symmetric diagonally dominant M-matrix, it is unitarily diagonalizable,

\[ G = P\Lambda P^T, \]

where \( P \) is a unitary matrix of eigenvectors and \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \).

\[ POA = \frac{4w^T w}{w^T(G + G^{-1} + 2I)w} \]

\[ = \frac{4w^T w}{w^T(P\Lambda P^T + P\Lambda^{-1}P^T + 2I)w} \]

\[ = \frac{4w^T w}{w^T(\Lambda + \Lambda^{-1} + 2I)w}. \quad (5.9) \]

We obtain the last equality because \( P \) is a unitary matrix, that is, the Euclidean length of \( P^T w \) is the same as that of \( w \).

Let \( \lambda \) denote the eigenvalues of the matrix, using the Rayleigh-Ritz Theorem, for any vector \( w \),

\[ \frac{w^T(\Lambda + \Lambda^{-1} + 2I)w}{w^T w} \leq \max_{\lambda_i(G)} \left[ \frac{\lambda_i(G)}{\lambda_i(G) + 2}. \right. \]
Therefore, the price of anarchy is bounded by

\[ POA \geq \frac{4}{\max_{\lambda_i(G)} [\frac{\lambda_i(G)}{\lambda(G)} + \frac{1}{\lambda(G)} + 2]} . \]  

(5.10)

By Proposition 5.2.2, we obtain a lower bound for the price of anarchy in terms of the minimum eigenvalue of \( G \),

\[ POA \geq \frac{4}{\lambda_{\min}(G) + \frac{1}{\lambda_{\min}(G)} + 2} \]

\[ = \frac{4\lambda_{\min}(G)}{(1 + \lambda_{\min}(G))^2} . \]  

(5.11)

\[ \square \]

**Proposition 5.2.3.** \( POA(\lambda_{\min}(G)) \) and the actual price of anarchy value coincide when \( d = (B + \Gamma)\Gamma^{-\frac{1}{2}}kv \), i.e., \( w = kv \), where \( v \) is the eigenvector corresponding to \( \lambda_{\min}(G) \), \( k \in \mathbb{R} \).

*Proof.* Let \( w = kv \) such that \( Gw = kGv = k\lambda_{\min}(G)v \).

From Equation (5.9),

\[ POA = \frac{4(kv)^T(kv)}{(kv)^T(\Lambda + \Lambda^{-1} + 2I)(kv)} \]

\[ = \frac{4(kv)^T(kv)}{(kv)^Tk(\lambda_{\min}(G) + \frac{1}{\lambda_{\min}(G)} + 2)v} \]

\[ = \frac{4(kv)^T(kv)}{(\lambda_{\min}(G) + \frac{1}{\lambda_{\min}(G)} + 2)(kv)^T(kv)} \]

\[ = \frac{4}{\lambda_{\min}(G) + \frac{1}{\lambda_{\min}(G)} + 2} \]

\[ = \frac{4\lambda_{\min}(G)}{(1 + \lambda_{\min}(G))^2} \]

\[ = POA(\lambda_{\min}(G)) . \]

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5.2.2 POA in terms of $r_{\text{max}}(G)$

**Corollary 5.2.4.** Under Assumptions 2 and 3,

$$POA \geq \frac{4(1 - r_{\text{max}}(G))}{(2 - r_{\text{max}}(G))^2}.$$ 

**Proof.** Using Gersgorin’s Theorem, it follows that the eigenvalues of $G$ are located in at least one of the disks:

$$|\lambda_{\text{min}}(G) - g_{i,i}| \leq \sum_{j=1}^{n} |g_{i,j}|, \quad 1 \leq i \leq n$$

$$\lambda_{\text{min}}(G) - g_{i,i} \geq -\sum_{j=1}^{n} |g_{i,j}|, \quad 1 \leq i \leq n$$

Since $g_{i,i} = 1 \quad \forall i,$

$$\lambda_{\text{min}}(G) \geq 1 - \sum_{j=1}^{n} |g_{i,j}|, \quad 1 \leq i \leq n.$$ 

The price sensitivity ratio for seller $i$ is defined as:

$$r_i(G) = \sum_{j \neq i} \frac{|g_{i,j}|}{g_{i,i}} = \sum_{j \neq i} |g_{i,j}|.$$ 

We reach the last equality because all the diagonal elements of $G$ are equal to 1.

Since $G$ is an M-matrix and diagonally dominant (refer to Appendix A), $r_i(G) \leq 1,$

$$\lambda_{\text{min}}(G) \geq 1 - r_i(G), \quad \text{for some } i \quad \Rightarrow \lambda_{\text{min}}(G) \geq 1 - r_{\text{max}}(G). \quad (5.12)$$
Substitute Equation (5.12) into Equation (5.11) to obtain another lower bound which is in terms of $r_{\text{max}}(G)$.

$$POA \geq \frac{4}{1 - r_{\text{max}}(G) + \frac{1}{1 - r_{\text{max}}(G)} + 2}$$

$$= \frac{4(1 - r_{\text{max}}(G))}{(1 - r_{\text{max}}(G))^2 + 1 + 2(1 - r_{\text{max}}(G))}$$

$$= \frac{4(1 - r_{\text{max}}(G))}{(2 - r_{\text{max}}(G))^2}. \quad (5.13)$$

Remark 5.2.5. Notice that the lower bound in (5.12) is easily computable.

Corollary 5.2.6. For the uniform model with respect to own sensitivities (2.2), the lower bound can be further simplified to:

$$POA \geq \frac{4(1 - r_{\text{max}}(B))}{(2 - r_{\text{max}}(B))^2}.$$

Proof. In the uniform demand model with respect to own sensitivities, the price sensitivity matrix $B$ has the same diagonal elements denoted by $\alpha$ while the off-diagonal elements $\beta_{i,j}$ are allowed to vary across all sellers. Each off-diagonal element of $G$ is then given by

$$g_{i,j} = \frac{-\beta_{i,j}}{\sqrt{\alpha_i \alpha_j}} = \frac{-\beta_{i,j}}{\alpha}.$$

$$\sum_{j=1}^{n} |g_{i,j}| = \sum_{j=1}^{n} \frac{\beta_{i,j}}{\alpha} = r_i(B).$$
Corollary 5.2.4 implies
\[ r_i(B) = r_i(G). \]
Therefore,
\[ POA \geq \frac{4(1 - r_{\max}(B))}{(2 - r_{\max}(B))^2}. \tag{5.15} \]

Remark 5.2.7. The lower bound in Corollary 5.2.6 generalizes the result with the general uniform demand case. Notice that when \( B \) is a general uniform matrix (2.1), \( r_{\max}(G) = r \), and the result coincides with that in Theorem 4.1.1.

5.3 Upper Bound for Symmetric Matrix \( B \)

Theorem 5.3.1. Let \( \bar{\lambda} \) and \( \check{\lambda} \) denote the two eigenvalues of \( G \) which are the closest to 1 (from above and below). Under Assumptions 1 to 5, \( POA \leq \min_{\lambda_i \in \{\bar{\lambda}, \check{\lambda}\}} \frac{4\lambda_i}{(1 + \lambda_i)^2} \).

Proof. Applying the Rayleigh-Ritz Theorem,
\[ POA \leq \frac{4}{\min_{\lambda_i(G)}(G + G^{-1} + 2I)} \]
\[ = \frac{4}{\min_{\lambda_i(G)} \left[ \lambda_i(G) + \frac{1}{\lambda_i(G)} + 2 \right]}. \]

From Figure 5-1, \( x + \frac{1}{x} + 2 \) is a convex function whose minimum occurs at 1. Thus, we only need to restrict to the 2 eigenvalues that are the closest to the 1.

Arrange all the eigenvalues of \( G \) in a non-decreasing order, \( \lambda_1(G) \leq \lambda_2(G) \ldots \leq \lambda_n(G) \).
Proposition 5.3.2. For the uniform model with respect to competitors' sensitivities,

\[ POA \leq \frac{4\lambda_2(G)}{(1 + \lambda_2(G))^2}. \]

To prove Proposition 5.3.2, we first establish the following Lemma 5.3.3 and Proposition 5.3.4.

Lemma 5.3.3. For the general uniform demand model,

\[ POA \leq \frac{4\lambda_{\text{max}}(G)}{(1 + \lambda_{\text{max}}(G))^2} = \frac{4\lambda_2(G)}{(1 + \lambda_2(G))^2}. \]

Proof. For the general uniform demand model,

\[ B_u = \begin{bmatrix} \alpha & -1 \\ & \ddots \\ -1 & \alpha \end{bmatrix}, \quad G_u = \begin{bmatrix} 1 & -\frac{1}{\alpha} \\ & \ddots \\ -\frac{1}{\alpha} & 1 \end{bmatrix}, \quad R_u = \begin{bmatrix} 0 & \frac{1}{\alpha} \\ \frac{1}{\alpha} & 0 \end{bmatrix}. \]

The matrix \( G_u \) only has 2 distinct eigenvalues, one at \( \frac{n-1}{\alpha} = \lambda_1(G) = \lambda_{\text{min}}(G) \) and \( n-1 \) repeated eigenvalues at \( \frac{1}{\alpha} = \lambda_2(G) = \lambda_{\text{max}}(G) \). Since \( \lambda(G) = 1 - \lambda(R) \), \( R_u \) only has 2 distinct eigenvalues, one positive eigenvalue at \( \frac{n-1}{\alpha} \), and \( n-1 \) negative eigenvalues at \( -\frac{1}{\alpha} \).

Using similar arguments as in Theorem 5.2.1, due to the convexity of function \( x + \frac{1}{x} \), it follows that

\[ \min_{\lambda_i(G)} \left[ \frac{1}{\lambda_i(G)} + \frac{1}{\lambda_i(G)} \right] = \lambda_{\text{max}}(G) + \frac{1}{\lambda_{\text{max}}(G)} = \lambda_2(G) + \frac{1}{\lambda_2(G)}. \]

Therefore,

\[ POA \leq \frac{4\lambda_{\text{max}}(G)}{(1 + \lambda_{\text{max}}(G))^2} = \frac{4\lambda_2(G)}{(1 + \lambda_2(G))^2}. \]
Proposition 5.3.4. For the uniform demand model with respect to competitors’ sensitivities, $R$ is congruent to $R_u$.

**Proof.** Given a price sensitivity matrix $B$ which is uniform with respect to competitors’ sensitivities,

$$B = \begin{bmatrix} \alpha_1 & \cdots & -1 \\ \vdots \\ -1 & \alpha_n \end{bmatrix} \quad G = \begin{bmatrix} 1 & -\frac{1}{\sqrt{\alpha_1 \alpha_2}} & \cdots & -\frac{1}{\sqrt{\alpha_1 \alpha_n}} \\ -\frac{1}{\sqrt{\alpha_2 \alpha_1}} & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & -\frac{1}{\sqrt{\alpha_{n-1} \alpha_n}} \\ -\frac{1}{\sqrt{\alpha_n \alpha_1}} & \cdots & -\frac{1}{\sqrt{\alpha_{n-1} \alpha_{n-1}}} & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & \frac{1}{\sqrt{\alpha_1 \alpha_2}} & \cdots & \frac{1}{\sqrt{\alpha_1 \alpha_n}} \\ \frac{1}{\sqrt{\alpha_2 \alpha_1}} & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \frac{1}{\sqrt{\alpha_{n-1} \alpha_n}} \\ \frac{1}{\sqrt{\alpha_n \alpha_1}} & \cdots & \frac{1}{\sqrt{\alpha_{n-1} \alpha_{n-1}}} & 0 \end{bmatrix}$$

Sylvester’s law of inertia is stated as the following: let $A, B \in M_n$ be Hermitian matrices. There is a nonsingular matrix $S \in M_n$ such that $A = SBS^*$ if and only if $A$ and $B$ have the same inertia, that is, the same number of positive, negative, and zero eigenvalues.

We can use a diagonal matrix $D$ whose diagonal element equals to $d_i = \sqrt{\frac{\alpha}{\alpha_i}}$ to transform $R_u$ into $R$, where $\alpha$ is the diagonal element of matrix $B_u$. 

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With the results obtained in Lemma 5.3.3 and Proposition 5.3.4, we can now prove Proposition 5.3.2.

Proof for Proposition 5.3.2. We have shown that $R$ and $R_u$ are congruent, and thus, $R$ has the same number of positive and negative eigenvalues as $R_u$ (i.e., 1 positive eigenvalue and $n-1$ negative ones as shown in Figure 5-2).

Since $\lambda(G) = 1 - \lambda(R)$, we can conclude that for the uniform model with respect to competitors' sensitivities, $G$ has only 1 eigenvalue that is less than 1 and $n-1$ of them
that are greater than 1.

If we arrange all the eigenvalues of $G$ in a non-decreasing order, i.e., $\lambda_1(G) \leq \lambda_2(G) \ldots \leq \lambda_n(G)$, where $\lambda_1(G)$ is smaller than 1 and the rest of them are larger than 1.

As we have shown earlier $\lambda_{\min}(G) = \lambda_1(G)$ is further away from 1 than any other eigenvalue therefore, $\lambda_2(G)$ is the eigenvalue which is the closest to 1. In addition, $f(x)$ is a monotonically increasing function after 1, and thus $f(x)$ is at its minimum at $\lambda_2(G)$.

Therefore,

$$POA \leq \frac{4}{\min_{\lambda_i(G)} \left[ \lambda_i(G) + \frac{1}{\lambda_i(G)} \right]}$$

$$= \frac{4}{\lambda_2(G) + \frac{1}{\lambda_2(G)} + 2}$$

$$= \frac{4\lambda_2(G)}{(1 + \lambda_2(G))^2}.$$

$\square$

Remark 5.3.5. Proposition 5.3.2 is a special case of Theorem 5.3.2. Notice that it does not necessarily apply in general. Consider the following example,

$$G = \Gamma^{-\frac{1}{2}}B\Gamma^{-\frac{1}{2}} = \begin{bmatrix}
1.0000 & -0.0713 & -0.1060 & -0.0617 & -0.0162 \\
-0.0713 & 1.0000 & -0.0181 & -0.0672 & -0.1778 \\
-0.1060 & -0.0181 & 1.0000 & -0.1369 & -0.1715 \\
-0.0617 & -0.0672 & -0.1369 & 1.0000 & -0.1743 \\
-0.0162 & -0.1778 & -0.1715 & -0.1743 & 1.0000
\end{bmatrix}$$

Values 0.5776, 0.9909, 1.0209, 1.1438, 1.2668 represent the eigenvalues of matrix $G$. Notice that more than one eigenvalue is smaller than 1. As a result, it is not apparent
(a) Eigenvalues of a general uniform demand model

(b) Eigenvalues of a uniform demand model with respect to competitors’ sensitivities

Figure 5-2: Eigenvalues of a general uniform demand model and a uniform demand model with respect to competitors’ sensitivities
which of the two eigenvalues (the one below one, i.e. 0.9909 or the one above one, i.e., 1.0209) achieves the minimum of $\lambda(G) + \frac{1}{\lambda(G)}$. Nonetheless, Theorem 5.3.1 still applies.

5.4 Tightness of the Bounds

In this section, we analyze the tightness of the bounds derived in the earlier sections by varying vector $d$ and comparing the bounds with the actual price of anarchy value. First we consider the lower bounds in terms of $\lambda_{\min}(G)$ and $r_{\max}(G)$, which we denote as $POA(\lambda_{\min}(G))$ and $POA(r_{\max}(G))$ respectively. Next, in Section 5.4.2, we evaluate the tightness of the upper bound, namely, $POA(\lambda_2(G))$.

5.4.1 Lower Bound

In this simulation, we use a general price sensitivity matrix $B$ that satisfies Assumptions 1 to 5. We vary the number of sellers from 2 to 20 and randomly generate the matrix $B$ (each generation of its elements is referred to as iteration). Figures 5-3 to 5-6 depict the simulation results when we vary vector $d$ according to the following scenarios,

1. $d = (B + \Gamma)\Gamma^{-\frac{1}{2}}kv$, i.e., $w = kv$ where $v$ is the eigenvector corresponding to $\lambda_{\min}(G)$ and $k \in \mathbb{R}$.

2. $d = (d, d, \ldots, d)$, it corresponds to the case when there is no quality difference among sellers. As we have shown the coefficient of variation is at its minimum, i.e., $CV^2 = 0$.

3. $d$ is a random vector where $d = (d_1, d_2, \ldots, d_n)$ and $d_i$ differs across the sellers.

4. The vector $d$ has only one non-zero element. For this case, $CV^2$ is at its maximum, i.e., $CV^2 = n - 1$, where $n$ is the number of sellers in the market.
Lower bounds when \( w = v \) corresponding to \( \lambda_{\min}(G) \)

Figure 5-3: Lower bound when \( w \) is the eigenvector corresponding to the minimum eigenvalue of \( G \)

Note that in both scenarios 3 and 4, quality differences exist among the sellers.

**Discussions**

1. \( POA(\lambda_{\min}(G)) \) is fairly tight to the actual value of the price of anarchy. For example, in Figures 5-3, 5-4 and 5-5, the lower bound is either tight or very close to the actual value.

2. \( POA(r_{\max}(G)) \) is not as tight as \( POA(\lambda_{\min}(G)) \). This result is expected as \( POA(r_{\max}(G)) \) bounds \( POA(\lambda_{\min}(G)) \) from below. Nonetheless, it is much easier to compute \( r_{\max}(G) \) than the minimum eigenvalue of \( G \).

3. The lower bound in terms of \( \lambda_{\min}(G) \) can be tight as compared to the actual price of anarchy values as shown in Figure 5-3. We have shown in Proposition 5.2.3 that the two values coincide when \( w \) is the eigenvector corresponding to the minimum eigenvalue of \( G \).
Figure 5-4: Lower bound when \( d \) is the same for all sellers. i.e., no quality differences

Figure 5-5: Lower bound when \( d \) is a random vector
4. When the coefficient of variation is not large, the lower bounds in terms of both $\lambda_{\text{min}}(G)$ and $r_{\text{max}}(G)$ are rather tight to the actual price of anarchy value as shown in Figure 5-3, 5-4 and 5-5.

5. The worst case for the lower bound occurs when $CV^2 = n - 1$ as depicted in Figure 5-6. This is because the lower bound only involves the minimum eigenvalue of $G$ and does not take $d$ into consideration, and as shown in the previous chapter, if $n$ and $r$ is fixed, the price of anarchy is at its maximum when $CV^2 = n - 1$.

### 5.4.2 Upper Bound

In this section, we analyze the tightness of the upper bound. We consider a uniform matrix $B$ with respect to competitors’ sensitivities, as a result, the bound is expressed in terms of the second largest eigenvalue of matrix $G$, i.e., $POA(\lambda_2(G))$. 
Upper bounds when \( w = v_2 \)

Figure 5-7: Upper bound when \( w \) is the eigenvector corresponding to the second eigenvalue of \( G \)

For the simulation, we vary \( d \) for almost the same scenarios as in the lower bound analysis, except for Figure 5-7, where \( w = \varphi u \), \( u \) is the eigenvector corresponding to the \( \lambda_2(G) \) (which is closest to 1) and \( \varphi \in \mathbb{R} \).

Discussions

1. From the simulation results, we see that the upper bound is not very tight. The result is not surprising, because given Equation (5.16) and \( 1 < \lambda_2(G) \leq 2 \), the minimum value of \( POA(\lambda_2(G)) \) is 0.888.

2. The upper bound is tight in Figure 5-7, i.e., when \( w \) is the scaled eigenvector which corresponds to \( \lambda_2(G) \). The same reasoning in Proposition 5.2.3 that explains when the lower bound is tight can be applied here.

3. When the values of \( d_i \) are the same, i.e., when quality differences do not exist among sellers, the upper bound is not tight as compared to the actual price of
Figure 5-8: Upper bound when $d$ is the same for all sellers. i.e., no quality differences

Figure 5-9: Upper bound when $d$ is a random vector
Figure 5-10: Upper bound when \( d \) has only one non-zero term, i.e., \( CV^2 = n - 1 \) anarchy as shown in Figure 5-8.

4. When the values of \( d_i \) are randomly generated, i.e., quality differences exist among sellers, the upper bound is not very tight as depicted in Figure 5-9.

5. When only one \( d_i \) has non-zero value, (i.e., the coefficient of variation is maximum), the upper bound is tighter (i.e., closer to the to the actual price of anarchy) in shown in Figure 5-10. This is because the actual price of anarchy is maximum when coefficient of variation is at its maximum, i.e., \( CV^2 = n - 1 \).

### 5.5 Lower bound for Asymmetric Matrix B

In general, the lower bound (5.11) only holds for a symmetric M-matrix \( B \). For instance, Figure 5-11 shows a simulation result when an asymmetric matrix \( B \) is used. We see that the lower bound that uses only the minimum eigenvalue of matrix \( G \) does not hold anymore.
5.5.1 A Measure of Asymmetry

In this section, we briefly introduce the concept of measuring the asymmetry of a matrix in this section. We refer the reader to Perakis [24] for more insights on this topic.

For an asymmetric matrix $B$, we consider the symmetrized matrix

$$ S = \frac{B + B^T}{2}. $$

Let $E = S^{-1}B$. We introduce the asymmetry constants, $c_1$ and $c_2$, which measure the degree of asymmetry of the matrix $B$. 

Figure 5-11: The lower bound (5.11) fails to hold for an asymmetric matrix $B$
Definition 5.5.1.

\[ c_1^2 = \|E\|_S^2 = \sup_{x \neq 0} \frac{\|Ex\|_S^2}{\|x\|_S^2} = \|S^{-\frac{1}{2}}BS^{-\frac{1}{2}}\|_2^2 = \lambda_{\max}(S^{-\frac{1}{2}}BS^{-\frac{1}{2}}) \]

\[ c_2^2 = \|E\|_2^2 = \sup_{x \neq 0} \frac{\|Ex\|_2^2}{\|x\|_2^2} = \lambda_{\max}(E^TSE) = \lambda_{\max}(B^TS^{-1}S^{-1}B). \]

When the matrix \( B \) is symmetric, that is, \( B = B^T \) (and therefore, \( S = B \)), then \( c_1 = c_2 = 1 \).

Notice \( c_1 = \|S^{-\frac{1}{2}}BS^{-\frac{1}{2}}\|_2 \), thus,

\[ c_1 \leq \|S^{\frac{1}{2}}\|_2\|S^{-\frac{1}{2}}\|_2c_2. \]

But note that numerically there seems to be a tighter connection.

5.5.2 A Lower Bound for Asymmetric Matrix \( B \)

When the price sensitivity matrix \( B \) becomes, the price of anarchy is as follows,

\[ \text{POA} = \frac{4((B + \Gamma)^{-1}d)\Gamma((B + \Gamma)^{-1}d) d^T \Gamma^{-1}d}{d^T \Gamma^{-1}d}. \] (5.16)

Theorem 5.5.1. Given an asymmetric matrix \( B \) which is uniform with respect to the seller's own sensitivities, we define

\[ G = \Gamma^{-\frac{1}{2}}S\Gamma^{-\frac{1}{2}}. \]

A lower bound for the price of anarchy is given as

\[ \text{POA} \geq \frac{4}{c_2^2 \lambda_{\min}(G) + \frac{1}{\lambda_{\min}(G)} + 2c_2} \]

\[ = \frac{4\lambda_{\min}(G)}{c_2^2 \lambda_{\min}(G)^2 + 2c_2 \lambda_{\min}(G) + 1}. \]

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where $c_1 = \|E\|_s$, and, $c_2 = \|E\|_2$.

To prove Theorem 5.5.1, we first need to establish the following proposition.

**Proposition 5.5.2.**

\[
\frac{w^T E^T G E w}{w^T w} \leq c_1^2 \frac{w^T G w}{w^T w},
\]

where $c_1 = \|E\|_s$.

**Proof.**

\[
\frac{w^T E^T G E w}{w^T w} = \frac{w^T B^T S^{-\frac{1}{2}} S^{-\frac{1}{2}} B w}{w^T w} = \frac{\Gamma^{-\frac{1}{2}} w^T B^T S^{-1} S S^{-1} B \Gamma^{-\frac{1}{2}} w}{w^T w} = \frac{\|S^{-1} B \Gamma^{-\frac{1}{2}} w\|_S^2}{w^T w} \leq \|S^{-1} B\|_S^2 \frac{\|\Gamma^{-\frac{1}{2}} w\|_S^2}{w^T w} = \|E\|_S^2 \frac{w^T \Gamma^{-\frac{1}{2}} S \Gamma^{-\frac{1}{2}} w}{w^T w} = c_1^2 \frac{w^T G w}{w^T w}
\]

**Proof for Theorem 5.5.1.** When matrix $B$ is uniform with respect to own price sensitivities, the diagonal elements are equal and we can write $\Gamma = \alpha I$, where $I$ is an identity matrix. Also,

\[
G = \Gamma^{-\frac{1}{2}} S \Gamma^{-\frac{1}{2}} = \frac{S}{\alpha}.
\]

Let $w = \Gamma^{\frac{1}{2}} (B + \Gamma)^{-1} d$. From Equation (5.16),

\[
POA = \frac{4w^T w}{w^T \Gamma^{-\frac{1}{2}} (B^T + \Gamma) S^{-1} (B + \Gamma) \Gamma^{-\frac{1}{2}} w} = \frac{4w^T w}{w^T \Gamma^{-\frac{1}{2}} (B S^{-1} B + B^T S^{-1} \Gamma + \Gamma S^{-1} B + \Gamma S^{-1} \Gamma) \Gamma^{-\frac{1}{2}} w}.
\]
Let $E = S^{-1}B$, and $\Gamma = \alpha I$,

$$
POA = \frac{4w^T w}{w^T(\frac{1}{\alpha}B^T S^{-1}B + 2E + \alpha S^{-1})w} \\
= \frac{4w^T w}{w^T(E^TGE + G^{-1} + 2E)w}.
$$

(5.17)

Applying Proposition 5.5.2, Equation (5.17) is bounded from below by

$$
POA \geq \frac{4w^T w}{w^T(c_1^2G + G^{-1} + 2E)w}.
$$

Using similar arguments as in the symmetric case, it follows that,

$$
POA \geq \frac{4}{c_1^4 \lambda_{\min}(G) + \frac{1}{\lambda_{\min}(G)} + 2c_2} \\
= \frac{4\lambda_{\min}(G)}{c_1^4 \lambda_{\min}(G)^2 + 2c_2 \lambda_{\min}(G) + 1}. 
$$

(5.18)

where $c_1 = \|E\|_S$, $c_2 = \|E\|_2$. □

### 5.5.3 Tightness of the Lower Bound under Asymmetry

In this section, we analyze the tightness of the lower bound derived for an asymmetric matrix $B$. We use the same scenarios as in Section 5.4.1. The results for each scenario are shown from Figure 5-12 to 5-15.

**Discussions**

1. Figure 5-13 shows the result when the values of $d_i$ are the same, i.e., when quality differences do not exist among the sellers. The lower bound is fairly close to the actual value of price of anarchy.

2. Figure 5-14 depicts the result when quality differences exist among the seller.
Figure 5-12: Asymmetric case: Lower bound when \( w = v_{\min}(G) \)

Figure 5-13: Asymmetric case: Lower bound when \( d = \text{ones}(n,1) \)

Figure 5-13: Asymmetric matrix: Lower bound when \( d \) is the same for all sellers. i.e. no quality differences
Figure 5-14: Asymmetric matrix: Lower bound when \( d \) is a random vector

Figure 5-15: Asymmetric matrix: Lower bound when \( d \) has only one non-zero term, i.e. \( CV^2 = n - 1 \)
The values of $d_i$ are randomly generated. In Figure 5-14, the lower bound is fairly tight.

3. In Figure 5-12, the lower bound is not exact as compared to the actual value of price of anarchy when $w$ is the eigenvector corresponding to the minimum eigenvalue of $G$. Recall that the lower bound is tight when the price sensitivity matrix is symmetric (see Proposition 5.2.3). This is because the denominator of the closed-form solution for $POA$ also includes $E^T G E$, and in general, the eigenvectors of $G$ and $E^T G E$ are not the same.

4. Similar to the symmetric case, the worst-case for the lower bound happens when only one seller has non-zero demand as shown in Figure 5-15. This is because the price of anarchy is at its maximum when $CV^2$ is maximum, but the lower bound does not take the demand variation into account.

5. In general, the lower bound in (5.18), which is in terms of $\lambda_{\min}(G)$ and the asymmetry constants $c_1$ and $c_2$ is not as tight as that for the symmetric matrix as shown in Section 5.4.1.

In summary, we notice the following: including the asymmetry constants $c_1$ and $c_2$ gives a valid lower bound to the price of anarchy when the price sensitivity matrix $B$ is asymmetric. In addition, we observe that the lower bound for an asymmetric matrix is not as tight as that for the symmetric case (nevertheless, it is still fairly close). The worst-case for the lower bound happens when the coefficient of variation is maximum, i.e., when one seller has the non-zero demand.
Chapter 6

Nonlinear Demand Model

In this chapter, we study the price of anarchy for the nonlinear demand model. The analysis in this chapter is carried out under Assumptions 6 and 7 described in Section 2.2. We show that the price of anarchy for the nonlinear model can be expressed in a form similar to the affine case. Subsequently, we derive a lower and an upper bound for the price of anarchy using arguments similar to the affine demand model. The results discussed in this chapter can also be found in Farahat, Perakis and Sun [13].

6.1 Jacobian Similarity Property

In this section, we briefly introduce the concept of the Jacobian similarity property. We refer the reader to J. Sun [31] for more information on this concept.

**Definition 6.1.1 (The Jacobian similarity property).** Given $-J_h(p)$ is positive definite, there exist $A \geq 1$ such that for all $w$ and $p$ and $p'$:

$$Aw^T(-J_h(p))w \geq w^T(-J_h(p'))w \geq \frac{1}{A}w^T(-J_h(p))w. \quad (6.1)$$

**Proposition 6.1.1 (The inverse Jacobian similarity property).** If a matrix $-J_h(p)$
satisfies Equation (6.1), so does its inverse with the same constant $A$:

$$Aw^T(-Jh(p))^{-1}w \geq w^T(-Jh(p'))^{-1}w \geq \frac{1}{A}w^T(-Jh(p))^{-1}w. \quad (6.2)$$

**Proof.** Let $A = \max_p \frac{\lambda_{\max}(-Jh(p))}{\lambda_{\min}(-Jh(p))}$. Considering the inverse matrix $-Jh(p)^{-1}$, the Jacobian similarity constant for the inverse remains the same since

$$\lambda_{\max}(-Jh(p)^{-1}) = \frac{1}{\lambda_{\min}(-Jh(p))}, \quad \text{and}$$
$$\lambda_{\min}(-Jh(p)^{-1}) = \frac{1}{\lambda_{\max}(-Jh(p))}.$$ 

Thus,

$$A = \max_p \frac{\lambda_{\max}(-Jh(p)^{-1})}{\lambda_{\min}(-Jh(p)^{-1})} = \max_p \frac{\lambda_{\max}(-Jh(p))}{\lambda_{\min}(-Jh(p)).}$$

\[\Box\]

### 6.2 User and System Optimality Conditions

To shorten notation, we use $h_{UO}$, $h_{SO}$, $B_{UO} = -Jh(p_{UO})$, $B_{SO} = -Jh(p_{SO})$, $\Gamma_{UO} = \text{diag}(B_{UO})$ and $\Gamma_{SO} = \text{diag}(B_{SO})$ to denote the demand function, the Jacobian matrix and the diagonal elements of that Jacobian matrix at user and system optimum respectively.

The user optimization problem is given as:

$$\text{UO(Nonlinear)} : \max_{p_{UO,i}} p_{UO,i} \cdot h_i(p_{UO,i}, \tilde{p}_{UO,i}) \quad \text{s.t.} \quad p_i \geq 0$$

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At optimality,

\[ h_{\text{uO}} - T_{\text{uO}p_{\text{uO}}} = 0 \]
\[ \Rightarrow h_{\text{uO}} = T_{\text{uO}p_{\text{uO}}}. \tag{6.3} \]

The system optimization problem is shown as the following:

\[
\text{SO(Nonlinear)} : \begin{array}{c}
\max_{p_{\text{SO}}} \ p_{\text{SO}}^T h_{\text{SO}} \\
\text{s.t.} \quad p_{\text{SO}} \geq 0
\end{array}
\]

The optimality conditions ensure that Equation (6.4) holds.

\[ h_{\text{SO}} - B_{\text{SO}p_{\text{SO}}} = 0 \]
\[ \Rightarrow h_{\text{SO}} = B_{\text{SO}p_{\text{SO}}}. \tag{6.4} \]

6.3 Lower Bound

**Theorem 6.3.1.** Under Assumptions 6 and 7, given a nonlinear symmetric demand model, a lower bound for the price of anarchy is given as

\[
\text{POA} \geq \frac{(A + 1)^2}{A^3} \frac{\lambda_{\text{min}}(G_{\text{uO}})}{(1 + \lambda_{\text{min}}(G_{\text{uO}}))^2}.
\]

**Proof.** First note that

\[
\text{POA} = \frac{Z_{\text{uO}}}{Z_{\text{SO}}} = \frac{h_{\text{uO}p_{\text{uO}}}}{h_{\text{SO}p_{\text{SO}}}} = \frac{p_{\text{uO}}^T T_{\text{uO}p_{\text{uO}}}}{p_{\text{SO}}^T B_{\text{SO}p_{\text{SO}}}}. \tag{6.5}
\]

The goal in the analysis below is to bound the denominator \( p_{\text{SO}}^T B_{\text{SO}p_{\text{SO}}} \).
Since \( h_{uo} \) is a concave function,
\[
-h_{so} \geq -h_{uo} + B_{uo}(p_{so} - p_{uo}). \tag{6.6}
\]

Substitute Equation (6.3) and (6.4) into Equation (6.6) to obtain
\[
-B_{so}p_{so} \geq -\Gamma_{uo}p_{uo} + B_{uo}(p_{so} - p_{uo})
-(B_{so} + B_{uo})p_{so} \geq -(\Gamma_{uo} + B_{uo})p_{uo}
(B_{so} + B_{uo})p_{so} \leq (\Gamma_{uo} + B_{uo})p_{uo}. \tag{6.7}
\]

Similarly, since \( h_{so} \) is also a concave function, we obtain Equation (6.8). We substitute Equation (6.3) and (6.4) into Equation (6.8).
\[
-h_{uo} \geq -h_{so} + B_{so}(p_{uo} - p_{so}). \tag{6.8}
\]
\[
-\Gamma_{uo}p_{uo} \geq -B_{so}p_{so} + B_{so}(p_{uo} - p_{so})
-(\Gamma_{uo} + B_{so})p_{uo} \geq -(B_{so} + B_{so})p_{so}
(\Gamma_{uo} + B_{so})p_{uo} \leq 2B_{so}p_{so}. \tag{6.9}
\]

We multiply Equation (6.7) by \( p_{so} \geq 0 \) to obtain Equation (6.10).
\[
 p_{so}^T(B_{so} + B_{uo})p_{so} \leq p_{so}^T(\Gamma_{uo} + B_{uo})p_{uo}
 p_{so}^T(B_{so})p_{so} + p_{so}^T(B_{uo})p_{so} \leq p_{so}^T(\Gamma_{uo} + B_{uo})p_{uo}. \tag{6.10}
\]

Using the Jacobian similarity property, we observe that
\[
 p_{so}^T B_{uo} p_{so} \geq \frac{1}{A} p_{so}^T B_{so} p_{so}. \tag{6.11}
\]
Thus, Equation (6.10) becomes
\[ p_{so}^T(B_{so})p_{so} + \frac{1}{A} p_{so}^T(B_{so})p_{so} \leq p_{so}^T(T_{uo} + B_{uo})p_{uo} \]
\[ \left( 1 + \frac{1}{A} \right) p_{so}^T(B_{so})p_{so} \leq p_{so}^T(T_{uo} + B_{uo})p_{uo}. \quad (6.12) \]

We replace Equation (6.12) by \( I = B_{uo}B_{uo}^{-1} \), and apply the Cauchy’s inequality to obtain Equation (6.14).
\[ \left( 1 + \frac{1}{A} \right) \| p_{so} \|^2_{B_{so}} \leq \| p_{so} \|_{B_{so}} \| B_{so}^{-1}(T_{uo} + B_{uo})p_{uo} \|_{B_{so}}. \quad (6.13) \]

After canceling the term \( \| p_{so} \|_{B_{so}} \) from both sides of the inequality, we obtain
\[ \left( 1 + \frac{1}{A} \right) \| p_{so} \|_{B_{so}} \leq \| B_{so}^{-1}(T_{uo} + B_{uo})p_{uo} \|_{B_{so}} \]
\[ \| p_{so} \|_{B_{so}} \leq \left( \frac{A}{A+1} \right) \| B_{so}^{-1}(T_{uo} + B_{uo})p_{uo} \|_{B_{so}}. \]

We square both sides of the inequality to obtain
\[ \| p_{so} \|^2_{B_{so}} \leq \left( \frac{A}{A+1} \right)^2 \| p_{uo}^T(T_{uo} + B_{uo})B_{so}^{-1}(T_{uo} + B_{uo})p_{uo} \|_{B_{so}}. \quad (6.14) \]

Substituting inequality (6.14) into the denominator of Equation (6.5),
\[ PQA \geq \left( \frac{A+1}{A} \right)^2 \frac{p_{uo}^T(T_{uo} + B_{uo})^TB_{so}^{-1}(T_{uo} + B_{uo})p_{uo}}{p_{uo}^T(T_{uo} + B_{uo})^TB_{so}^{-1}(T_{uo} + B_{uo})p_{uo}}. \quad (6.15) \]

Applying the Jacobian similarity property to the inverse matrix, and using Proposi-
tion 6.2,
\[ p_{uo}^T (T_{uo} + B_{uo})^T B_{so}^{-1} (T_{uo} + B_{uo}) p_{uo} \leq A p_{uo}^T (T_{uo} + B_{uo})^T B_{uo}^{-1} (T_{uo} + B_{uo}) p_{uo}. \]

The lower bound for the price anarchy function becomes
\[
POA \geq \frac{(A + 1)^2}{A^3} \frac{p_{uo}^T T_{uo} p_{uo}}{p_{uo}^T (T_{uo} + B_{uo})^T B_{uo}^{-1} (T_{uo} + B_{uo}) p_{uo}}.
\]

We can use the same argument as in the affine demand model. Let \( w = \Gamma_{uo}^{-\frac{1}{2}} p_{uo} \) and \( G_{uo} = \Gamma_{uo}^{-\frac{1}{2}} B_{uo} \Gamma_{uo}^{-\frac{1}{2}} \), the lower bound can be further simplified as follows:
\[
POA \geq \frac{(A + 1)^2}{A^3} \frac{w^T w}{w^T \Gamma_{uo}^{-\frac{1}{2}} (\Gamma_{uo} + B_{uo})^T B_{uo}^{-1} (\Gamma_{uo} + B_{uo}) \Gamma_{uo}^{-\frac{1}{2}} w} = \frac{(A + 1)^2}{A^3} \frac{w^T w}{w^T (G_{uo} + G_{uo}^{-1} + 2I) w} \geq \frac{(A + 1)^2}{A^3} \frac{1}{\lambda_{\max}(G_{uo} + G_{uo}^{-1} + 2I)} = \frac{(A + 1)^2}{A^3} \frac{1}{\lambda_{\min}(G_{uo}) + \frac{1}{\lambda_{\min}(G_{uo})} + 2} = \frac{(A + 1)^2}{A^3} \frac{\lambda_{\min}(G_{uo})}{(1 + \lambda_{\min}(G_{uo}))^2}.
\]

(6.16)

\[ \square \]

**Corollary 6.3.2.** When matrix \( B_{uo} \) is asymmetric, we can use the symmetrized matrix, \( S_{uo} = \frac{B_{uo} + B_{uo}^T}{2} \), and \( G_{uo} = \Gamma_{uo}^{-\frac{1}{2}} S_{uo} \Gamma_{uo}^{-\frac{1}{2}} \),
\[
POA \geq \frac{(A + 1)^2}{A^3} \frac{\lambda_{\min}(G_{uo})}{c_1^2 \lambda_{\min}(G_{uo})^2 + 2c_2 \lambda_{\min}(G_{uo}) + 1},
\]

where \( c_1 = \|S_{uo}^{-1} B_{uo}\|_{s_{uo}} \), and \( c_2 = \|S_{uo}^{-1} B_{uo}\|_2 \).
Remark 6.3.3. When the demand function is affine, (i.e., $A = 1$), the lower bounds (for both the symmetric and asymmetric demand cases) coincide with the results we obtained in Chapter 5.

6.4 Upper Bound

Proposition 6.4.1. Under Assumptions 6 and 7, given a nonlinear, symmetric demand model, an upper bound for the price of anarchy is given as

$$POA < \frac{(2A)^2 p_{uo}^T \Gamma_{uo} p_{uo}}{p_{uo}^T (\Gamma_{uo} + B_{so}) B_{so}^{-1} (\Gamma_{uo} + B_{so}) p_{uo}}.$$

Proof. Recall (6.9),

$$(\Gamma_{uo} + B_{so}) p_{uo} \leq 2B_{so} p_{so}.$$ 

Since $B_{so}$ is an M-matrix, $B_{so}^{-1} \geq 0$. Hence,

$$(\Gamma_{uo} + B_{so}) B_{so}^{-1} = \Gamma_{uo} B_{so}^{-1} + I \geq 0. \quad (6.17)$$

Multiply (6.9) with $p_{uo}^T (\Gamma_{uo} + B_{so}) B_{so}^{-1} \geq 0$, we obtain

$$p_{uo}^T (\Gamma_{uo} + B_{so}) B_{so}^{-1} (\Gamma_{uo} + B_{so}) p_{uo} \leq 2p_{uo}^T (\Gamma_{uo} + B_{so}) p_{so}$$

Applying Cauchy's inequality,

$$\|B_{so}^{-1} (\Gamma_{uo} + B_{so}) p_{uo}\|_B^{2} \leq 2\|p_{uo}\|_B \|B_{so}^{-1} (\Gamma_{uo} + B_{so}) p_{uo}\|_B$$

$$\|B_{so}^{-1} (\Gamma_{uo} + B_{so}) p_{uo}\|_B \leq 2\|p_{so}\|_B \quad (6.18)$$

Square both sides and apply the Jacobian similarity property,

$$p_{uo}^T (\Gamma_{uo} + B_{so}) B_{so}^{-1} (\Gamma_{uo} + B_{so}) p_{uo} \leq 4A p_{so}^T B_{uo} p_{so} \quad (6.19)$$
In what follows, we show that \( \|B_{SO}^{-1}(\Gamma_{UO} + B_{SO})puo\|_{B_{SO}}^2 \) is lower bounded by \( \frac{1}{A} \|B_{UO}^{-1}(\Gamma_{UO} + B_{UO})puo\|_{B_{UO}}^2 \).

\[
puo^T(\Gamma_{UO} + B_{SO})B_{SO}^{-1}(\Gamma_{UO} + B_{SO})puo \\
= puo^T(\Gamma_{UO} + B_{SO})(I + B_{SO}^{-1}\Gamma_{UO})puo \\
= puo^TB_{SO}puo + 2puo^T\Gamma_{UO}puo + puo^T\Gamma_{UO}B_{SO}^{-1}\Gamma_{UO}puo \\
\geq \frac{1}{A} puo^TB_{UO}puo + 2puo^T\Gamma_{UO}puo + \frac{1}{A} puo^T\Gamma_{UO}B_{UO}^{-1}\Gamma_{UO}puo \\
\geq \frac{1}{A} puo^T(\Gamma_{UO} + B_{UO})B_{UO}^{-1}(\Gamma_{UO} + B_{UO})puo
\]

We obtain the first inequality by the Jacobian similarity property and the second inequality since \( A \geq 1 \),

\[
puo^T\Gamma_{UO}puo \geq \frac{1}{A} puo^T\Gamma_{UO}puo.
\]

From (6.19), we conclude that,

\[
\frac{1}{A} \|B_{UO}^{-1}(\Gamma_{UO} + B_{UO})puo\|_{B_{UO}}^2 \leq 4A_{PSO}^TBUOPSO \\
\Rightarrow \quad pso^TBUOPSO \geq \frac{1}{(2A)^2} \|B_{UO}^{-1}(\Gamma_{UO} + B_{UO})puo\|_{B_{UO}}^2 \\
\Rightarrow \quad pso^TBUOPSO \geq \frac{1}{(2A)^2} puo^T(\Gamma_{UO} + B_{UO})B_{UO}^{-1}(\Gamma_{UO} + B_{UO})puo (6.20)
\]

Substitute (6.20) into the closed-form solution for the price of anarchy (6.5),

\[
POA = \frac{puo^T\Gamma_{UO}puo}{pso^TBUOPSO} \\
\leq \frac{(2A)^2 puo^T\Gamma_{UO}puo}{puo(\Gamma_{UO} + B_{UO})B_{UO}^{-1}(\Gamma_{UO} + B_{UO})puo}
\]

\( \square \)

\textit{Remark 6.4.2.} Notice that if we let \( w = \Gamma_{UO}^{\frac{1}{2}}puo \) and

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Guo = o-uo BuoIo -o , then we can simplify the upper bound as follows,

\[ \text{POA} \leq \frac{(2A)^2 w^T w}{w^T (G_{UO} + G_{UO}^{-1} + 2I) w}. \]

Therefore, versions of Theorem 5.3.1 and Proposition 5.3.2 for the upper bound also apply in the nonlinear demand case.

Remark 6.4.3. When the demand function is affine, (i.e., \( A = 1 \)), the upper bound also coincides with the result we obtained in Chapter 5.

6.5 Tightness of the Lower Bound under Nonlinear Demand

Figure 6-1 depicts the simulation result of the lower bound for the price of anarchy when nonlinear demand models are used. At every iteration, we randomly generate the parameters for the nonlinear demand functions that satisfy Assumptions 6 and 7, e.g., concave quadratic functions of the form: \( q_i = d_i - \alpha_i p_i^2 - \gamma_i p_i + \beta_{-i} \bar{p}_{-i} \).

Next, we solve both the user and the system optimization problems for this nonlinear demand function and compute the optimal solutions respectively. We substitute the user optimal prices into the Jacobian matrix to obtain \( B_{UO} \). Lastly, making use of \( G_{UO} = \Gamma_{UO}^{-\frac{1}{2}} B_{UO} \Gamma_{UO}^{-\frac{1}{2}} \) and Equation (6.16), we compute the lower bound for the price of anarchy.

From Figure 6-1, we see that the lower bound is not very tight. One explanation is the choice of the constant \( A \), as it determines the tightness of the bound. In the analysis in the previous section, one example for this constant \( A \) is,

\[ A = \max_p \frac{\lambda_{\max}(-Jh(p))}{\lambda_{\min}(-Jh(p))}. \]

However, it is hard to find such a constant as we have to search through all the feasible values of \( p \). Therefore, we use the optimal prices of the user optimization problem.
and $A$ becomes the condition number of the Jacobian matrix, i.e.,

$$A = \frac{\lambda_{\max}(-Jh(p_{uo}))}{\lambda_{\min}(-Jh(p_{uo}))}.$$ 

By doing so, we trade off the tightness of the lower bound for constant $A$ which is easier to compute.
Chapter 7

Conclusions and Future Work

In this thesis, we studied the price of anarchy in a Bertrand oligopoly market by comparing the total profits generated by the user and the system optimization setting respectively. The most notable result of this thesis is a lower and an upper bound for the price of anarchy depending on the eigenvalues of the price sensitivity matrix or a price sensitivity ratio. We first apply the analysis to affine symmetric demand models. We compare and discuss the tightness of the bounds. We would like to bring to the attention of the reader the fact that the bounds do not depend on quality differences among sellers. The same analysis has also been extended to the affine asymmetric demand model where we include two constants that measure the degree of asymmetry of the price sensitivity matrix. In addition, we extend the analysis to the nonlinear demand model with ideas similar to that used in the affine demand model.

For the general uniform demand model, we have shown that $POA_u$ is the worst-case value for the price of anarchy, when quality differences do not exist. When the quality differences are allowed to vary, we have shown that the price of anarchy increases when the number of sellers in the market increases and/or the coefficient of variation increases.

There are several extensions to the current model that could be proposed for future
research. These include:

1. Price of anarchy bounds in a Cournot rather than a Bertrand oligopoly market.

2. Incorporating multiple products with common resource constraints:
   It would be interesting to extend this work to consider multiple substitutable,
   differentiated products being sold by sellers, where each seller has his own port-
   folio of products. In addition, the total quantity available for these products
   might be tied by capacity or other common resource constraints.

3. More sophisticated demand functions under fewer assumptions:
   One could use a more complicated demand function, e.g., use the logit demand
   function to capture a more realistic setting.

4. Examine the price of anarchy in a multi-period setting for perishable or non-
   perishable products with various capacity or other constraints.
Appendix A

M-matrix

M-matrix is a very important special class of real positive stable matrices. We briefly introduce some of the key concepts of M-matrices that are drawn upon in the thesis.

Definition A.0.1. A matrix \( A \) is called an M-matrix if \( A \in \mathbb{Z} \), and \( A \) is positive stable (if every eigenvalue has positive real part), where \( \mathbb{Z} = \{ [a_{ij}] \in M_n(\mathbb{R}) : a_{ij} \leq 0 \text{ if } i \neq j, i, j = 1, \ldots, n \} \)

In order to recognize M-matrices in the immense variety of ways in which they arise, it is useful to list several necessary and sufficient conditions for a given matrix in \( \mathbb{Z} \) to be an M-matrix. The reader who is interested in the proof for these properties can refer to Horn and Johnson [15].

Theorem A.0.1. If \( A \in \mathbb{Z}_n \), the following statements are equivalent:

1. \( A \) is positive stable, that is, \( A \) is an M-matrix.

2. \( A = aI - P, P \geq 0, a > \rho(P) \).

3. Every real eigenvalue of \( A \) is positive.

4. \( A \) is nonsingular and \( A^{-1} \geq 0 \).
5. The diagonal entries of $A$ are positive and there exist positive diagonal matrices $D, E$ such that $DAE$ is both strictly row diagonally dominant and strictly column diagonally dominant.

Theorem A.0.2. Let $A, B \in \mathbb{Z}_n$ and assume that $A = [a_{ij}]$ is an M-matrix and $B \geq A$. Then

1. $B$ is an M-matrix.

2. $A^{-1} \geq B^{-1} \geq 0$.

3. $\det B \geq \det A > 0$
Appendix B

Proof for Proposition 2

Proof. From Equation (4.8),

\[
J(r, n, CV^2) = \frac{1 + \left(\frac{(n-1)(2-r)}{2(n-1)+r}\right)^2 CV^2}{1 + \frac{(n-1)(1-r)}{n-1+r} CV^2} = \frac{A(r, n, CV^2)}{B(r, n, CV^2)}.
\]

Differentiating \( J(r, n, CV) \) with respect to \( n \),

\[
\frac{\partial J(r, n, CV^2)}{\partial n} = \frac{B \frac{\partial A}{\partial n} - A \frac{\partial B}{\partial n}}{B^2}. \tag{B.1}
\]

Because

\[
\frac{\partial A}{\partial n} = \frac{2r(n-1)(2-r)^2}{(2(n-1) + r)^3} CV^2,
\]

and

\[
\frac{\partial B}{\partial n} = \frac{r(n-1)}{(n-1+r)} CV^2.
\]

We substitute \( \frac{\partial A}{\partial n} \) and \( \frac{\partial B}{\partial n} \) into Equation (B.1). Because \( B(r, n, CV^2) \) is always great

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than or equal to 1, it is sufficient to just look at the numerator.

\[ B \frac{\partial A}{\partial n} - A \frac{\partial B}{\partial n} \]

\[ = \frac{\partial A}{\partial n} - \frac{\partial B}{\partial n} + \frac{\partial A}{\partial n} \frac{(n-1)(1-r)}{n-1+r} CV^2 - \frac{\partial B}{\partial n} \left( \frac{(n-1)(2-r)}{2(n-1)+r} \right)^2 CV^2 \]

To prove \( \frac{BJ(r,n,CV)}{\partial n} \geq 0 \), we need to prove \( \frac{A}{\partial n} - \frac{B}{\partial n} \geq 0 \) and \( \frac{A}{\partial n} \frac{(n-1)(1-r)}{n-1+r} - \frac{B}{\partial n} \left( \frac{(n-1)(2-r)}{2(n-1)+r} \right)^2 \geq 0 \).

It can be shown as follows:

\[ \frac{\partial A}{\partial n} - \frac{\partial B}{\partial n} \]

\[ = \left( \frac{2r(n-1)(2-r)^2}{(2(n-1)+r)^3} - \frac{r(n-1)}{(n-1+r)^2} \right) CV^2 \]

\[ = \frac{2r(n-1)(2-r)^2(n-1+r)^2 - r(n-1)(2n-1+r)^3}{(2(n-1)+r)^3(n-1+r)^2} CV^2 \]

\[ = \frac{r(2(n-1+r)^2(n-1)(2-r)^2 - (1-r)(2(n-1)+r)^3)}{(2(n-1)+r)^3(n-1+r)^2} CV^2 \]

\[ = \frac{8r((n-1+r)^2(n-1)(1-\frac{r}{2})^2 - (1-r)(n-1+\frac{1}{2})^3)}{(2(n-1)+r)^3(n-1+r)^2} CV^2 \]

Note that \( \frac{\partial A}{\partial n} - \frac{\partial B}{\partial n} = 0 \) iff \( r = 0 \). Thus,

\[ \frac{\partial A}{\partial n} - \frac{\partial B}{\partial n} \geq 0, \text{ for } r \geq 0 \text{ and } n \geq 2 \]

\[ \frac{\partial A}{\partial n} \frac{(n-1)(1-r)}{n-1+r} - \frac{\partial B}{\partial n} \left( \frac{(n-1)(2-r)}{2(n-1)+r} \right)^2 CV^2 \]

\[ = \frac{2r(n-1+r)(n-1)(2-r)^2 - r(1-r)(2n-1+r)^3}{(n-1+r)^3(2(n-1)+r)^3} CV^2 \]

\[ = \frac{r(n-1)^2(1-r)(2-r)^2}{(n-1+r)^3(2(n-1)+r)^3} CV^2(2(n-1+r) - (2(n-1)+r)) \]

\[ = \frac{r^2(n-1)^2(1-r)(2-r)^2}{(n-1+r)^3(2(n-1)+r)^3} CV^2 \geq 0, \text{ for } n \geq 2 \]
Thus, we have shown that \( \frac{\partial J(r,n,CV^2)}{\partial n} \geq 0 \) for \( n \geq 2 \), that is, \( J(r,n,CV^2) \) increases with \( n \). \( \square \)
Bibliography


