A Study of the Dynamics of Shells with Boundary Layers and a Study of the MITC3 Shell Element

by

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Abstract

Shell structures are frequently used for their aesthetic appeal, and material efficiency. They can be found everywhere, in all sorts of contexts. For example, they can serve as roofs to encapsulate large areas (think Millennium Park in Chicago); as fuselage for an airplane; as the frame of a car; as chemical containers (such as pressure vessels); even submarines.

One of the most challenging phenomena in shells is boundary layers. Boundary layers are high concentrations of energy due to either irregular loadings, or incompatible boundary conditions, or geometric discontinuities. Previous studies have determined the location of this concentrated energy and how that location varies depending on the thickness of the shell; up to now however, any correlation between the boundary layers and the vibrations of the shell has not yet been examined. In our first study, we review the boundary layer behavior, and we investigate the dependence of the shell's natural frequencies on the boundary layer phenomenon for two shell benchmark problems: the Scordelis-Lo Roof and the Clamped Hyperboloid.

In our second study, we shortly review the MITC3 shell element. This element is known to lock in plate bending cases for a specific mesh pattern. Finally, we formulate a new MITC triangular, four node element and investigate its plate bending behavior in detail.

Our results show that including an extra node in the center of the element does not improve the plate bending behavior. We discuss our results and make suggestions for future investigations.

Thesis Supervisor: Klaus-Jürgen Bathe
Title: Professor
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Introduction

Thesis Outline

Part I – Asymptotic Behavior in the Dynamics of Shells

Chapter 1 discusses the “finicky” behavior of shell structures in that small perturbations in geometry or loading can produce high gradients in strain. A specific phenomenon where these gradients are concentrated, called the “boundary layer,” is discussed. An investigation into the correlation between a shell’s natural frequencies and the existence of a boundary layer follows in Chapter 2.

Chapter 2 explores the existence of the boundary layer in the natural vibrations of two benchmark shell problems: the Scordelis-Lo roof and the Clamped Hyperboloid.

Part II – Study of the MITC3 Shell Element

Chapter 3 provides a brief background on finite element shell models and explains in particular the formulation of the MITC3, a triangular shell finite element. This chapter addresses the previously observed locking of the MITC3 in a plate bending problem. Attempts to improve the element for this given mesh pattern are discussed in Chapter 4.

Chapter 4 explains the formulation of the new triangular element, the MITC3i. The element is formulated specifically for plate bending. Different formulations are
investigated based on implementing three separate integral tying conditions. These formulations are tested with the two-sided clamped plate case study that exhibited locking in Chapter 3.
Part I – Asymptotic Behavior in the Dynamics of Shells

Chapter 1

Shell Structures

1.1 Introduction
1.1.1 Historic Shell Structures

In 1418, the great Cathedral of Santa Maria del Fiore in Florence, Italy, stood incomplete without a dome. The design of the dome had presented such an engineering challenge that many contemporary architects and engineers had been asked to solve the problem. After many unsuccessful attempts by other architects, the Italian architect Filippo Brunelleschi stepped forth, claiming only he could build the dome. To prove his worth, he put forth a riddle to his colleagues. He challenged them to make an egg stand on a marble table, saying that only a person who could accomplish such a feat would be capable of designing the dome. After many engineers and architects failed to make the egg stand on the marble table, Brunelleschi seized the egg and smashed it on the table! He was left with half the egg standing on its own skin. Brunelleschi won the commission to build the dome, and after 16 years of construction the great Il Duomo was completed.
While shells can be found in nature, such as Brunelleschi’s egg, man has also mimicked nature’s solution in impressive structures since as long ago as the construction of the Pantheon (75-138 AD).

Shell structures were first employed as domes because they allowed for a large encapsulation of an area while being aesthetically pleasing. New shell shapes were explored, and as their high membrane stiffness capabilities became apparent, shell structures evolved into new shapes such as cylinders, spheres, and hyperboloids. These structures provided new forms of building roofs such as the Sydney Opera House and the Kresge Dome of MIT.
Shell structures are employed not only for their aesthetics and material efficiency, but also for their ability to carry an extreme amount of transverse loading. The transverse loading is translated to the membrane forces in the shell skin, providing the stiffness [1].

1.1.2 Shell Attributes
A shell is classified by its geometry and not by its function. Here we will review how the curvature defines the geometry. The curvature of a shell is defined as being either positive, negative, or zero. Positive curvature is defined as a convex lens, negative curvature is a concave lens, and zero curvature is a flat surface.

![Figure 7 - Curvature](image)

The sign of the curvature is further used to define the shell’s shape by calculating the Gaussian curvature. The Gaussian curvature is the product of the minimum and maximum curvatures, the principal radii of curvatures. Positive Gaussian curvature is a result of having two negative or two positive principal radii of curvatures producing an elliptical shape. A negative Gaussian curvature implies one principal radius of curvature to be positive while the other is negative. These are hyperbolic structures which are also known as structures with “double curvature.” These shapes are used in eccentric roofs and cooling towers.
The third possibility is zero Gaussian curvature which results in one of the principal radii of curvature being zero. This is seen in cylindrical roof like structures. In Chapter 2, we will consider two shell geometries, one of zero Gaussian curvature and the other of negative Gaussian curvature. One advantage obtained from the curvature of shells is that the curvature provides continuity (smoothness) in the structure. This continuity facilitates the transmission of forces in multiple directions.

While the aesthetics and load carrying capabilities of these curved structures are highly appealing, the curvature also presents a disadvantage – a small perturbation in the geometry can lead to high stress concentrations. More detailed work demonstrating these shell sensitivities are discussed in [2]. One manifestation of this instability is illustrated in the phenomenon of boundary layers which we discuss in the following section.
1.2 Boundary Layer
1.2.1 Sources of Boundary Layer

A boundary layer is a section of a shell that contains large concentrations in strain energy. These concentrations in energy tend to be located along the edge of a surface and at a distance from the edge.

The large gradient in energy can be attributed to either discontinuities in the shell geometry, incompatible boundary conditions, or irregular loadings [4]. Discontinuities in curvature imply a kink in the geometry leading to a stress jump. In practice, such discontinuities would require the structure to have reinforcement. Consider, for example, pressure vessels. Though they have smooth curvature, most pressure vessels require reinforcement at the junction of the cylindrical tank to the dome cover.

The second source of boundary layers is incompatible boundary conditions. These are more common than discontinuities in geometry, and they arise when the structure requires more support beyond those of the membrane force in the shell skin. For example, a clamped boundary condition uses an axial and shear force and a moment to support the structure. Clearly the membrane forces alone cannot equilibrate these support forces.
The third cause of boundary layers is irregular loading, such as a concentrated force on a shell body as shown for example in Figure 12.
layer can then be measured. Lee and Bathe performed the above process to calculate the boundary layer width for two specific case studies: the Scordelis-Lo Roof and the partly clamped hyperbolic paraboloid. We use their numerical calculation for the boundary layer width for the Scordelis-Lo provided by equation (1.1) in degrees. This information tells us how to properly mesh our case study later in Chapter 2.

\[ d = 60t^{0.4} \]  

(1.1)

Hiller and Bathe [6] also stress the importance of using a graded mesh for the boundary layer in order to accurately calculate the energy norm. They consider two case studies, the Clamped and Free Hyperboloid. We will also be considering the Clamped Hyperboloid in Chapter 2. We use the boundary layer width for the Clamped Hyperboloid provided in [6] and given by equation (1.2).

\[ d = 6t^{0.2} \]  

(1.2)

Beirão da Veiga, and Chinosi [5, 7], also propose a numerical procedure for obtaining the boundary layer behavior for the Scordelis-Lo roof and the pinched cylinder.

All these studies have calculated the total strain energy as well as the contributions from the membrane and bending energies. In doing so, they found that the boundary layer depended on the thickness of the shell and calculated the width of the boundary layer.
While these studies have looked at the asymptotic energy behavior of shells, no research has focused on the existence of the boundary layer in the natural vibrations of the structure or on the frequencies dependence on the boundary layer. We expect there to exist mode shapes which contain high levels of energy in the boundary layer zone. This study investigates which mode shapes of the Scordelis-Lo roof and the clamped hyperboloid contain the boundary layer, and the correlation between those frequencies and thickness.

In general, to prove the existence of the boundary layer, we can calculate the membrane and bending energies and show that it does not equate the external energy as shown in [2]. Bathe and Chapelle explain the asymptotic behavior by considering the mathematical model of a shell. First, given the boundary conditions or geometry of the shell, the shell can be classified as either of the two limiting cases, membrane dominated or bending dominated, or a mixed problem where both membrane and bending interplay. In a pure bending inhibited case, we consider only membrane action and so the stiffness of the structure is only proportional to $t$. Also the total strain energy is dominated by the membrane strain energy. The second case pure bending non-inhibited is when the structure can bend and so the stiffness is proportional to the bending stiffness $t^3$. In this case the total strain energy is dominated by the bending strain energy.

In Chapter 2 we will consider two problems, Scordelis-Lo Roof, a mixed shell problem and the clamped hyperboloid, a membrane dominated problem. For the Scordelis-Lo roof, the existence of the boundary layer is shown in [2] and uses the following process.
First, create a series of displacement fields, then calculate the membrane and external energy. Second, check if the membrane energy does not equate the external energy. In the Scordelis-Lo roof, the membrane energy does not equal the external energy thus proving that there is a singularity and that the singularity will appear near the free boundary and magnify as the thickness decreases. In our second case study, the clamped hyperboloid, the boundary layers is known to exist along the clamped edge.

We are considering later the natural frequencies and mode shapes of the structure. The natural vibrations of a structure do not depend on the loading, yet some of the causes of these boundary layers occur only with loading. The question remains why then search for boundary layers in the mode shapes of the structure? Given a certain loading or initial displacement, dynamics tells us those conditions will excite some natural behavior in the structure. This behavior will be some superposition of the natural vibrations of the structure. And so, if a boundary layer is excited in the response of a structure then there exist specific mode shapes that contain these high energy concentrations.
Chapter 2

Case Studies

2.1 Overview

We study two shell benchmark problems, the Scordelis-Lo Roof and the Clamped Hyperboloid, to investigate the following: the magnitude of the natural frequencies with respect to the thickness change, the influence of the boundary layer mesh in calculating the frequencies, and the mode shapes that contain the boundary layer.

We are interested in both the Scordelis-Lo roof and the clamped hyperboloid since these problems are known to exhibit boundary layers. For the Scordelis-Lo roof we know the boundary layer exists along the free edge of the panel, and for the clamped hyperboloid the boundary layer exists along the clamped edge. We also know the boundary layer width for each structure and so can focus on studying the natural vibrations.

2.2 Scordelis-Lo Roof - Problem Description

The Scordelis-Lo roof, shown in Figure 13, is a panel of a cylindrical shell that is supported at its ends by diaphragms, inhibiting the x and z translations.
Using symmetry and applying the correct boundary conditions, we can model the roof as $\frac{1}{4}$ of the actual structure. Figure 14 and Table 1 show the material, geometrical parameters, and boundary conditions for the quarter model. Table 2 includes the boundary layer widths for varying thickness [4].
Figure 14 - The Scordelis-Lo Roof – quarter model

Table 1 - Scordelis-Lo Roof Parameters – quarter model

<table>
<thead>
<tr>
<th>Edge</th>
<th>Boundary Conditions</th>
<th>Material Properties</th>
<th>Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>$u_y = \alpha = 0$</td>
<td>$E = 4.32e8 \text{ lb/ft}^2$</td>
<td>$x = R \cos \theta$</td>
</tr>
<tr>
<td>DC</td>
<td>$u_x = u_u = 0$</td>
<td>$\nu = 0$</td>
<td>$y = y$</td>
</tr>
<tr>
<td>AD</td>
<td>Free</td>
<td>$\rho = 4.66 \text{ slugs/ft}^3$</td>
<td>$z = R \sin \theta$</td>
</tr>
<tr>
<td>BC</td>
<td>$u_x = \beta = 0$</td>
<td></td>
<td>$R = 25$</td>
</tr>
</tbody>
</table>

Table 2 - The Scordelis-Lo Roof, Boundary Layer Width – quarter model

<table>
<thead>
<tr>
<th>$t$</th>
<th>Boundary Layer Width (Degrees – Measured from D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>40</td>
</tr>
<tr>
<td>0.025</td>
<td>23.78</td>
</tr>
<tr>
<td>0.0025</td>
<td>13.375</td>
</tr>
</tbody>
</table>
2.3 Results – Scordelis-Lo Roof
2.3.1 Mesh Convergence Study
The objectives are achieved by first completing a mesh convergence of the first six natural frequencies using the MITC4. Two meshes are used; the first is a uniform mesh of \( N \times N = 18 \times 18, 36 \times 36, 72 \times 72, \) and 144x144 elements. The second mesh is a graded mesh that consists of \( \frac{N}{2} \times N \) elements in the region of the boundary layer and \( \frac{N}{2} \times N \) density in the rest of the mesh.

Figure 15 plots the first 6 natural frequencies, for all three \( t/L \) ratios. The dashed lines represent a mesh that does not use a graded mesh in the boundary layer. As can be inferred from the plots, the boundary layer mesh is not necessary for obtaining the correct frequencies for the first 6 natural frequencies. Figure 15 also illustrates that the 36x36 mesh is adequate in determining the frequencies, thus, the following analyses use the 36x36 mesh. However, in future studies we should consider expanding our mesh convergence study for calculating the first 200 frequencies.
Figure 15 - Frequencies versus Mesh Size
2.3.2 Frequencies

After selecting a proper mesh, the frequencies are plotted as a function of the thickness. The frequencies for the three different thickness to length ratios are plotted below in Figure 16. The squared frequency of the structure is the ratio between its stiffness and mass. For highly constrained structures, the particles are not able to “breathe” or move as they prefer. Since the structure is stiffer it takes a higher frequency to excite any motion.

Therefore we can expect, and we observe, that as the structure gets thinner it loses stiffness and, as a result, the frequencies decrease. In essence, it takes less effort to excite the vibrations of a thin structure, such as a sheet of paper, than excite the vibrations of a slightly thicker structure, such as twenty sheets of paper combined.
2.3.3 Mode Shapes
The mode shapes for a cylinder will be either bending, circumferential, longitudinal, radial, or torsional. [8] The first four modes observed for all three thickness to length ratios vary between longitudinal bending and circumferential bending. No membrane modes are observed in the earlier modes. For the first 200 frequencies, membrane (stretching) modes are observed only for the largest \( t/L \) ratio and for the higher frequencies (24, 97, 161, and 192). Examples of these three mode shapes, longitudinal bending, circumferential bending and membrane, are shown in the Figures 17 to 19 for \( t/L = 1e-2 \).
Figure 17 Bending Mode, $t/L=1e^{-2}$.

Figure 18 - Circumferential Mode, $t/L=1e^{-2}$.

Figure 19 - Membrane Mode, $t/L=1e^{-2}$. 
The mode shapes do vary between the different thicknesses. There are more bending modes observed in the thinner shells for the lower frequencies. This is apparent in Figures 20 to 22 which compare the first 11 mode shapes for all three thickness to length ratios. The third column represents the thinnest case and when compared to the thicker shells, we clearly observe more circumferential bending modes. This behavior is expected if we consider the following.

Mechanics tells us that a structure will take the form that will cause it the least amount of energy. We can estimate the contributions of the bending and membrane behavior to the total strain energy. We know that the bending and membrane stiffness will change of the order $O(t^3)$ and $O(t)$, respectively. We thus know that the strain energy due to bending will change of the order $O(t^3)$ and the membrane energy will change of the order $O(t)$. Thus, the least amount of energy stored in the structure, as the structure get thinner, is the bending energy. Therefore the structure will take the form of bending since it will cause the structure the least amount of energy.
| \( \varphi_1 \) | t/\( L \) = 1e-2 | t/\( L \) = 1e-3 | t/\( L \) = 1e-4 |
|---|---|---|
| ![Mode Shape 1](image1.png) | ![Mode Shape 2](image2.png) | ![Mode Shape 3](image3.png) |
| ![Mode Shape 1](image4.png) | ![Mode Shape 2](image5.png) | ![Mode Shape 3](image6.png) |
| ![Mode Shape 1](image7.png) | ![Mode Shape 2](image8.png) | ![Mode Shape 3](image9.png) |
| ![Mode Shape 1](image10.png) | ![Mode Shape 2](image11.png) | ![Mode Shape 3](image12.png) |
| ![Mode Shape 1](image13.png) | ![Mode Shape 2](image14.png) | ![Mode Shape 3](image15.png) |
| ![Mode Shape 1](image16.png) | ![Mode Shape 2](image17.png) | ![Mode Shape 3](image18.png) |

Figure 20 – The Scordelis-Lo Roof, Mode Shapes 1-4
Figure 21 – The Scordelis-Lo Roof, Mode Shapes 5-8
<table>
<thead>
<tr>
<th>( \Phi_9 )</th>
<th>( t/L = 1e-2 )</th>
<th>( t/L = 1e-3 )</th>
<th>( t/L = 1e-4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /></td>
<td><img src="image4.png" alt="Image" /></td>
</tr>
<tr>
<td>( \Phi_{10} )</td>
<td><img src="image5.png" alt="Image" /></td>
<td><img src="image6.png" alt="Image" /></td>
<td><img src="image7.png" alt="Image" /></td>
</tr>
<tr>
<td>( \Phi_{11} )</td>
<td><img src="image8.png" alt="Image" /></td>
<td><img src="image9.png" alt="Image" /></td>
<td><img src="image10.png" alt="Image" /></td>
</tr>
</tbody>
</table>

Figure 22 – The Scordelis-Lo Roof, Mode Shapes 9-11
A question remains if all the minimum frequencies are that of bending modes and if they then proceed to membrane modes. As Figure 23 shows, the first 400 frequencies do not distinguish between zones of bending or membrane. Instead the frequencies exhibit a steady increase. Future work should consider checking if the frequencies for all three thickness to length ratios will converge to the same point. These results can be obtained by calculating the next, for example, 600 frequencies and plotting the results.

It is interesting to note that the computational time to calculate the first 400 frequencies for $t/L=1e-2$ took 17 minutes to complete for a computer with 2GB of RAM. This indicates an even greater need for continuing research in formulating efficient finite elements.

![Scordelis-Lo Roof: Frequency Spectra](image)

Figure 23 – Frequency Spectra
2.3.4 Boundary Layer Frequencies
The frequencies that excite the boundary layer have two distinctive mode shapes illustrated in Figures 24 and 25. The first shape, illustrated in Figure 24, contains displacement jumps, concentrated only at the edge. This shape is observed in all thickness to length ratios. The second shape is a series of circumferential bends within the boundary layer width. This shape is only observed in the thinnest case, \( t/L = 1 \times 10^{-4} \).

Figure 24 – Boundary Layer Frequency – Edge Concentration, mode shape a, \( t/L = 1 \times 10^{-3} \)

Figure 25 – Boundary Layer Frequency – Width Concentration, mode shape b, \( t/L = 1 \times 10^{-4} \)
However, the process for finding the boundary layer mode shapes should be reconsidered for future work. Although Figure 24 and 25 exhibit what looks like boundary layer excitation, Figure 24 could just be an edge concentration, a localization of the curvature.

The shape of Figure 25 can also be explained differently. Since the boundary layer edge is free, and the center line has some fixed degrees of freedom, we can expect that the free edge would be more easily excited than the centerline. This already explains why the area of the boundary layer will exhibit more excitation. As for the wave propagating to the center line, we should keep in mind that since the center line will stop all motion, the excited shapes must eventually die out as they approach the edge. Finally, since the boundary layer shapes are shown for $t/L=1e-3$, we must keep in mind that the width is half the structure allowing ample time for the excitation to die out. For example, if the width of the boundary layer was only one tenth the length and we observed that the displacements did not go past the boundary layer, then we could confidently consider calling these boundary layer mode shapes.

Figure 26 plots the number of observed boundary layer frequencies within the first 200 calculated frequencies. In this figure we observe that the number of boundary layer frequencies increase as the thickness decreases. This is expected since the boundary layer becomes more prominent as the structure gets thinner.
Future work necessary to understand the relationship between the boundary layer and frequencies includes calculating the higher frequencies past two hundred and checking for a convergence for all three thickness to length ratios. It also includes explaining theoretically why the boundary layer exists in the structure’s natural vibrations.

For the Scordelis-Lo Roof, as touched upon in the Frequencies section, we should reconsider how we determine a boundary layer mode shape. Note that the modes shapes are not limits of either pure bending or pure membrane actions. The mode shapes can have coupling behavior between both. Perhaps it would be better to plot the membrane and bending energy distributions. We can use scalar energy plots to see if, for example, the mode shapes of Figures 24 and 25 are really with high concentrations of energy in the
boundary layer. We must keep in mind that this is a mixed problem and so we need to look more carefully at the different energy contributions.

2.4 Clamped Hyperboloid - Problem Description

The second case study is the Clamped Hyperboloid. The objective of this study is to investigate the presence of the boundary layer in the mode shapes. The first 6 natural frequencies are recorded for a total of 3 thickness to length ratios (1e-2, 1e-3, 1e-4). The mesh densities are MITC4 24x24, 32x32, 64x64, 96x96. As in the Scordelis-Lo roof we can use symmetry to model only a portion of the structure.

Using symmetry, the clamped hyperboloid can be modeled using one-eighth of the structure (one half along the y axis and one fourth along the circumference). The boundary conditions, material and geometrical parameters and the boundary layer thickness for the one eighth model are listed in Figure 27 and Tables 3 and 4.

![Figure 27 - The Clamped Hyperboloid](image)
Table 3 - Clamped Hyperboloid Parameters

<table>
<thead>
<tr>
<th>Boundary Conditions</th>
<th>Material Properties</th>
<th>Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Edge</td>
<td>Fixed-Fixed</td>
<td>E 2e11 N/m²</td>
</tr>
<tr>
<td>AB</td>
<td>Fixed</td>
<td>ν 0.33</td>
</tr>
<tr>
<td>DC</td>
<td>u_y = α = 0</td>
<td>ρ 7850 kg/m³</td>
</tr>
<tr>
<td>AD</td>
<td>u_x = β = 0</td>
<td></td>
</tr>
<tr>
<td>BC</td>
<td>u_z = δ = 0</td>
<td></td>
</tr>
</tbody>
</table>

Table 4 - The Clamped Hyperboloid - Boundary Layer Width

<table>
<thead>
<tr>
<th>t</th>
<th>Boundary Layer Width</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.6</td>
</tr>
<tr>
<td>0.001</td>
<td>0.19</td>
</tr>
<tr>
<td>0.0001</td>
<td>0.06</td>
</tr>
</tbody>
</table>

2.5 Results – Clamped Hyperboloid

2.5.1 Mesh Convergence Study

As in the Scordelis-Lo Roof, a mesh convergence is performed to determine the necessity in using a graded mesh for frequency convergence. Figure 28 plots the mesh convergence for the lowest 6 frequencies. We conclude that the 32x32 mesh is adequate for calculating the lowest 6 frequencies. In future studies we suggest completing a mesh convergence study for the first 200 frequencies.
Hyperboloid - Clamped-Clamped Case: Frequencies vs Mesh Size

Figure 28 - The Clamped Hyperboloid, Mesh Convergence
2.5.2 Frequencies
As in the Scordelis-Lo roof, the natural frequencies of the Clamped Hyperboloid, shown in Figure 29, decrease as the thickness decreases. Again, this is expected since the bending stiffness decreases significantly of $O(r^3)$, resulting in lower frequencies.

![Clamped Hyperboloid Natural Frequencies](image)

*Figure 29 - The Clamped Hyperboloid, Natural Frequencies*

2.5.3 Mode Shapes
One distinct mode shape, displayed in Figure 30 for $t/L$ of $1e^{-2}$, contains the concentrated energy of the boundary layer. Unlike for the Scordelis-Lo roof, there is only one distinctive mode shape for the clamped hyperboloid. Also, as in the Scordelis-Lo roof, we may want to reconsider how we observe these boundary layer mode shapes. Again the width of the boundary layer shown in Figure 30 is more than half the length of the
structure. So the structure is behaving such that the displacement is diminishing towards the constrained edge.

![Figure 30 - The Clamped Hyperboloid, Example of boundary layer mode shape, t/L=1e-2.](image)

### 2.5.4 Boundary Layer Frequencies

The boundary layer frequencies are more evident in the thickest case for hyperboloid shell. Figure 31 plots the frequency number of those frequencies that excited the boundary layer. There are more boundary layer frequencies in the thickness to length ratio of 1e-2 than 1e-4. This behavior deviates from what is expected and what we observed in the Scordelis-Lo Roof results. However, the structure is so thin and unstable that it is difficult to view a concentrated behavior.
Boundary Layer Frequencies

Figure 31 - Boundary layer Frequencies

2.5.5 Future Work
Further work includes a comparison of the free hyperboloid, a bending dominated problem, with the clamped hyperboloid. Also, as suggested with the Scordelis-Lo Roof, it would be of great value to plot the membrane and bending energies for the different mode shapes. This will help confirm if what looks like a bending mode shape is indeed pure bending without any membrane action. It will also be easier to visualize if the energy of these mode shapes is really concentrated in the boundary layer, especially for such a structure with such a thin boundary layer.

Also it would be important to explain the behavior of the clamped hyperboloid mathematically. We would like to analytically calculate the frequencies that contain the boundary layer, as well as estimate how many will exist. After which, we could also
calculate the mode shapes. We should also consider expanding our mesh convergence study to not just that of the first 6 frequencies, but to that of the first 200 frequencies.

As indicated earlier, the amount of time to run fine meshes for complex problems such as our two case studies can get costly and so it is beneficial to have effective elements. In these case studies, we used the MITC4, a heavily researched element. Because it will be difficult to improve such an element, we will focus our efforts on another attractive element, the triangular, 3-node MITC3. We will discuss in the next section the development of the MITC3 and possible improvements to make the element more effective.
Part II – Study of the MITC3 Element

Chapter 3

MITC3

3.1 FEM of Shells

The finite element method of shells projects a thin 3-dimensional body onto a two-dimensional surface. Due to the thinness of the structure relative to the length, the transverse normal strain and stress are negligible. Reissner kinematics are used, and the general shell element has a total of 5 degrees of freedom: two rotations and 3 translations. The degrees of freedom are measured from the midsurface of the shell.

There is a need to further research specifically in shells, since the modeling of thin structures are desired in upcoming fields like biomechanics. In particular, this body of work is on the study of a triangular shell, three node finite element. Triangular elements are powerful tools since they make meshing complex geometry more feasible. These elements are highly attractive since they are lower order elements, involving less degrees of freedom and less computational effort.

One common problem of shell finite elements is shear locking. Shear locking occurs in bending dominated problems when the displacement-based formulation cannot represent zero shear strain. To represent zero shear strain, the displacement-based formulation
enforces all displacements to be zero, and thus the element “locks”. For example, consider the cantilever beam with an applied end moment.

![Figure 32 - Cantilever Beam](image)

The shear strain formulation for a 2 node isoparametric beam is:

![Figure 33 - Isoparametric Beam – Finite element model](image)

\[ \gamma = \frac{w_2}{L} - \frac{(1 + r)}{2} \theta_2 \]  

(3.1)

To satisfy a shear strain of zero no deformation can occur.

\[ \gamma = 0 \forall r, w_2 = \theta_2 = 0 \]  

(3.2)

Clearly, having an element that enforces the nodal displacements to be zero is undesirable. To alleviate this locking phenomenon, the Mixed Interpolated Tensorial Component (MITC) technique [9] evaluates the strain at a point other than a node, called a “tying point”, and equates this strain with the assumed strain.

\[ \gamma^{AS} = \gamma|_{r=0} \]  

(3.3)
This technique is employed in the element we are studying, the MITC3, as well as the proposed new element, the MITC3i. In the following sections we discuss the formulation for these two elements.

3.2 MITC3

3.2.1 Overview

The MITC3 element is a three node, spatially isotropic triangular element, developed by Lee et al [10]. It is a MITC element that assumes constant transverse shear strains along all three edges. Figure 34 illustrates the element geometry, the tying point and node locations, and the transverse shear strain conditions.

![Image of the MITC3 element](image)

Figure 34 – The MITC3

The MITC3 uses three tying points, each one located at the midpoint on each edge. The formulation results in the following strains:

\[
\begin{align*}
\tilde{e}_{rr} &= e_{rr}^{(1)} + cs \\
\tilde{e}_{ss} &= e_{ss}^{(2)} - cr \\
\epsilon &= e_{ss}^{(2)} - e_{ss}^{(3)} + e_{rr}^{(3)} - e_{rr}^{(1)}
\end{align*}
\]  

(3.4)
The numerical integration scheme for the MITC3 is 3 point Gauss over a triangular domain.

3.2.2 Comparison between Displacement-based and MITC3 element

The formulation for the displacement-based, 3 node element is given by equation (3.5).

\[
\tilde{e}_n^{(\alpha)} = \frac{1}{2} \left( -\frac{a}{2} \sum_{i=1}^{3} h_i \delta_i + \frac{a}{2} \sum_{i=1}^{3} h_i \omega_i \right) \\
\tilde{e}_s^{(\alpha)} = \frac{1}{2} \left( -\frac{a}{2} \sum_{i=1}^{3} h_i \alpha_i + \frac{a}{2} \sum_{i=1}^{3} h_i \omega_i \right) 
\]

Figures 35 to 38 also graphically display the strain fields for 9 different cases. Columns one to three consider a unit displacement for each of the three degrees of freedom. The rows represent at which node the unit displacements are applied.

We can use these figures to compare the expected and unexpected strain calculations. Consider for example a nodal rotation about the x-axis on node 2 ($\alpha_2=1$). The displacement-based formulation, as well as intuition, would yield a zero $\tilde{e}_n$ transverse shear strain. However, due to the fact that the MITC3 element maintains spatial isotropy, there does exist a $\tilde{e}_n$ transverse shear strain which is equal to the $\tilde{e}_s$ resulting from a rotation about the y-axis at node 3 ($\beta_3=1$). The relationships of all spatial isotropic strains are listed below in equation (3.6).

\[
-\tilde{e}_n^{(\alpha_1)} = \tilde{e}_s^{(\beta_1)} \\
\tilde{e}_n^{(\alpha_2)} = \tilde{e}_s^{(\beta_2)} \\
\tilde{e}_n^{(\alpha_3)} = \tilde{e}_s^{(\beta_3)} \\
\tilde{e}_n^{(\alpha_1)} = -\tilde{e}_s^{(\alpha_1)} \\
\tilde{e}_n^{(\alpha_2)} = -\tilde{e}_s^{(\alpha_2)} \\
\tilde{e}_n^{(\alpha_3)} = -\tilde{e}_s^{(\alpha_3)} \\
\tilde{e}_n^{(\alpha_1)} = \tilde{e}_s^{(\alpha_1)} \\
\tilde{e}_n^{(\alpha_2)} = \tilde{e}_s^{(\alpha_2)} \\
\tilde{e}_n^{(\alpha_3)} = \tilde{e}_s^{(\alpha_3)} 
\]

45
Figure 35 - Transverse Shear Strain $\varepsilon_{rt}$ Displacement-Based
Figure 36 - Transverse Shear Strain $\varepsilon_{rt}$, Mixed Formulation
Figure 37 - Transverse Shear Strain $\varepsilon_{st}$, Displacement-Based
Figure 38 - Transverse Shear Strain $\varepsilon_{st}$, Mixed Formulation
3.2.3 Prior Case Study

Lee et al. [10] demonstrated that the MITC3 locked specifically for a two-sided clamped plate shown in Figure 39.

\[
\begin{align*}
\bar{x} &= \begin{bmatrix} 
\sum_{i=1}^{3} h_i x_i \\
\sum_{i=1}^{3} h_i y_i \\
\frac{t}{2} a
\end{bmatrix} \\
\bar{u} &= \begin{bmatrix} 
\frac{t}{2} a \sum_{i=1}^{3} h_i \beta_i \\
-\frac{t}{2} a \sum_{i=1}^{3} h_i \alpha_i \\
\sum_{i=1}^{3} h_i w_i
\end{bmatrix}
\end{align*}
\]  

(3.7)

A plate is basically a flat shell. There are only three degrees of freedom per node: two rotations \((\alpha_x, \beta_y)\) and one vertical translation \((w)\). Equation (3.7) describes the motion of the element.

The MITC3 locked for the two-sided clamped plate for the mesh pattern A. Since this is a pure bending problem, the transverse shear strains must be zero. After formulating the transverse shear strains for the given boundary conditions and element connectivity, it is shown that the nodal displacements do not satisfy pure bending and so the element locks.
Since the MITC3 locked for a plate bending case study, our motive is to alleviate this locking and so we focus only on the "plate" behavior of the shell. The motivation of the study that follows in Chapter 4 is to prevent transverse shear strain locking specifically for plate bending.
Chapter 4

MITC3i

4.1 Proposed Solutions

The proposed new element, the MITC3i, is built from the MITC3, keeping properties such as spatial isotropy, linear transverse shear strains, and three corner nodes. To improve the MITC3, two options were considered: use the enhanced strain technique, or add a node to the center of the element.

The enhanced strain technique was developed by Simo and Rifai [11]. This technique enriches the space of interpolation function by adding terms that are missing from the original formulation. These terms do not need to be used for all strains but can be used to specifically enhance certain strains. For example, given a 4 node displacement-based element, the normal and shear strain are given in terms of equation (4.1).

\[
\begin{bmatrix}
\varepsilon_{rr} \\
\varepsilon_{ss}
\end{bmatrix}
= \begin{bmatrix}
1 & s \\
1 & r \\
1 & r & s
\end{bmatrix}
\]

(4.1)

Enhancing the strains would add an \( r \) term to \( \varepsilon_{rr} \) and an \( s \) term to \( \varepsilon_{ss} \), completing the basis as so:
\[
\begin{bmatrix}
\varepsilon_{rr} \\
\varepsilon_{ss} \\
\varepsilon_{rs}
\end{bmatrix} = 
\begin{bmatrix}
1 & r & s \\
1 & r & s \\
1 & r & s
\end{bmatrix}
\] (4.2)

While the enhanced strain technique can be considered for future work, we will first implement the second option, adding a node to the center.

### 4.2 Formulation

To improve the MITC3 a node is added to the center of the element. To maintain spatial isotropy the fourth node is located at the centroid, and is used to describe only the rotational degrees of freedom \((\alpha, \beta)\) and not the vertical translation \((w)\).

![Figure 39 - The MITC3i](image)

The interpolation function for the middle node gives a weight of one at itself and zero at the other three nodes. The interpolation function selected is in equation (4.4).

\[
h_4 = 27rs(1 - r - s)
\] (4.4)

A 3-D and contour plot of \(h_4\) are shown in Figure 40 (a) and (b).
Figure 40 - 3D (a) and Contour Plot (b) of the Center Node's Interpolation Function
Since the middle node only describes the rotational degrees of freedom, the original interpolation functions of the MITC3 are used to describe the translational motion of the three corner nodes. Table 5 below lists the differences between the interpolation functions for the MITC3 and the MITC3i

<table>
<thead>
<tr>
<th>MITC3</th>
<th>MITC3i</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_1 = (1 - r - s)$</td>
<td>$h_1 = (1 - r - s) - \frac{h_4}{3}$</td>
</tr>
<tr>
<td>$h_2 = r$</td>
<td>$h_2 = r - \frac{h_4}{3}$</td>
</tr>
<tr>
<td>$h_3 = s$</td>
<td>$h_3 = s - \frac{h_4}{3}$</td>
</tr>
<tr>
<td>$h_4 = 27 rs (1 - r - s)$</td>
<td>$\tilde{h}_1 = (1 - r - s)$</td>
</tr>
<tr>
<td>$\tilde{h}_2 = r$</td>
<td>$\tilde{h}_2 = r$</td>
</tr>
<tr>
<td>$\tilde{h}_3 = s$</td>
<td>$\tilde{h}_3 = s$</td>
</tr>
</tbody>
</table>

The displacement-based formulations of the transverse shear strains for the MITC3i are given in equation (4.5).

$$
\tilde{\varepsilon}_{n}^{\alpha} = \frac{1}{2} \left( -\frac{a}{2} \sum_{i=1}^{4} h_i \alpha + \frac{a}{2} \sum_{i=1}^{3} \tilde{h}_i \omega_i \right)
$$

$$
\tilde{\varepsilon}_{n}^{\alpha} = \frac{1}{2} \left( -\frac{a}{2} \sum_{i=1}^{4} h_i \alpha + \frac{a}{2} \sum_{i=1}^{3} \tilde{h}_i \omega_i \right)
$$

(4.5)

As compared to equation (3.5), the displacement-based formulation for the 3 node triangle, the 4 node triangle uses different interpolation functions for the nodal rotations. However, since the interpolation function for the middle node is zero along all three edges, and, thus, zero at the tying points, this additional node would not improve our objective, the transverse shear strains. Therefore to incorporate a contribution of the middle node in the transverse shear strains, the tying points must be relocated to inside
the element. In order to maintain spatial isotropy, the tying points are moved to the
centroid of each partitioned triangle, as shown in Figure 41.

![Figure 41 - The MITC3i](image)

The locations of the three tying points are:

<table>
<thead>
<tr>
<th>Tying Point</th>
<th>Isoparametric Coordinate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(4/9, 1/9)</td>
</tr>
<tr>
<td>2</td>
<td>(1/9, 4/9)</td>
</tr>
<tr>
<td>3</td>
<td>(4/9, 4/9)</td>
</tr>
</tbody>
</table>

Since the tying points have moved, we must consider new tying conditions and
reformulate the transverse shear strains. It is important to remember that what we are
trying to find is the best tying conditions that can represent the strain field – a
complicated surface – and represent it by the values at the three tying points. The first
strain condition is the same as that of the MITC3; we maintain that the transverse shear
strains must be constant along the edges. As explained earlier, this is to prevent shear
locking and results in the following equations:

\[
\begin{align*}
\tilde{e}_n &= a_s + c_s s \\
\tilde{e}_u &= a_s + c_s r
\end{align*}
\] (4.6)
Using our tying point locations, equation (4.6) yields the following condition (4.7).

\[
\begin{align*}
\tilde{\varepsilon}_n(0, \frac{1}{9}) &= \tilde{\varepsilon}_n(\frac{8}{9}, \frac{1}{9}) = \tilde{\varepsilon}_n^{(1)} \\
\tilde{\varepsilon}_n(\frac{1}{9}, 0) &= \tilde{\varepsilon}_n(\frac{1}{9}, \frac{8}{9}) = \tilde{\varepsilon}_n^{(2)}
\end{align*}
\] (4.7)

Equation (4.6) requires two sets of equations to solve for the unknowns, thus requiring another tying condition. The following three tying conditions are considered:

**Case A:** the average of the strains at the three tying points is equal to the strain at the middle node.

\[
\frac{1}{3} \sum_{j=1}^{3} \tilde{\varepsilon}_n^{(j)} |^{D_A} = \tilde{\varepsilon}_n |^{A} 
\] (4.8)

**Case B:** the integral of the displacement-based strain must equal that of the assumed strain.

\[
\int \tilde{\varepsilon}_n |^{D_A} dA = \int \tilde{\varepsilon}_n |^{A} dA \\
\int \tilde{\varepsilon}_n |^{D_A} dA = \int \tilde{\varepsilon}_n |^{A} dA
\] (4.9)

**Case C:** the moment calculated from the displacement-based strain is equal to that of the assumed strain.

\[
\int r\tilde{\varepsilon}_n |^{D_A} dA = \int r\tilde{\varepsilon}_n |^{A} dA \\
\int s\tilde{\varepsilon}_n |^{D_A} dA = \int s\tilde{\varepsilon}_n |^{A} dA
\] (4.10)
Applying the conditions in equation (4.10) results in the following transverse shear strains.

\[
\tilde{\varepsilon}_n = a_r + c_r s \\
\tilde{\varepsilon}_s = a_s + c_s r
\]  \hspace{1cm} (4.11)

where,

\[
a_r = \frac{a}{4} \left[ (x_2 - x_1) \sum_{i=1}^{4} k_i \beta_i + (y_2 - y_1) \sum_{i=1}^{4} l_i \alpha_i + \sum_{i=1}^{3} m_i w_i \right] \\
c_r = \frac{a}{4} \left[ (x_2 - x_1) \sum_{i=1}^{4} f_i \beta_i + (y_2 - y_1) \sum_{i=1}^{4} g_i \alpha_i + \sum_{i=1}^{3} h_i w_i \right] \\

(4.12)

\[
a_s = \frac{a}{4} \left[ (x_3 - x_1) \sum_{i=1}^{4} k_i \beta_i + (y_3 - y_1) \sum_{i=1}^{4} l_i \alpha_i + \sum_{i=1}^{3} m_i w_i \right] \\
c_s = \frac{a}{4} \left[ (x_3 - x_1) \sum_{i=1}^{4} f_i \beta_i + (y_3 - y_1) \sum_{i=1}^{4} g_i \alpha_i + \sum_{i=1}^{3} h_i w_i \right] \\

(4.13)

Please note that the coefficients \( k_i, l_i, m_i \) are constants and are not the same for \( \tilde{\varepsilon}_n \) and \( \tilde{\varepsilon}_s \); we use different variables here for clarity. Tables 7 to 9 list the values of the coefficients for the three different integral tying cases. For the MITC3i a higher order integration scheme must be used for accuracy and so we employ the 7 point Gauss numerical integration over a triangular domain.
### Table 7 - Case A Coefficients

<table>
<thead>
<tr>
<th></th>
<th>$a_r$</th>
<th></th>
<th>$a_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$ coefficients</td>
<td>$\alpha$ coefficients</td>
<td>$w$ coefficients</td>
<td>$\beta$ coefficients</td>
</tr>
<tr>
<td>$k_1$</td>
<td>0.302</td>
<td>$l_1$</td>
<td>-0.302</td>
</tr>
<tr>
<td>$k_2$</td>
<td>0.302</td>
<td>$l_2$</td>
<td>-0.302</td>
</tr>
<tr>
<td>$k_3$</td>
<td>-0.198</td>
<td>$l_3$</td>
<td>0.198</td>
</tr>
<tr>
<td>$k_4$</td>
<td>0.593</td>
<td>$l_4$</td>
<td>-0.593</td>
</tr>
</tbody>
</table>

### Table 8 - Case B Coefficients

<table>
<thead>
<tr>
<th></th>
<th>$a_r$</th>
<th></th>
<th>$a_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$ coefficients</td>
<td>$\alpha$ coefficients</td>
<td>$w$ coefficients</td>
<td>$\beta$ coefficients</td>
</tr>
<tr>
<td>$k_1$</td>
<td>0.279</td>
<td>$l_1$</td>
<td>-0.279</td>
</tr>
<tr>
<td>$k_2$</td>
<td>0.279</td>
<td>$l_2$</td>
<td>-0.279</td>
</tr>
<tr>
<td>$k_3$</td>
<td>-0.221</td>
<td>$l_3$</td>
<td>0.221</td>
</tr>
<tr>
<td>$k_4$</td>
<td>0.664</td>
<td>$l_4$</td>
<td>-0.664</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$c_r$</th>
<th></th>
<th>$c_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$ coefficients</td>
<td>$\alpha$ coefficients</td>
<td>$w$ coefficients</td>
<td>$\beta$ coefficients</td>
</tr>
<tr>
<td>$f_1$</td>
<td>-0.500</td>
<td>$g_1$</td>
<td>0.500</td>
</tr>
<tr>
<td>$f_2$</td>
<td>-0.500</td>
<td>$g_2$</td>
<td>0.500</td>
</tr>
<tr>
<td>$f_3$</td>
<td>1.000</td>
<td>$g_3$</td>
<td>-1.000</td>
</tr>
<tr>
<td>$f_4$</td>
<td>0.000</td>
<td>$g_4$</td>
<td>0.000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$c_t$</th>
<th></th>
<th>$c_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$ coefficients</td>
<td>$\alpha$ coefficients</td>
<td>$w$ coefficients</td>
<td>$\beta$ coefficients</td>
</tr>
<tr>
<td>$f_1$</td>
<td>-0.286</td>
<td>$g_1$</td>
<td>0.286</td>
</tr>
<tr>
<td>$f_2$</td>
<td>-0.286</td>
<td>$g_2$</td>
<td>0.286</td>
</tr>
<tr>
<td>$f_3$</td>
<td>1.214</td>
<td>$g_3$</td>
<td>-1.214</td>
</tr>
<tr>
<td>$f_4$</td>
<td>-0.642</td>
<td>$g_4$</td>
<td>0.642</td>
</tr>
</tbody>
</table>
Table 9 - Case C Coefficients

<table>
<thead>
<tr>
<th>Case C</th>
<th>(a_r)</th>
<th>(a_s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta) coefficients</td>
<td>(\alpha) coefficients</td>
<td>(w) coefficients</td>
</tr>
<tr>
<td>(k_1)</td>
<td>0.364</td>
<td>(-0.364)</td>
</tr>
<tr>
<td>(k_2)</td>
<td>0.164</td>
<td>(-0.164)</td>
</tr>
<tr>
<td>(k_3)</td>
<td>-0.236</td>
<td>(l_3)</td>
</tr>
<tr>
<td>(k_4)</td>
<td>0.707</td>
<td>(l_4)</td>
</tr>
</tbody>
</table>

As expected, the values for the coefficients do exhibit isotropy between the transverse shear strains \(\bar{e}_{nt}\) and \(\bar{e}_{st}\). For example, for Case C we observe that the \(\beta\)-coefficient for node 2 of \(a_r\) is the same as the \(\beta\)-coefficient for node 3 of \(a_s\). Likewise, the \(\beta\)-coefficient for node 3 of \(a_r\) is the same as the \(\beta\)-coefficient for node 2 of \(a_s\). This property is repeated for all three cases. We also note that the \(\beta\)-coefficients are equal but opposite the \(\alpha\)-coefficients.

We are trying to incorporate the behavior of the middle node and the results show that in all three cases the fourth node has the largest weight for the \(a_r\) and \(a_s\) terms. However, the influence of the middle node in the \(c_r\) and \(c_s\) terms is negligible for Case A, and has weights comparable to the other degrees of freedom for Case B and Case C.

We can compare the weights of the nodal degrees of freedom with the original MITC3. If we consider the \(a_r\) and \(a_s\) terms listed in Table 10 we can see that for Case B and C
have the same weights for the vertical translation as the MITC3. We can gain further insight to the different formulations by recalculating the results for the two-sided clamped plate problem in the following section.

Table 10 – Original MITC3 Coefficients \( a_r \) and \( a_s \) terms

<table>
<thead>
<tr>
<th>( k ) coefficients</th>
<th>( r ) coefficients</th>
<th>( w ) coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_1 ) 0.5</td>
<td>( l_1 ) -0.5</td>
<td>( m_1 ) -1</td>
</tr>
<tr>
<td>( k_2 ) 0.5</td>
<td>( l_2 ) -0.5</td>
<td>( m_2 ) 1</td>
</tr>
<tr>
<td>( k_3 ) 0</td>
<td>( l_3 ) 0</td>
<td>( m_3 ) 0</td>
</tr>
</tbody>
</table>

\[
\begin{array}{cccc}
\beta \text{ coefficients} & \alpha \text{ coefficients} & w \text{ coefficients} \\
\hline
k_1 & 0.5 & l_1 & -0.5 \\
k_2 & 0.5 & l_2 & -0.5 \\
k_3 & 0 & l_3 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
\beta \text{ coefficients} & \alpha \text{ coefficients} & w \text{ coefficients} \\
\hline
k_1 & 0.5 & l_1 & -0.5 \\
k_2 & 0 & l_2 & 0 \\
k_3 & 0.5 & l_3 & -0.5 \\
\end{array}
\]

4.3 Case Study – Two-Sided Clamped Plate

The effect of the new formulation is tested by running the two-sided clamped plate that illustrated locking problems for the MITC3 [10].

After reformulating the strains for the three different integral tying conditions, the transverse shear strains for the two-sided clamped plate are recalculated and are checked for locking. Table 11 lists the values the strain energy for each of the three different integral tying conditions. These values are compared to Table 12 which lists the strain energies of MITC3 and the expected strain energy reported in [10]. The tying conditions of Case A yielded very similar strain energies to that of the original MITC3.

We also observe that Case B and Case C yield similar results. Possible explanation for this is that both cases have the same \( w \) coefficients in both \( a_r \) and \( c_r \). Also the weights for \( a_r \) and for the nodal rotations (\( \beta, \alpha \)) are very close. The main difference between Case B and C is in the \( c_r \) coefficients for node 3; Case B has more weight on node 3.
Table 11 - MITC3i Strain Energies for the Two-Sided Clamped Plate

<table>
<thead>
<tr>
<th>Formulation</th>
<th>MITC3i - Strain Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>t/L = 1/100</td>
</tr>
<tr>
<td>Case A</td>
<td>4.1123e-004</td>
</tr>
<tr>
<td>Case B</td>
<td>2.1038e-004</td>
</tr>
<tr>
<td>Case C</td>
<td>2.3556e-004</td>
</tr>
</tbody>
</table>

Table 12 - MITC3 Strain Energies for Two-Sided Clamped Plate

<table>
<thead>
<tr>
<th>Mesh Pattern</th>
<th>MITC3 - Strain Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>t/L = 1/100</td>
</tr>
<tr>
<td>Mesh Pattern A</td>
<td>4.11903e-4</td>
</tr>
<tr>
<td>Expected</td>
<td>6.86937e-1</td>
</tr>
</tbody>
</table>

4.4 Future Work

As the results show, none of the three tying conditions improved the results of the two-sided clamped plate. In the end, placing a node in the middle created a steep interpolation function forcing the weight to be zero along all three edges. One option to consider would be a smoother interpolation function. It appears that the MITC3 element is formulated to its optimal capability.
References


