Linear Optimization I

A Sample of Successful Applications

- The San Francisco Police Department improved the way it scheduled its patrol officers & saved $11 million per year, improved response time by 20%, & increased revenue from traffic citations by $3 million.

- Digital used a Global Supply Chain Management model to locate its facilities, & plan its sourcing, production, & distribution networks. The restructuring has reduced cumulative costs by $1 billion & assets by $400 million.

- Kodak Pty. Ltd. uses a new system for diagramming small customer rolls from large bulk rolls of photographic color paper, saving $2 million in the first year.

- **American Airlines**, three years of improvement to the TRIP crew scheduling model have yielded annual savings of **$20 million!** Optimization models of overbooking, discount allocation, and traffic management contributes **$500 million** per year to American Airlines.

- **Reynolds Metals Company** uses a central dispatch model to assign shipments to 14 trucking firms for its over 200 locations; it has improved delivery time and reduced annual freight costs by over **$7 million.**

- **GTE** uses NETCAP to plan its **$300 million** annual investment in new telephone lines and other customer access facilities.

Terminology

***Decision Variable*:** Describes a decision that needs to be made, e.g. how many items to produce.

***Objective Function*:** An expression (in terms of the variables) that needs to be minimized or maximized.

***Constraint*:** An expression that restricts the values of the variables.

Steps in Writing A Formulation

1. Define the variables

2. Write the objective as a function of these variables

3. Write the constraints as functions of these variables

4. Determine the variable restrictions, e.g. non-negative, integer

2. Gemstone Tool Company

Gemstone Tool Company (GTC) is a privately-held firm that competes in the consumer and industrial market for construction tools.
In addition to its main manufacturing facility in Seattle, Washington, GTC operates several other manufacturing plants located in the United States, Canada, and Mexico.

For the sake of simplicity, let us suppose that the Winnipeg, Canada plant only produces wrenches and pliers. Wrenches and pliers are made from steel, and the process involves molding the tools on a molding machine and then assembling the tools on an assembly machine.

The amount of steel used in the production of wrenches and pliers and the daily availability of steel is given in the first line of Table I.
Also the machine utilization rates needed in the production of wrenches and pliers are given as well as the capacity of these machines.
Finally, the last two rows of the table indicate the daily market demand for these tools, and their variable (per unit) contribution to earnings.

Table I

	Wrenches	Pliers	Availability/Capacity
Steel (lbs.)	1.5	1.0	27,000 lbs./day
Molding Machine (hours)	1.0	1.0	21,000 hours/day
Assembly Machine (hours)	0.3	0.5	9,000 hours/day
Demand Limit (tools/day)	15,000	16,000	
Contribution to earnings ($/1,000 units)	$ 130	$ 100	

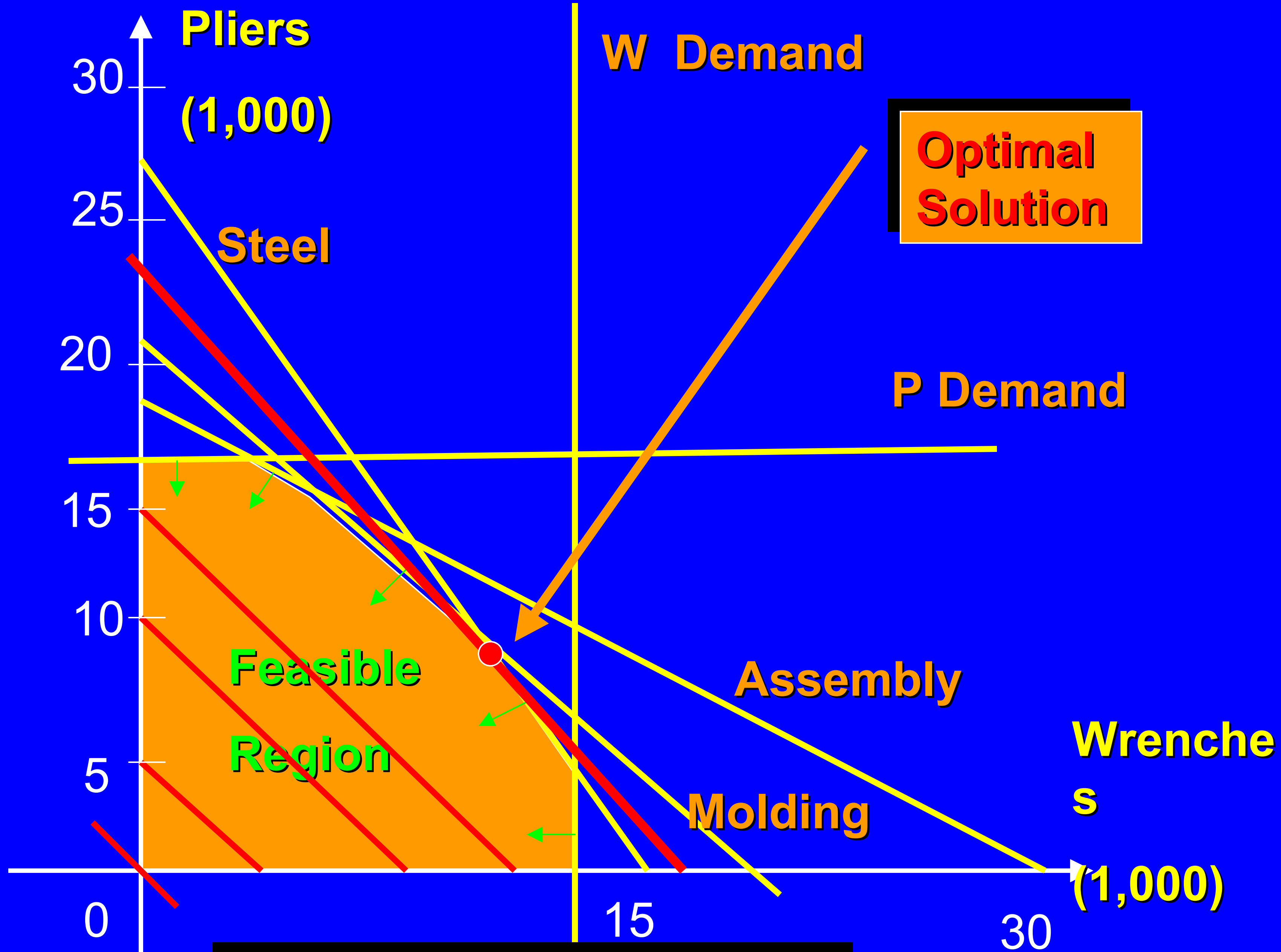
GTC would like to plan for the daily production of pliers and wrenches at its Winnipeg plant so as to maximize the contribution to earnings.

As Before...

W : # of wrenches produced per day (1,000s)

P : # of pliers produced per day (1,000s)

Maximize:		$130\,W + 100\,P$
Subject to:	steel:	$1.5\,W + P \leq 27$
	molding:	$W + P \leq 21$
	assembly:	$0.3\,W + 0.5\,P \leq 9$
	W-demand:	$W \leq 15$
	P-demand:	$P \leq 16$
	nonnegativity:	$W,\ P \geq 0$



Graphical Solution of GTC

Optimal solution of this linear optimization model is :

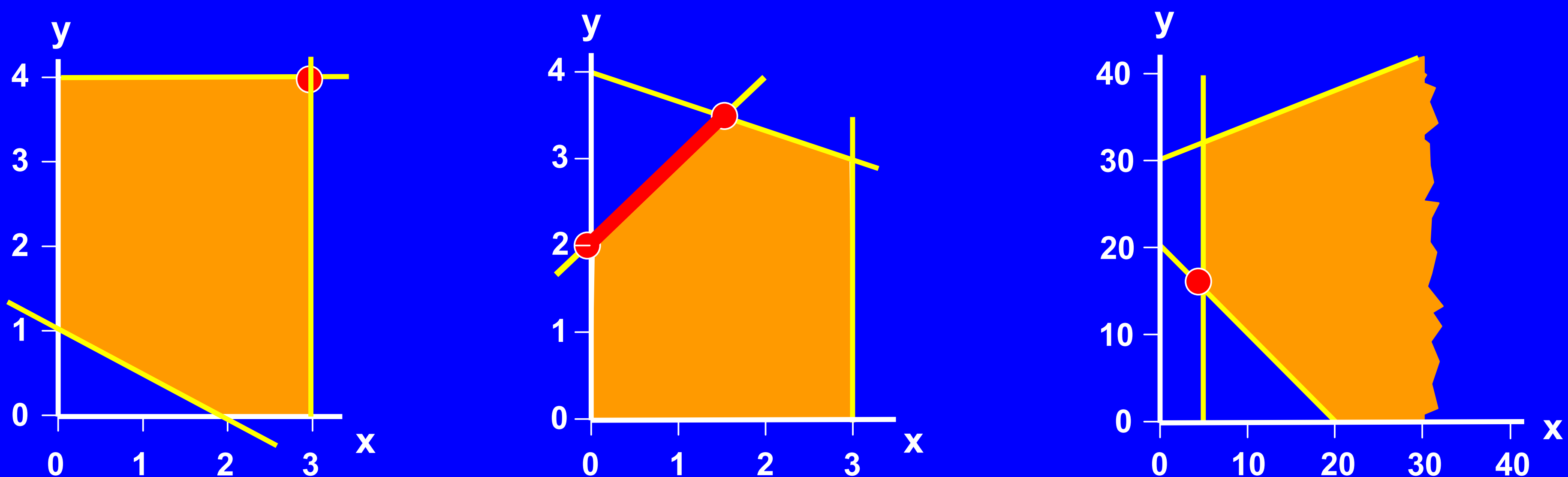
steel supply + molding machine constraints

i.e., solve:

$$1.5\,W + P = 27$$
$$W + P = 21$$

W = 12   P = 9   CONTRIBUTION = 130*12 + 100*9 = $2,460

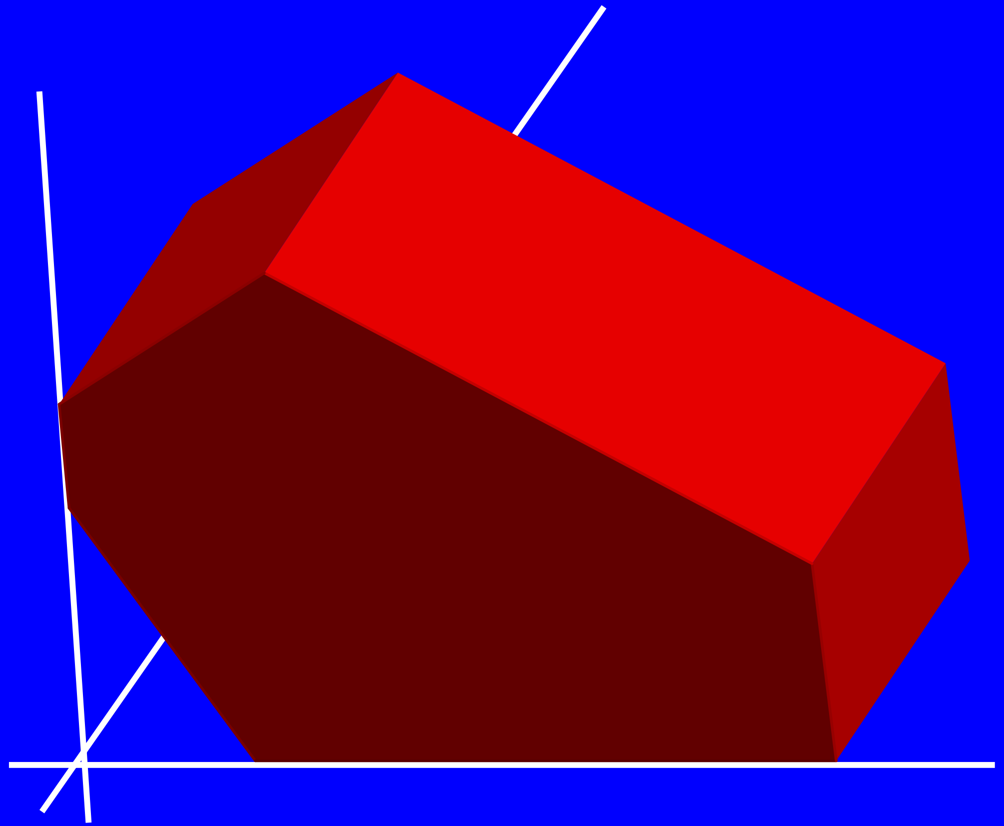
A Fundamental Point



If an optimal solution exists, there is always a corner point optimal solution!

Gemstone Tool Company			
	Wrenches	Pliers	Available
Steel	1.5	1	27
Molding	1	1	21
Assembly	0.3	0.5	9
Demand Limit	15	16	
Contribution	130	100	
Amounts	12	9	
Earnings:	2460		
Constraints			
	Actual	Limit	
steel	27	27	
molding	21	21	
assembly	8.1	9	
wdemand	12	15	
pdemand	9	16	

And We Can Extend this to Higher Dimensions



How Might We Solve an LP?

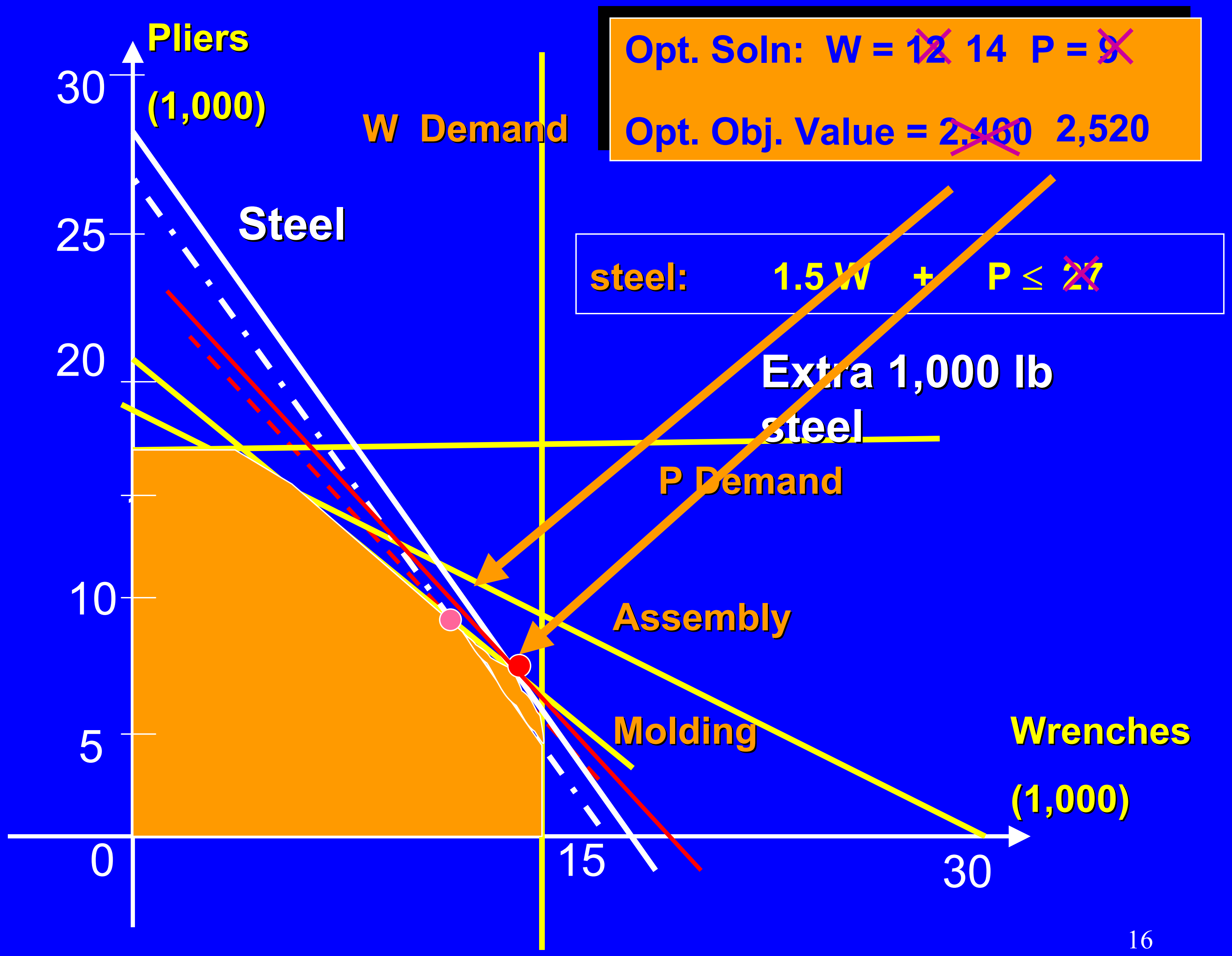
- The constraints of an LP give rise to a geometrical shape - we call it a **polyhedron**.
- If we can determine all the **corner points** of the polyhedron, then we can calculate the objective value at these points and take the best one as our optimal solution.
- The ***Simplex Method*** intelligently moves from **corner to corner** until it can prove that it has found the optimal solution.

Sensitivity Analysis
"Playing the What-If Game"

In business applications, we often want to know how our solution is effected by changes to our problem data.

- **What if our resource levels change?**
- **What if our cost structures change?**
- **What if our data is not precise? How much room for error do we have?**

We will first look graphically and then extend the intuition.


Opt. Soln: W = ~~12~~ 14 P = ~~9~~
Opt. Obj. Value = ~~2,460~~ 2,520
steel: 1.5 W + P ≤ ~~27~~
Pliers
(1,000)
30
25
20
10
5
0
15
30
W Demand
Steel
Extra 1,000 lb steel
P Demand
Assembly
Molding
Wrenches
(1,000)

We are given an extra 1,000 lb. of steel.

RHS of steel constraint would change: 27 to 28 = 27 + 1
our new optimal solution :

$$1.5\,W + P = 28$$
$$W + P = 21.$$

W = 14 P = 7 CONTRIBUTION = 130*14 + 100*7 = $ 2,520

The shadow price of the steel constraint
from extra 1,000 lb. of steel:

$60 = $2,520 – $2,460

GTC willing to pay up to $60/1,000 lb. for additional steel

About Shadow Prices

- Associated with each constraint is a shadow price.
- The ***shadow price*** is the change in the objective value per unit change in the right hand side, given all other data remain the same.
- Associated with each shadow price is a range over which this shadow price holds.
- Most solvers provide shadow prices and ranges as part of the solution information.
- Shadow prices are also called **dual values**.

What is the shadow price for the following constraints:

- **assembly machine capacity?**
- **wrench demand limit?**
- **pliers demand limit?**

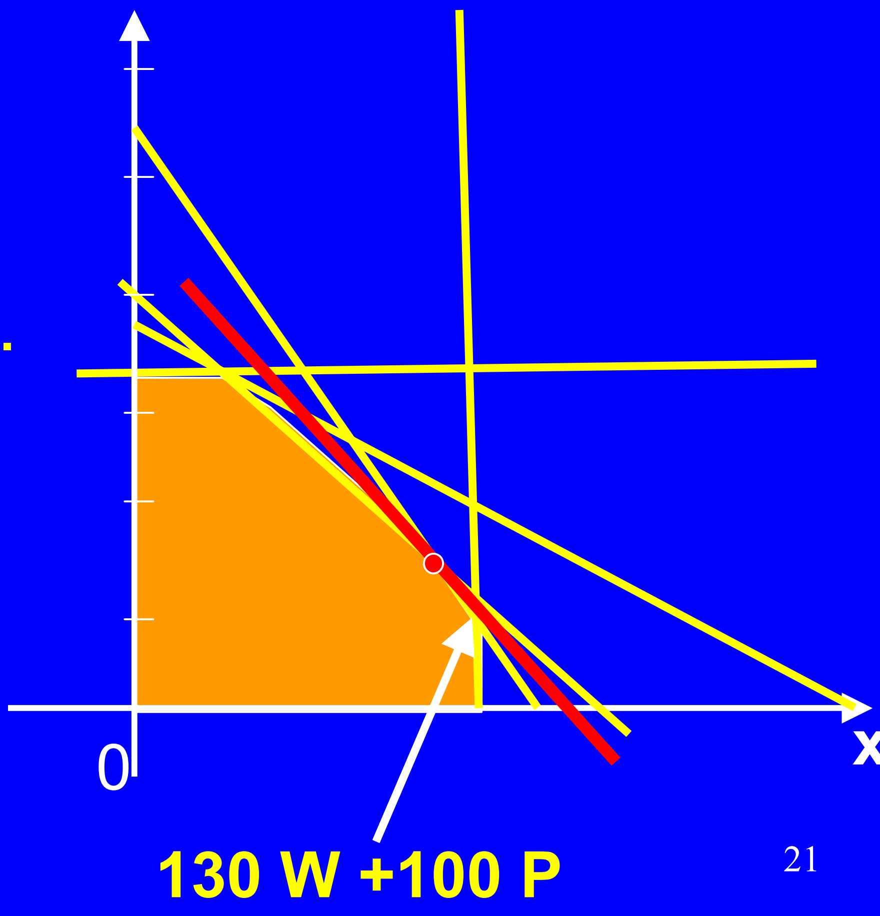
What About the Shadow Prices on the Non-Negativity Constraints?

- The shadow prices for non-negativity constraints are called ***reduced costs***.
- The ***reduced cost*** of a variable is the change in the objective function if we require that variable to be greater than or equal to one (rather than zero), assuming a feasible solution still exists.

So What If We Change One of Our Objective Coefficients?

Lets look graphically,

- The feasible region does not change.
- The slope of the objective function changes.
- There is a range over which the optimal solution will not change (of course the objective value will).
- This range is determined by the slope of the constraints that are *active* at the optimal solution.
- In this case, the range is [100, 150] on W and [86.66, 130] on P.



A More Intuitive Interpretation...

- If, at an optimal solution, a variable has a positive value, the reduced cost for that variable will be 0.

- If, at an optimal solution, a variable has a value of zero the reduced cost for that variable is:

  - the amount by which the objective value will change if we increase the value of this variable to one.

  OR

  - the amount by which the objective coefficient would have to change in order to have a positive optimal value for that variable.

  - CAVEAT: Assuming a feasible solution still exists!

Conclusions on Sensitivity Analysis

Q. What if we change the right hand side of a constraint by a small amount?

A. Look at the shadow price for that constraint.

Q. How much can we change a right hand side and still have the same result?

A. Look at the range on the shadow price.

Q. What if we change an objective function coefficient by a small amount?

A. If the amount is small enough, the same solution is optimal.

Q. How much can we change an objective function coefficient and still have the same result?

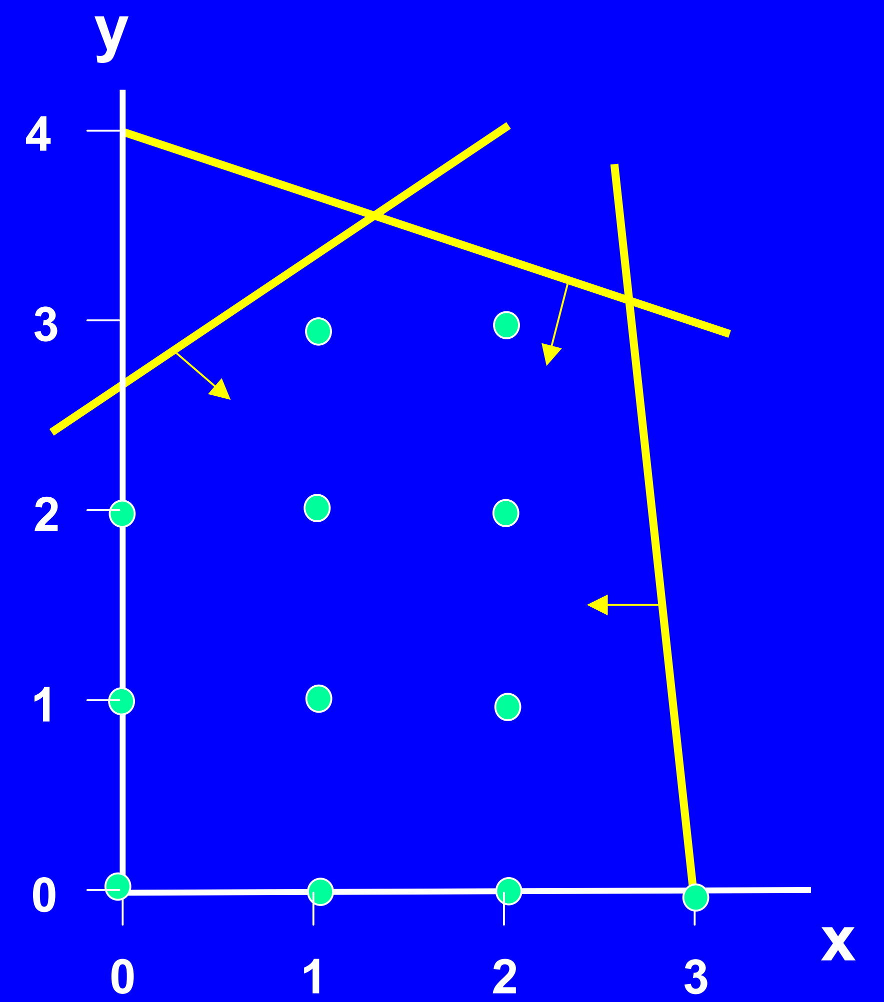
A. Look at the range on the objective value coefficient.

Q. How much would an objective value have to change before an activity is worthwhile?

A. Look at the reduced cost on the variable.

Integer Optimization

- Feasible region is a set of discrete points.
- Can't be assured a corner point solution.
- There are no "efficient" ways to solve an IP.
- Solving it as an LP provides a relaxation and a bound on the solution.



A "Partial Taxonomy" of Math Optimization

Linear Optimization (LP)
objective and constraints are linear expressions

Integer Optimization (IP)
variables are restricted to discrete (integer) values

Mixed - Integer Optimization (MIP)
some variables are continuous, some are discrete

Nonlinear Optimization (NLP)
objective and/or constraints are non-linear expressions

Examples of Application Areas

Production Planning: Given several products with varying production requirements and cost structures, determine how much of each product to produce in order to maximize profits.

Scheduling: Given a staff of people, determine an optimal work schedule that maximizes worker preferences while adhering to scheduling rules.

Network Installation: Given point-to-point demands on a network, install capacities on the edges so as to minimize installation and routing costs.

Portfolio Management: Determine bond portfolios that maximize expected return subject to constraints on risk levels and diversification.