Simulation Lecture Notes

and the

Gentle Lentil Case

General Overview of the Case

- What is the decision problem presented in the case?
- What are the issues Sanjay must consider in deciding among the alternative choices?

Accounting Model of Gentle Lentil's Monthly Earnings

What are the monthly fixed costs?

L = labor costs (between $5,040 and $6,860)

U = rent, utilities, other unavoidable costs = $3,995

What are the monthly variable costs?

F = food costs

M = number of meals served in month

F = $11 x M

What are the monthly total costs?

L + U + F = L + 3,995 + 11 x M

Gentle Lentil's Monthly Earnings,cont.

What are the monthly revenues?

R = monthly revenues
P = price of meal
R = P x M

What are the monthly earnings?

X = monthly earnings = revenues – costs
= P x M – (L + 3,995 + 11 x M)
= (P – 11) x M – L – 3,995

Which of these quantities are random variables?

P = price of prix fixe meal
M = number of meals sold
L = labor cost
X = monthly earnings

(X is a function of random variables, so it is a random variable)

Assumptions Regarding the Behavior of the Random Variables

M = number of meals sold per month

We assume that M obeys a Normal distribution with $\mu = 3{,}000$ and $\sigma = 1{,}000$

P = price of the prix fixe meal

We assume that P obeys the following discrete probability distribution

Scenario	Price of Prix Fixe Meal	Probability
Very healthy market	$20.00	0.25
Healthy market	$18.50	0.35
Not so healthy market	$16.50	0.30
Unhealthy market	$15.00	0.10

L = labor costs per month

We assume that L obeys a uniform distribution with a minimum of $5,040 and maximum of $6,860

The Behavior of the Random Variables, cont.

X = earnings per month

We do not know the distribution of X. We assume, however, that

$$X = (P - 11) \times M - L - 3{,}995$$

Always ask the following questions in any management analysis:

- **How realistic is this model?**
- **How good are the assumptions?**

What question do we want to ask & answer about the random variable X?

- **What is the shape of the probability distribution of X ?**
- **What is E(X) ?**
- **What is SD(X) ?**
- **What is P($X \leq \$5{,}000$) ?**
- **What is P($X \geq \$6{,}667$) ?**
- **Other questions to ask?**

And at the end:

What would you do if you were Sanjay?

Simulation of Gentle Lentil Monthly Earnings for "February"

Choose a value of M (number of meals served) that obeys the probability distribution N (3,000 , 1,000)

Choose a value of P (price of prix fixe meal) that obeys the discrete probability distribution for P

Choose a value of L (labor cost) that obeys the uniform distribution in the range [5,040 , 6,860]

Compute the monthly earnings X by computing

$$X = (P - 11) \times M - L - 3{,}995$$

Run this n times. We used n = 1,000

This will generate as output n numbers $x_1, x_2, \ldots, x_n$.

What shall we do with this output?

What to do with output created by the simulation?

We have as output the n numbers $x_1, x_2, ..., x_n$

- **Create an estimate of the shape of the underlying probability distribution of X (earnings).**
- **Create an estimate of the mean μ of X**
- **Create an estimate of the standard deviation σ of X**
- **Create an estimate of $P(X \leq \$5{,}000)$**
- **Create an estimate of $P(X \geq \$6{,}667)$**

Example: Let P denote the price of the prix fixe meal at Gentle Lentil

We assume that P obeys the following discrete probability distribution:

Price of Prix Fixe Meal ($)	
20.00	0.25
18.50	0.35
16.50	0.30
15.00	0.10

Create the following random number assignments:

Price of Prix Fixe Meal ($)	Probability	Random Number Assignment
20.00	.25	.00 -- .25
18.50	.35	.25 -- .60
16.50	.30	.60 -- .90
15.00	.10	.90 1.00

Illustration:

Trial	Random Number	Price of Prix Fixe Meal ($)
	.973	15.00
	.020	20.00
	.802	16.50
	.663	16.50
	.965	15.00
	.553	18.50

How to Generate Random Numbers from a Continuous Probability Distribution

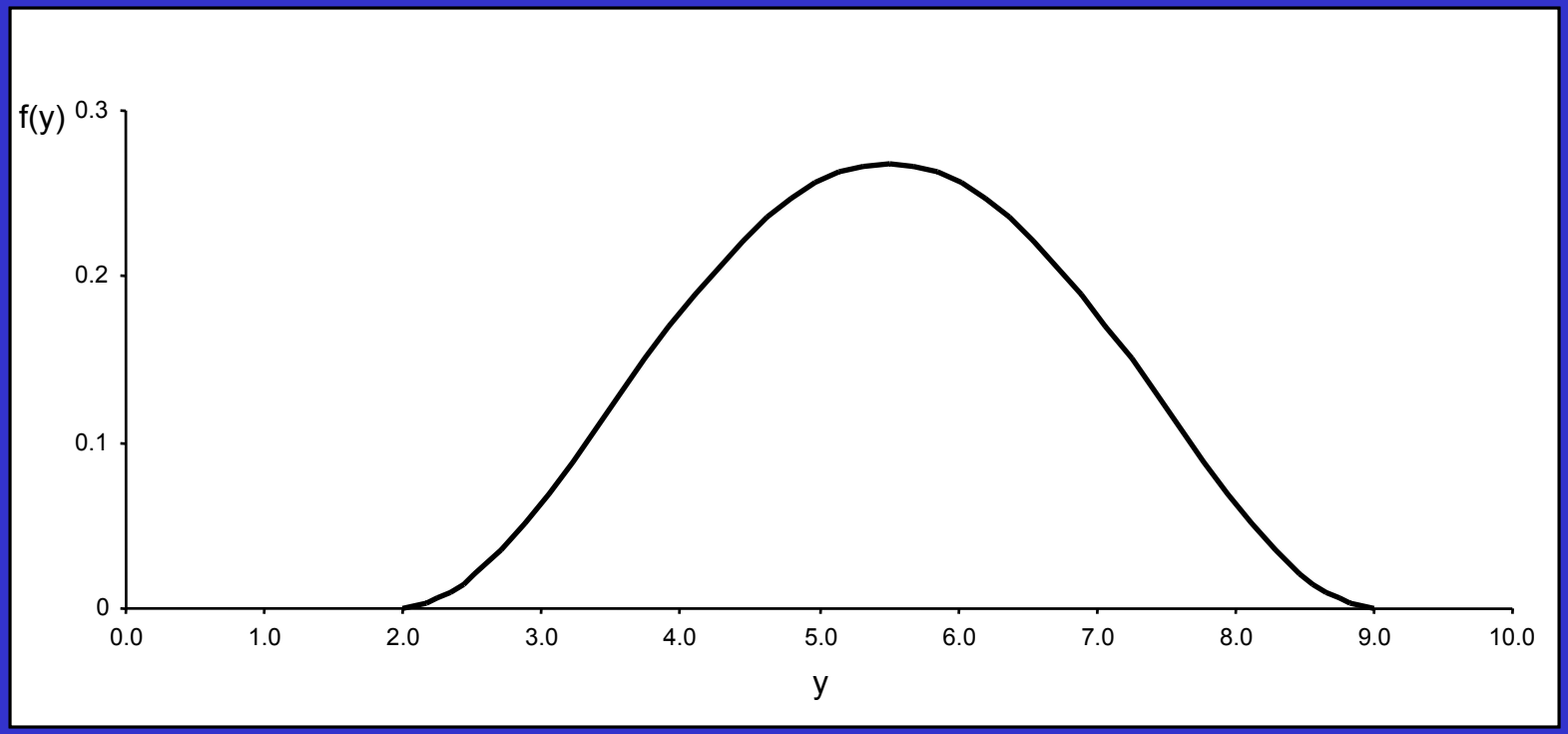
- **Most software programs that perform simulation have the capability to generate random numbers from a variety of standard continuous distributions, such as the Normal distribution, the uniform distribution, etc.**

- **The user need only specify the type of distribution and the parameters**

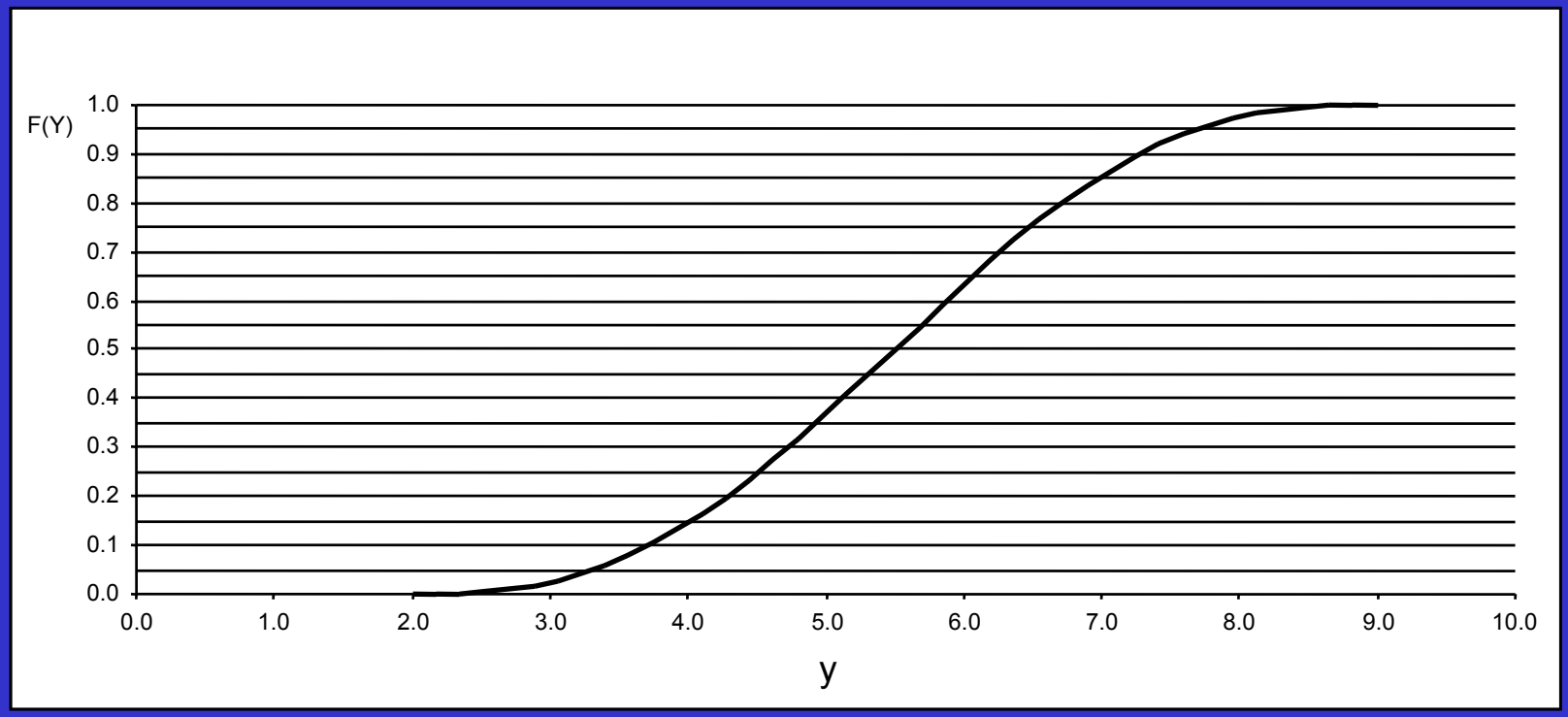
  (μ and σ for the Normal, a and b for the uniform, etc.)

However, it is worthwhile to point out how the computer accomplishes this task

The Probability Density Function f(y) of the Random Variable



Cumulative Distribution Function F(y) of the Random Variable

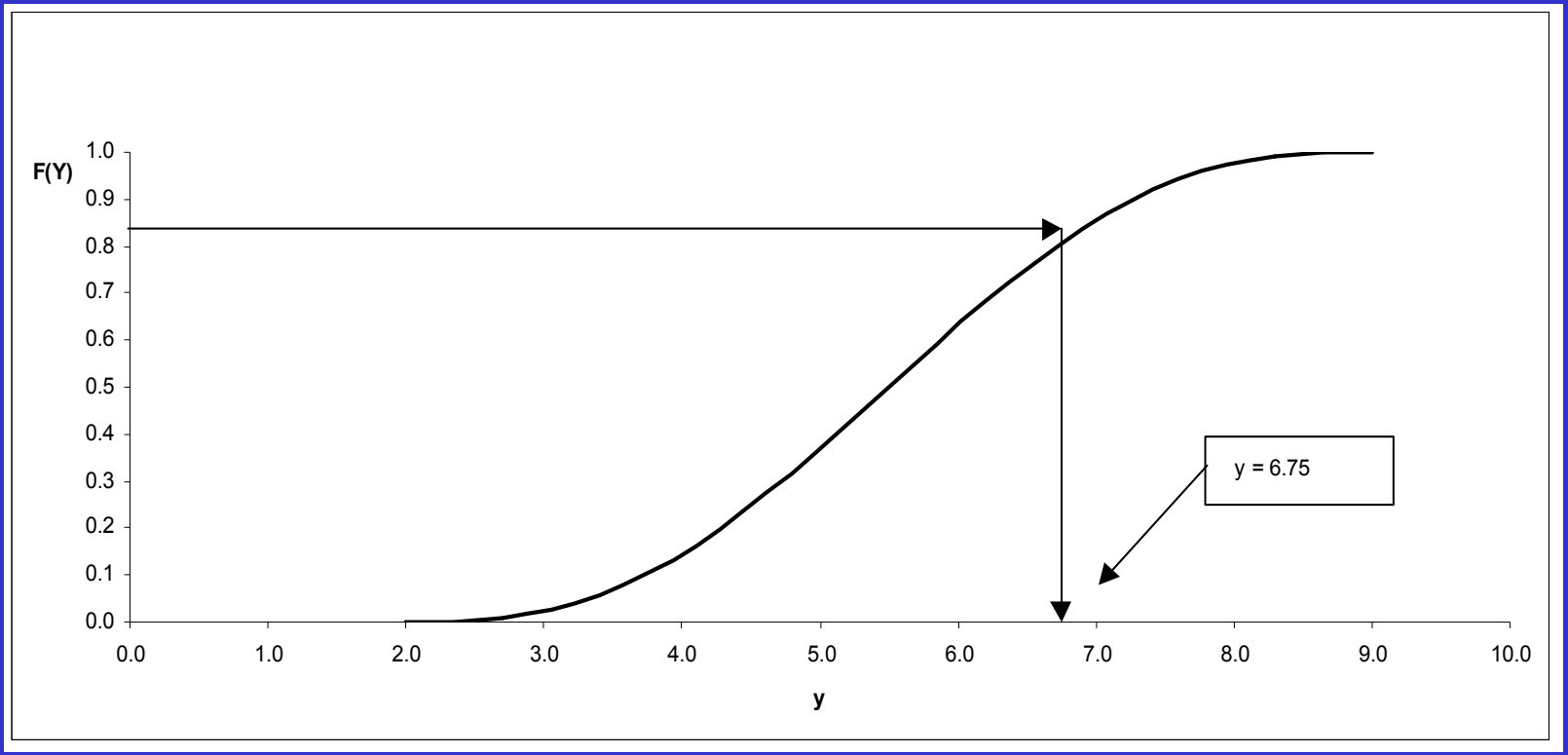


Creating Sample Data Drawn from a Continuous Probability Distribution

1. Use a random number generator to generate a number x that obeys a uniform distribution between 0.0 and 1.0

2. Place the number x on the vertical axis of the graph of the cdf F(y) of the distribution of interest. Then find the point y on the horizontal axis whose cdf value F(y) is equal to x

Cumulative Distribution Function F(y) of a Random Variable



1,000 Trials of Gentle Lentil Monthly Simulation ($)

Trial Number	Random Number	Price of Meal	Random Number	Number of Meals Served	Random Number	Labor Costs	Monthly Earnings
1	0.6616	16.50	0.1024	1,732.13	0.803	6,500.94	-969.23
2	0.9233	15.00	0.0971	1,701.95	0.649	6,221.59	-3,408.81
3	0.3003	18.50	0.1691	2,042.38	0.262	5,517.08	5,805.77
4	0.3774	18.50	0.2059	2,179.43	0.609	6,147.86	6,202.87
5	0.3621	18.50	0.8036	3,854.52	0.409	5,784.60	19,129.32
6	0.4994	18.50	0.2903	2,447.60	0.208	5,417.72	8,944.26
7	0.8688	16.50	0.6069	3,271.18	0.469	5,893.94	8,102.56
8	0.3809	18.50	0.3227	2,539.71	0.224	5,448.50	9,604.32
9	0.3605	18.50	0.9958	5,634.33	0.557	6,053.06	32,209.44
10	0.1199	20.00	0.8115	3,883.56	0.817	6,527.21	24,429.81
11	0.4668	18.50	0.2747	2,401.48	0.422	5,807.71	8,208.35
12	0.0229	20.00	0.4822	2,955.35	0.91	6,695.67	15,907.51
13	0.956	15.00	0.8403	3,995.79	0.446	5,851.35	6,136.81
14	0.9251	15.00	0.0298	1,115.85	0.384	5,738.51	-5,270.09
15	0.6625	16.50	0.6104	3,280.36	0.459	5,874.85	8,172.15
16	0.0778	20.00	0.9529	4,673.74	0.178	5,364.04	32,704.61
17	0.9002	15.00	0.732	3,618.72	0.773	6,446.92	4,032.97
18	0.4938	18.50	0.6949	3,509.67	0.187	5,380.98	16,946.53
19	0.6698	16.50	0.4064	2,763.19	0.171	5,350.51	5,852.02
20	0.669	16.50	0.4822	2,955.45	0.785	6,469.55	5,790.42
21	0.0554	20.00	0.2164	2,215.43	0.288	5,564.93	10,378.95
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
991	0.8662	16.50	0.3053	2,490.73	0.186	5,378.26	4,325.72
992	0.3161	18.50	0.363	2,649.49	0.537	6,017.32	9,858.83
993	0.2757	18.50	0.0123	751.69	8E-04	5,041.42	-3,398.72
994	0.6844	16.50	0.797	3,831.11	0.87	6,623.90	10,452.21
995	0.0572	20.00	0.9871	5,230.17	0.197	5,398.59	37,677.96
996	0.7899	16.50	0.8378	3,985.51	0.201	5,405.81	12,519.49
997	0.5202	18.50	0.3713	2,671.58	0.165	5,339.57	10,702.25
998	0.1155	20.00	0.7075	3,546.01	0.754	6,412.91	21,506.21
999	0.3012	18.50	0.2901	2,446.91	0.627	6,181.58	8,175.24
1,000	0.9151	15.00	0.292	2,452.37	0.365	5,704.71	109.75
Sample Mean							**10,303.59**
Sample S.D.							**8,491.70**

Discussion of Simulation Model Output

Let $x_1, x_2, ..., x_n$ denote the values of the monthly earnings obtained for each of the n = 1,000 trials

Then $x_1, x_2, ..., x_n$ are each observed values from the distribution of the random variable X

***Armed with these numbers, we want to estimate*:**

- the shape of the probability distribution of X
- the mean μ of X
- the standard deviation σ of X
- the probability that X will lie in a given range, i.e., $Pr(a \leq X \leq b)$ for given values of a and b

Discussion of Simulation Model Output, cont.

Estimating the shape of the distribution of the random variable X:

A histogram of the trial values $x_1, x_2, \ldots, x_n$ is a good estimate of the shape of the probability distribution of the random variable X

Estimating the probability that X will lie in a given range:
Suppose that we want to estimate Pr(a ≤ X ≤ b) for given values of a and b.

Let m denote the number of values among the n observations $x_1, x_2, \ldots, x_n$ that are in the range between a and b.

Let $\bar{p} = \frac{m}{n}$. Note that $\bar{p}$ is simply the fraction of the observations $x_1, x_2, \ldots, x_n$ that are in the range between a and b.

Then $\bar{p}$ is an estimate of the probability Pr(a ≤ X ≤ b).

Histogram of Monthly Earnings from the Gentle Lentil Simulation Model

Estimating the mean μ and standard deviation σ

Compute the observed sample mean of the random variable X

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

For example, from the spreadsheet we obtain

$$\bar{x} = \frac{-\$969.23 - \$3{,}408.81 + \ldots + \$109.75}{1{,}000} = \$10{,}303.59$$

Compute the observed sample standard deviation of the random variable X :

$$s = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n - 1}}$$

For example, from the spreadsheet we obtain

$$s^2 = \frac{(-969.23 - 10{,}303.59)^2 + (-3{,}408.81 - 10{,}303.59)^2 + \ldots + (109.75 - 10{,}303.59)^2}{999}$$

$$= 72{,}108{,}968.89$$

$$s = \$8{,}491.70$$

Simulation of Monthly Earnings at Gentle Lentil using Crystal Ball™

A random variable is modeled in Crystal Ball as an "assumption" cell

- **for each assumption cell, the user must choose and describe the parameters of the probability distribution of the random variable in the cell**

Suppose we are interested in finding out the unknown probability distribution of one or more quantities in the spreadsheet. In Crystal Ball, if we designate a cell as a "forecast" cell, then Crystal Ball will automatically compute :

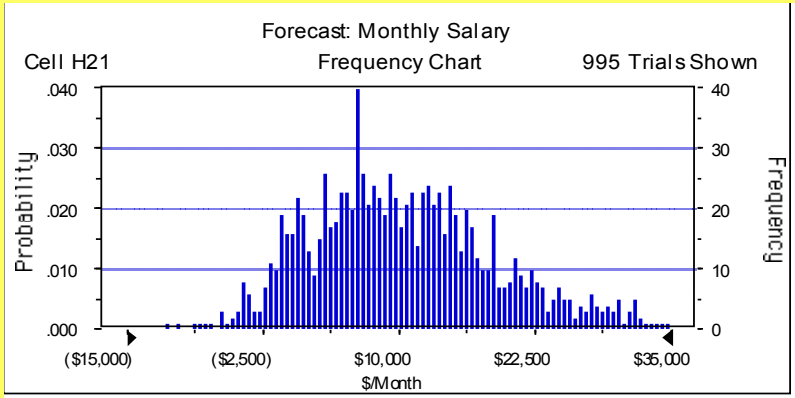
- **the histogram**
- **sample mean**
- **sample standard deviation**
- **all sorts of other useful information for the cell in the simulation**

Crystal Ball™ Output
Monthly Salary

Forecast: Monthly Salary **[GL.XLS]Monthly - Cell: H21**

Summary:
Display Range is from ($15,000) to $35,000 $/Month
Entire Range is from ($11,038) to $40,144 $/Month
After 1,000 Trials, the Std. Error of the Mean is $272

Statistics:	Value
Trials	1000
Mean	$10,526
Median (approx.)	$9,605
Mode (approx.)	$6,620
Standard Deviation	$8,602
Variance	$73,995,647
Skewness	0.51
Kurtosis	3.06
Coeff. of Variability	0.82
Range Minimum	($11,038)
Range Maximum	$40,144
Range Width	$51,183
Mean Std. Error	$272.02



Interpretation of Simulation Results: Monthly Earnings at Gentle Lentil

- What is the shape of the distribution of monthly earnings?
- What is an estimate of the expected monthly earnings?
- The standard deviation?

$\overline{x}$ = $10,526 s = $8,602

- What is an estimate of the probability that Sanjay will earn less than $5,000 in a given month?

27.6%

- What is an estimate of the probability that Sanjay will earn more than in consulting ($6,667 per month) in this particular month?

63.3%

What would you do if you were Sanjay?

Simulation of Annual Earnings at Gentle Lentil

Instead of looking just at monthly earnings at Gentle Lentil, it would be better to look at annual earnings

What should the annual model assume?

All assumptions for the monthly model, plus:

- **there are twelve distinct random variables M_1, ... , M_{12} for the twelve distinct number of meals served in each month**
- **these random variables will be independent and identically distributed (i.i.d.)**
- **the random variable P (price of prix fixed meal) remains the same over the entire year**
- **the random variable L (labor cost in month) remains the same over the entire year**

Simulation of Annual Earnings at Gentle Lentil, cont.

Use a random number generator to generate random values of P , L , and M_1, ... , M_{12} according to their respective distributions

$$X_i = (P - 11) \times M_i - L - 3{,}995$$

$$\text{for } i = 1, ..., 12$$

$$A = \text{annual earnings} = X_1 + X_2 + X_3 + ... + X_{12}$$

How is this model different from the monthly model?

Simulation Output from Crystal Ball™
Annual Proprietor Salary

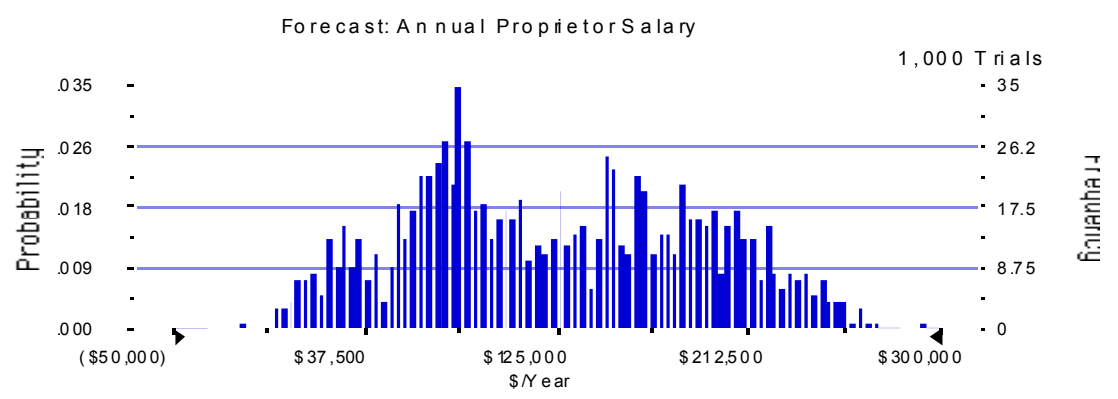
Forecast: Annual Proprietor Salary **[GL.XLS]Annual-Poca - Cell: D14**

Summary:

Display Range is from ($50,000) to $300,000 $/Year
Entire Range is from ($15,853) to $292,519 $/Year
After 1,000 Trials, the Std. Error of the Mean is $2,053

Statistics:

	Value
Trials	1000
Mean	$124,585
Median (approx.)	$121,030
Mode (approx.)	$81,284
Standard Deviation	$64,937
Variance	$4,216,752,610
Skewness	0.13
Kurtosis	2.05
Coeff. of Variability	0.52
Range Minimum	($15,853)
Range Maximum	$292,519
Range Width	$308,372
Mean Std. Error	$2,053.47



Interpretation of Simultaneous Results: Annual Earnings at Gentle Lentil/Proprietorship

- What is the shape of the distribution of annual earnings?
- What is an estimate of the expected annual earnings? The standard deviation? $\bar{x} = \$124,585, \quad s = \$64,937$
- How do these annual statistics compare to the monthly statistics?
- What is an estimate of the probability that Sanjay will earn less than $60,000?

16.2%

- What is an estimate of the probability that Sanjay will earn more than $80,000?

69.3%

What would you do if you were Sanjay?

Summary Table of Some Relevant Estimates

Choice	Estimate of Mean	Estimate of Std.Dev.	95% Confidence Interval of Mean	Estimate of Pr(Salary > $80.000)	Estimate of Pr(Salary≤ $60,000)
Consulting	$80,000	$ 0	$80,000± 0.0	100%	0%
Proprietorship	$124,585	$64,937	$124,585± 4,025	69.3%	16.2%

Simulation of Annual Earnings at Gentle Lentil Under a Partnership

What are the terms of the partnership?

Restaurant Earnings in month	Sanjay's Earnings in month
X ≤ $3,500	$3,500
$3,500 ≤ X ≤ $9,000	X
X > $9,000	$9,000 + 0.10 * (X – 9,000)

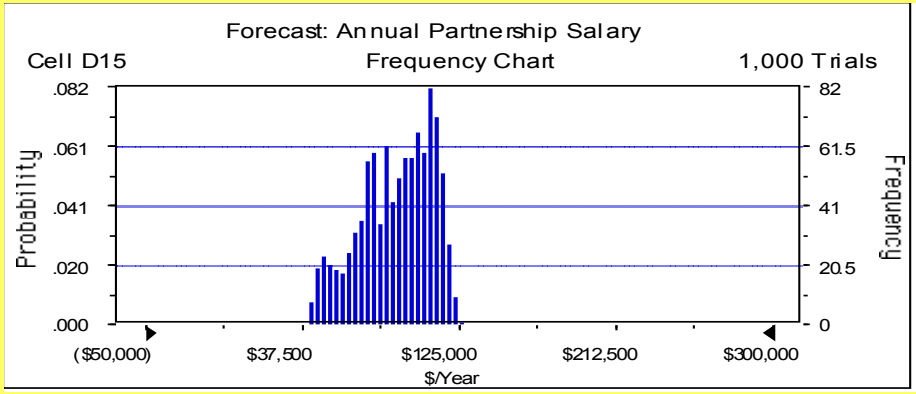
Sanjay's Monthly Salary Under Financial Partnership ($)

Simulation Output from Crystal Ball™
Annual Partnership Salary

Forecast: Annual Partnership Salary **[GL.XLS]Annual-Poca - Cell: D15**

Summary:
Display Range is from ($50,000) to $300,000 $/Year
Entire Range is from $42,000 to $126,452 $/Year
After 1,000 Trials, the Std. Error of the Mean is $637

Statistics:	Value
Trials	1000
Mean	$89,887
Median (approx.)	$92,530
Mode (approx.)	$99,005
Standard Deviation	$20,154
Variance	$406,203,025
Skewness	-0.44
Kurtosis	2.25
Coeff. of Variability	0.22
Range Minimum	$42,000
Range Maximum	$126,452
Range Width	$84,452
Mean Std. Error	$637.34



Interpretation of Simulation Results: Annual Earnings at Gentle Lentil / Partnership

What are the expected annual earnings and standard deviation under the partnership?

$\overline{x} = \$89,887$

$s = \$20,154$

In the partnership, what is an estimate of the probability that Sanjay will earn less than $60,000?

10.2%

In the partnership, what is an estimate of the probability that Sanjay will earn more than $80,000?

68.0%

Summary Table of Some Relevant Estimates

Choice	Estimate of Mean	Estimate of Std. Dev.	95% Confidence Interval of Mean	Estimate of Pr(Salary $\geq$ $80,000)	Estimate of Pr(Salary $\leq$ $60,000)
Consulting	$80,000	$ 0	$80,000 $\pm$ 0.0	100%	0%
Proprietorship	$124,585	$64,937	$124,585 $\pm$ 4,025	69.3%	16.2%
Partnership	$89,887	$20,154	$89,887 $\pm$ 1,249	68.0%	10.2%

Estimate of Pr(Proprietorship outperforms Partnership) = 71.7%

Comparison: Partnership versus Proprietorship

What is the likelihood that the proprietorship is better than the partnership?

71.7%

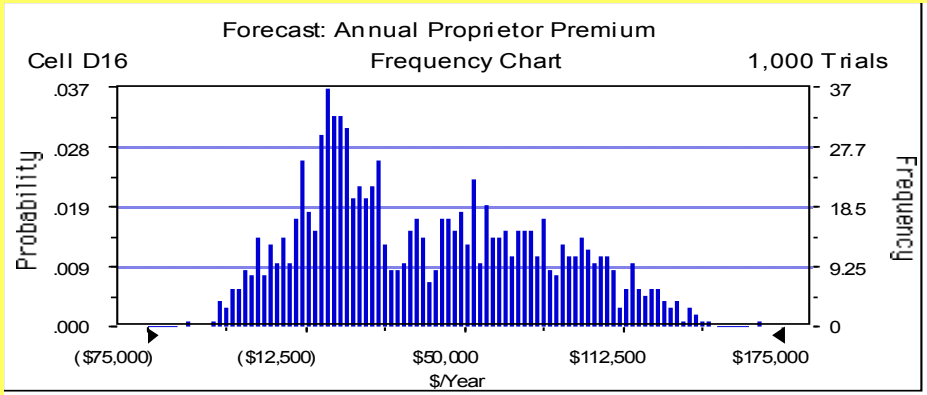
Should Sanjay go for Gentle Lentil alone, or in financial partnership with his aunt?

Simulation Output from Crystal Ball™ Annual Proprietor Premium

Forecast: Annual Proprietor Premium [GL.XLS]Annual-Poca - Cell: D16

Summary:
Display Range is from ($75,000) to $175,000 $/Year
Entire Range is from ($57,853) to $166,067 $/Year
After 1,000 Trials, the Std. Error of the Mean is $1,448

Statistics:	Value
Trials	1000
Mean	$34,698
Median (approx.)	$27,610
Mode (approx.)	($753)
Standard Deviation	$45,787
Variance	$2,096,427,661
Skewness	0.38
Kurtosis	2.15
Coeff. of Variability	1.32
Range Minimum	($57,853)
Range Maximum	$166,067
Range Width	$223,920
Mean Std. Error	$1,447.90



Discussion of the Estimates

How good are these estimates?

- **Intuition suggests that when the number of trials is very large, then the estimates are very reliable**
- **However, to give a more precise answer to this question, we will need to learn sampling theory**

Some Lessons of Simulation

- Simulation attempts to measure things that average case analysis and simple formulas cannot
- The successful application of a simulation model depends on the ability to generate random variables that obey a variety of discrete and continuous probability distributions
- Simulation can demonstrate effects in a system that cannot otherwise be “derived”

Some Lessons of Simulation, cont.

- The results that one can obtain in a simulation are not precise, due to the inherent randomness in a simulation. Care must be used in interpreting simulation results
- The typical conclusions that one can draw from a simulation model are:
  - estimates of the distributions of particular quantities of interest
  - means and standard deviations of these distributions
- From these distributions, one can derive confidence intervals and other inferences of statistical sampling

Some Lessons of Simulation, cont,

- The question of how many trials or runs of a simulation can become a complex statistical issue
- Fortunately, with today's computing power, this is not a paramount issue for most problems
- In practice, one should recognize that gaining managerial confidence in a simulation model will depend on at least three factors:
  1. A good understanding of the underlying management problem
  2. One's ability to use the concepts of probability and statistics correctly
  3. One's ability to communicate these concepts effectively