Stochastic processes

- t is time.
- $X()$ is a *stochastic process* if $X(t)$ is a random variable for every t.
- t is a scalar — it can be discrete or continuous.
- $X(t)$ can be discrete or continuous, scalar or vector.

- A *Markov process* is a stochastic process in which the probability of finding X at some value at time $t + \delta t$ depends only on the value of X at time t.

- Or, let $x(s), s \leq t$, be the history of the values of X before time t and let A be a possible value of X. Then

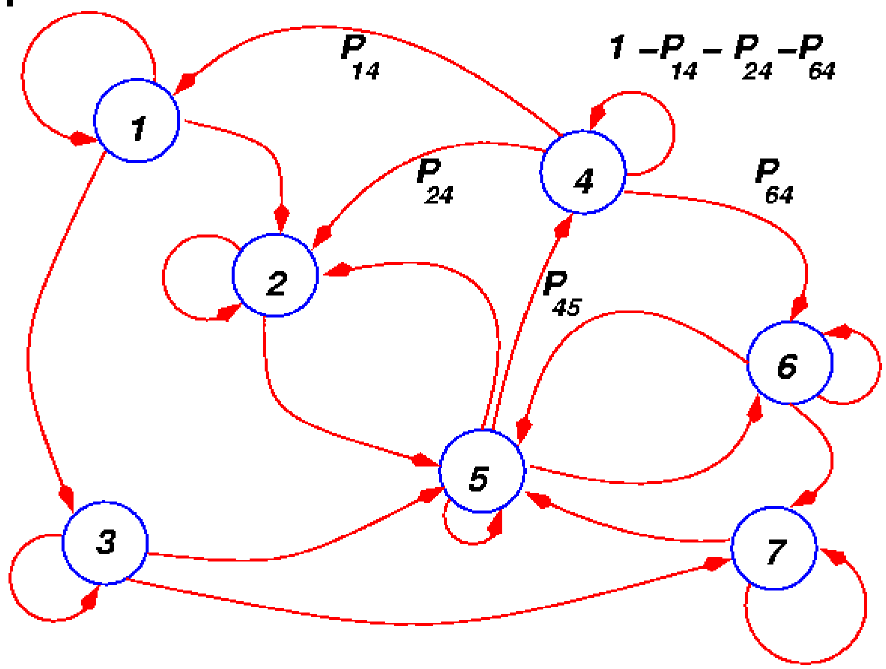
  prob$\{X(t + \delta t) = A | X(s) = x(s), s \leq t\}$ =
  prob$\{X(t + \delta t) = A | X(t) = x(t)\}$

- In words: if we know what X was at time t, we don't gain any more useful information about $X(t + \delta t)$ by *also* knowing what X was at any time earlier than t.

- States can be numbered 0, 1, 2, 3, ... *(or with multiple indices if that is more convenient)*.
- Time can be numbered 0, 1, 2, 3, ... *(or 0, Δ, 2Δ, 3Δ, ... if more convenient)*.
- The probability of a transition from j to i in one time unit is often written P_{ij}, where

$$P_{ij} = \text{prob}\{X(t+1) = i | X(t) = j\}$$

Transition graph



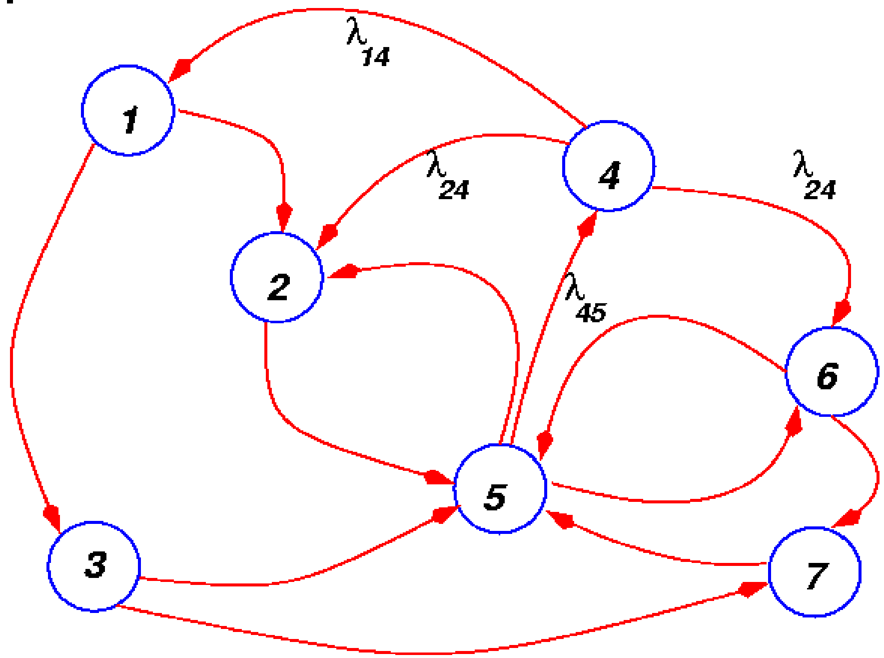
P_{ij} is a probability. Note that $P_{ii} = 1 - \sum_{m, m \neq i} P_{mi}$.

- Define $\pi_i(t) = \text{prob}\{X(t) = i\}$
- Transition equations: $\pi_i(t+1) = \sum_j P_{ij}\pi_j(t)$.
- *Steady state:* $\pi_i = \lim_{t\to\infty} \pi_i(t)$, if it exists.
- Steady-state transition equations: $\pi_i = \sum_j P_{ij}\pi_j$.
- Steady-state balance equations:
  $\pi_i \sum_{m,m\neq i} P_{mi} = \sum_{j,j\neq i} P_{ij}\pi_j$

- States can be numbered 0, 1, 2, 3, ... *(or with multiple indices if that is more convenient)*.
- Time is a real number, defined on $(-\infty, \infty)$ or a smaller interval.
- The probability of a transition from j to i during $[t, t + \delta t]$ is approximately $\lambda_{ij}\delta t$, where δt is small, and

$$\lambda_{ij}\delta t \approx \text{prob}\{X(t + \delta t) = i | X(t) = j\}$$

Transition graph

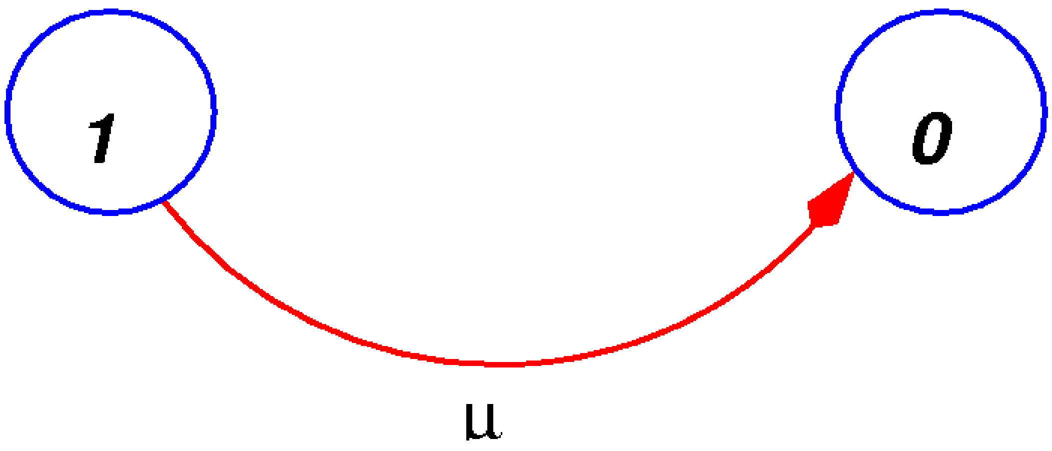


λ_{ij} is a probability *rate*. $\lambda_{ij}\delta t$ is a probability.

- Define $\pi_i(t) = \text{prob}\{X(t) = i\}$
- It is convenient to define $\lambda_{ii} = -\sum_{j \neq i} \lambda_{ji}$
- Transition equations: $\frac{d\pi_i(t)}{dt} = \sum_j \lambda_{ij} \pi_j(t)$.
- *Steady state:* $\pi_i = \lim_{t \to \infty} \pi_i(t)$, if it exists.
- Steady-state transition equations: $0 = \sum_j \lambda_{ij} \pi_j$.
- Steady-state balance equations:

  $\pi_i \sum_{m, m \neq i} \lambda_{mi} = \sum_{j, j \neq i} \lambda_{ij} \pi_j$

Exponential random variable: the time to move from state 1 to state 0.



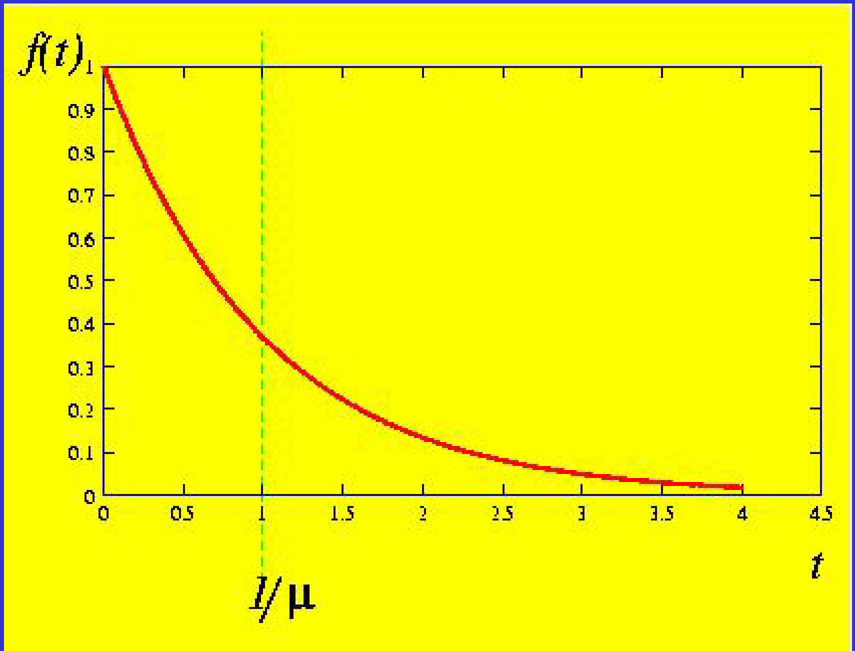
The Exponential Distribution

T~exp(μ)

$f(t) = \mu e^{-\mu t}, \; t \geq 0 \quad \mu > 0$

$F(t) = Pr(T \leq t) = 1 - e^{-\mu t}$

$E[T] = \frac{1}{\mu} \qquad Var(T) = \frac{1}{\mu^2}$



Properties

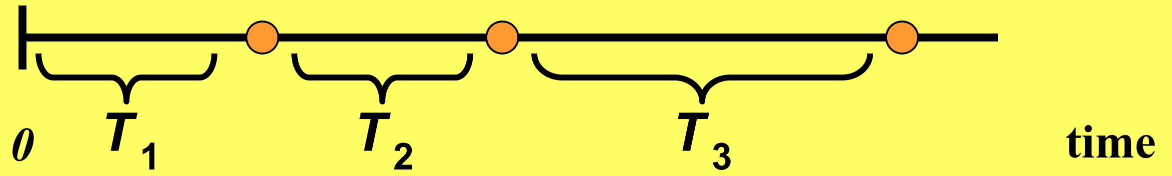
1) Memoryless (or Markov) Property $Pr(T > t + s \mid T > s) = Pr(T > t)$

2) If $T_1, \ldots, T_n$ independent and $T_i \sim \exp(\mu_i)$

then $Min\{T_1, \ldots, T_n\} \sim \exp(\mu_1 + \ldots + \mu_n)$

3) $Pr\{T \leq t + \Delta t \mid T > t\} \cong \mu \Delta t$

The Poisson Process



$$T_i \sim iid\ \exp(\lambda)$$

N(t) = number of events by time *t*

$$= \begin{cases} 0 & \text{if } T_1 > t \\ \sup\{n \geq 1 : T_1 + \ldots + T_n \leq t\} & \text{if } T_1 \leq t \end{cases}$$

N(t) is a Poisson Process

$$\Pr\{N(t) = n\} = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

N(t)~ Poisson random variable with mean λt

λ = Arrival rate

M/M/1 Queue

1) Customers arrive according to a Poisson Process

λ = arrival rate

2) There is one server

3) Service times are exponential

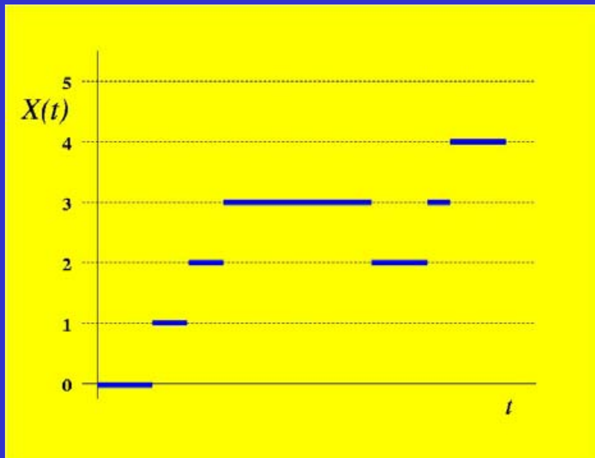
μ = service rate

4) First-come first-served (FCFS)

5) Infinite waiting room

6) Long-run behavior

Birth and Death Process



X(t) = # of customers in a queueing system

Continuous time

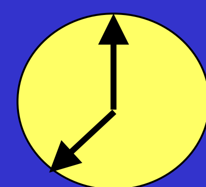
Discrete State Space = {0,1,2,…..}

λ_i , $i \geq 0$ birth (arrival) rates

μ_i , $i \geq 1$ death (service) rates

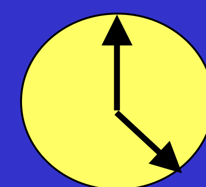
2 independent clocks when in state i

Arrival



$B_i \sim \exp(\lambda_i)$

Service



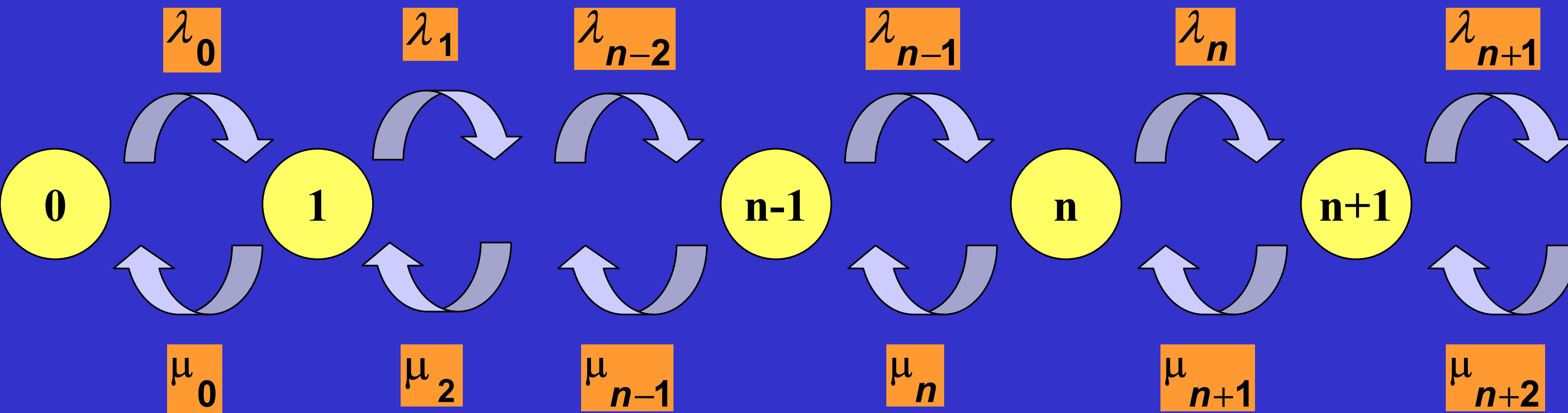
$D_i \sim \exp(\mu_i)$

H_i = Holding time in state *i* $= Min(B_i, D_i) \sim \exp(\lambda_i + \mu_i)$

$$\Pr(arrival) = \Pr(B_i < D_i) = \frac{\lambda_i}{\lambda_i + \mu_i}$$

$$\Pr(departure) = \Pr(D_i < B_i) = \frac{\mu_i}{\lambda_i + \mu_i}$$

Steady-State Solution to Birth and Death Process



Flow Balance Equations: Flow into state *n* = Flow out of state *n*

$P_n =$ Steady-state probability of being in state n

$$\lambda_0 P_0 = \mu_1 P_1$$

$$\lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1} = (\lambda_n + \mu_n) P_n$$

Steady-State Solution to Birth and Death Process... Continued

Solution:

$$P_1 = \frac{\lambda_0}{\mu_1} P_0$$

$$P_n = \frac{\lambda_{n-1}}{\mu_n} P_{n-1}$$

$$P_{n+1} = \frac{\lambda_n}{\mu_{n+1}} P_n + \frac{1}{\mu_{n+1}} \underbrace{\left(\mu_n P_n - \lambda_{n-1} P_{n-1}\right)}_{=0}$$

So:

$$P_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} P_0 \triangleq \pi_n P_0$$

$$\pi_0 = 1$$

$$\sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} \pi_n P_0 = 1$$

So:

$$P_0 = \left(\sum_{n=0}^{\infty} \pi_n\right)^{-1}$$

Solution to M/M/1 Queue

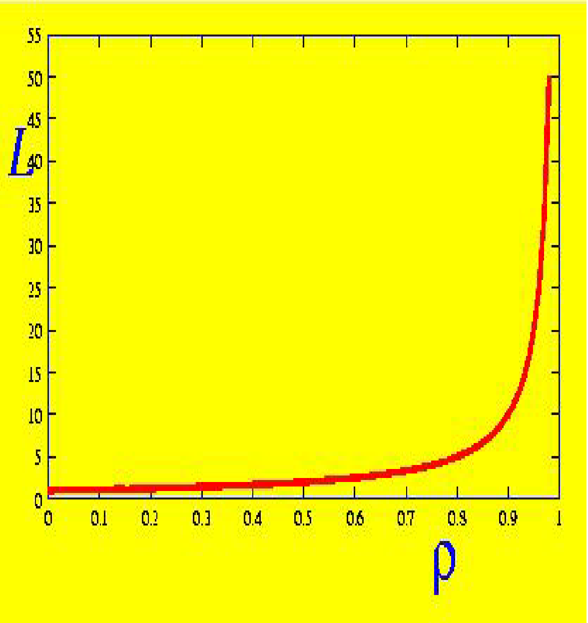
M/M/1 Queue = Birth - Death Process with $\lambda_n = \lambda, \mu_n = \mu$

$\rho = \frac{\lambda}{\mu}$ = utilization rate

$P_n = \rho^n P_0$

$P_0 = \left(\sum_{n=0}^{\infty} \rho^n\right)^{-1} = 1 - \rho$

If ρ < 1



$P_n = \Pr(n \text{ customers in system}) = \rho^n (1 - \rho)$

~ geometric distribution

$L = E[\text{number of customers in system}] = \frac{\rho}{1 - \rho}$

Little's Formula

W = Steady-state mean waiting time in the system

Little's Formula: $L = \lambda W$

$$W = \frac{1}{\mu(1-\rho)} \text{ for } M/M/1$$

L_q = steady - state mean number of customers in queue

W_q = steady - state mean waiting time in queue

$$W_q = W - \frac{1}{\mu} = \frac{\rho}{\mu(1-\rho)} \text{ for M/M/1}$$

$$L_q = \lambda W_q = \frac{\rho^2}{1-\rho} \text{ for M/M/1}$$

- Generalization of Model M/M/c System
- c identical serves in parallel, each with service rate μ
- See Exhibit 3 of Harvard note for exact performance measures

Approximations for Single-Station Models

M/M/c

$$W_q = \frac{\rho^{\sqrt{2(c+1)}-1}}{c\mu(1-\rho)}$$

G/G/1

$$scv = \text{squared coefficient of variation} = \frac{\text{Var}}{\text{mean}^2}$$

$$= 1 \text{ if exponential}$$

$$c_a^2 = \text{scv of interarrival time distribution}$$

$$c_s^2 = \text{scv of service time distribution}$$

$$W_q = \left(\frac{c_a^2 + c_s^2}{2}\right)\frac{\rho}{\mu(1-\rho)}$$

G/G/c

$$W_q = \left(\frac{c_a^2 + c_s^2}{2}\right)\frac{\rho^{\sqrt{2(c+1)}-1}}{c\mu(1-\rho)}$$